THE OLYMPIAD CORNER

No. 250

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We begin this number of the Corner with the problems of the 38th Mongolian Mathematical Olympiad, Final Round. We thank Bill Sands, Chair of the International Mathematical Olympiad Committee of the Canadian Mathematical Society, for obtaining them for the Corner.

38th MONGOLIAN MATHEMATICAL OLYMPIAD
Final Round, May 2002
10th Grade

1. Let \( n \) and \( k \) be natural numbers. Find the least possible value for the cardinality of a set \( A \) that satisfies the following condition: There exist subsets \( A_1, \ldots, A_n \) of \( A \) such that any union of \( k \) of the \( A_i \) is equal to \( A \), but any union of \( k - 1 \) of them is not equal to \( A \).

2. For a natural number \( p \), one can move between two integer points in a plane when the distance between the points is \( p \). Find all primes \( p \) for which the point \((2002, 38)\) can be reached from the point \((0, 0)\) using permitted moves.

3. The incircle of triangle \( ABC \) with \( AB \neq BC \) touches sides \( BC \) and \( AC \) at points \( A_1 \) and \( B_1 \), respectively. The segments \( AA_1 \) and \( BB_1 \) meet the incircle at \( A_2 \) and \( B_2 \), respectively. Prove that the lines \( AB, A_1 B_1, \) and \( A_2 B_2 \) are concurrent.

4. Given are 131 distinct natural numbers, each with prime divisors not exceeding 42. Prove that four of them can be chosen whose product is a perfect square.

5. Let \( a_0, a_1, \ldots \) be an infinite sequence of positive numbers. Show that \( 1 + a_n > \sqrt[2]{a_{n-1}} \) for infinitely many positive integers \( n \).

6. Let \( A_1, B_1, \) and \( C_1 \) be the respective mid-points of the sides \( BC, AC, \) and \( AB \) of triangle \( ABC \). Take a point \( K \) on the segment \( C_1 A_1 \) and a point \( L \) on the segment \( A_1 B_1 \) such that

\[
\frac{C_1K}{KA_1} = \frac{BC + AC}{AC + AB} \quad \text{and} \quad \frac{A_1L}{LB_1} = \frac{AC + AB}{AB + BC}.
\]

Let \( S = BK \cap CL \). Show that \( \angle C_1 A_1 S = \angle B_1 A_1 S \).
Next we present the problems of the 19th Balkan Mathematical Olympiad written in Antalya, Turkey, April 2002. My thanks again go to Bill Sands, chair of the International Olympiad Committee of the Canadian Mathematical Society for collecting these questions for our use.

19th BALKAN MATHEMATICAL OLYMPIAD

Antalya, Turkey
April 27, 2002

1. Let \( A_1, A_2, \ldots, A_n (n \geq 4) \) be points in the plane such that no three of them are collinear. Some pairs of distinct points among \( A_1, A_2, \ldots, A_n \) are connected by line segments in such a way that each point is connected to at least three others. Prove that there exists \( k > 1 \) and distinct points \( X_1, X_2, \ldots, X_{2k} \in \{ A_1, A_2, \ldots, A_n \} \) such that for each \( 1 \leq i \leq 2k - 1 \), \( X_i \) is connected to \( X_{i+1} \) and \( X_{2k} \) is connected to \( X_1 \).

2. The sequence \( a_1, a_2, \ldots, a_n, \ldots \) is defined by
\[
a_1 = 20, \quad a_2 = 30, \quad a_{n+2} = 3a_{n+1} - a_n, \quad \text{for } n > 1.
\]
Find all positive integers \( n \) for which \( 1 + 5a_n a_{n+1} \) is a perfect square.

3. Two circles with different radii intersect at two points \( A \) and \( B \). The common tangents of these circles are \( MN \) and \( ST \), where the points \( M \) and \( S \) are on one of the circles, and \( N \) and \( T \) are on the other.
Prove that the orthocentres of the triangles \( AMN \), \( AST \), \( BMN \), and \( BST \) are the vertices of a rectangle.

4. Find all functions \( f : \mathbb{N} \to \mathbb{N} \) such that for each \( n \in \mathbb{N} \)
\[
2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2003.
\]
(\( \mathbb{N} \) is the set of all positive integers.)

As a third set, we give the problems of the Bulgarian Mathematical Olympiad, Final Round, written May 17-18, 2003. My thanks go to Andy Liu of the University of Alberta, Canadian Team Leader to the IMO 2003 in Japan, for collecting them for our use.

BULGARIAN MATHEMATICAL OLYMPIAD


1. Let \( x_1, x_2, \ldots, x_5 \) be real numbers. Find the least positive integer \( n \) with the following property: if some \( n \) distinct sums of the form \( x_p + x_q + x_r \) (with \( 1 \leq p < q < r \leq 5 \)) are equal to 0, then \( x_1 = x_2 = \cdots = x_5 = 0 \).
2. Let $H$ be an arbitrary point on the altitude $CP$ of the acute triangle $ABC$. The lines $AH$ and $BH$ intersect $BC$ and $AC$ in $M$ and $N$, respectively.
(a) Prove that $\angle NPC = \angle MPC$.
(b) Let $O$ be the common point of $MN$ and $CP$. An arbitrary line through $O$ meets the sides of the quadrilateral $CNHM$ in $D$ and $E$. Prove that $\angle EPC = \angle DPC$.

3. Given the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_1 = y_2 = 1$ and

$$y_{n+2} = (4k-5)y_{n+1} - y_n + 4 - 2k, \quad n \geq 1,$$

find all integers $k$ such that every term of the sequence is a perfect square.

4. A set $A$ of positive integers is called uniform if, after any of its elements is removed, the remaining ones can be partitioned into two subsets with equal sums of their elements. Find the least positive integer $n > 1$ such that there exists a uniform set $A$ with $n$ elements.

5. Let $a, b, c$ be rational numbers such that $a + b + c$ and $a^2 + b^2 + c^2$ are equal integers. Prove that the number $a \cdot b \cdot c$ can be written as the ratio of a perfect cube and a perfect square which are relatively prime.

6. Determine all polynomials $P(x)$ with integer coefficients such that, for any positive integer $n$, the equation $P(x) = 2^n$ has an integer root.

To add to your puzzling pleasure over the seasonal break we give the problems of the Chinese Mathematical Olympiad 2003. Thanks again go to Andy Liu for collecting them for our use.

**CHINESE MATHEMATICAL OLYMPIAD 2003**

1. Let $I$ be the incentre of triangle $ABC$, and let $B_1$ and $C_1$ be the midpoints of $AC$ and $AB$, respectively. The extension of $B_1I$ cuts $C_1B$ at $B_2$, and the extension of $C_1I$ cuts the extension of $B_1C$ at $C_2$. Further, $B_2C_2$ cuts $BC$ at $K$. Let $H$ be the orthocentre of triangle $ABC$, and let $A_1$ be the circumcentre of triangle $HBC$. Prove that $A, I,$ and $A_1$ are collinear if and only if triangles $BKB_2$ and $CKC_2$ have equal area.

2. Determine the maximum size of a subset $S$ of $\{1, 2, \ldots, 100\}$ such that, for any $a$ and $b$ in $S$, there exist $c$ and $d$ in $S$, distinct from $a$ and $b$, with $c$ relatively prime to both $a$ and $b$, and $d$ relatively prime to neither $a$ nor $b$.

3. Let $n \geq 2$ be a fixed integer. For $i = 1, 2, \ldots, n$, let $\theta_i$ be such that $0 < \theta_i < \frac{\pi}{2}$ and

$$\tan \theta_1 \tan \theta_2 \cdots \tan \theta_n = 2^{n/2}.$$
Determine, in terms of $n$, the smallest positive number $\lambda$ such that
\[ \cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_n \leq \lambda. \]

4. Determine all triples $(a, m, n)$ of positive integers such that $a \geq 2$, $m \geq 2$, and $a^n + 203$ is a multiple of $a^m + 1$.

5. Ten candidates with different qualifications apply for a single job vacancy. The employer has decided on the following protocol. The candidates will be arranged in random order and interviewed one by one, at which time their qualifications will be assessed. The first three candidates will be rejected automatically. Thereafter, the first candidate better qualified than any of the first three will be hired. If there is no such candidate, then the tenth and last candidate will be hired.

Let the candidates be ranked from 1 to 10 in descending order of qualifications. For $1 \leq k \leq 10$, let $A_k$ denote the number of the 10 possible orders of interviews which would lead to the hiring of the $k^{th}$-ranked candidate. Prove that
(a) $A_1 > A_2 > \cdots > A_8 = A_9 = A_{10}$;
(b) the probability that the hired candidate is ranked 1, 2, or 3 is greater than $\frac{7}{16}$;
(c) the probability that the hired candidate is ranked 8, 9, or 10 is not greater than $\frac{1}{16}$.

6. Let $c_1, c_2, c_3,$ and $c_4$ be positive numbers such that $c_1c_2 + c_3c_4 = 1$. Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, and $(x_4, y_4)$ be points on the circle $x^2 + y^2 = 1$. Prove that
\[ (c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4)^2 + (c_1x_4 + c_2x_3 + c_3x_2 + c_4x_1)^2 \leq 2 \left( \frac{c_1^2 + c_2^2}{c_1c_2} + \frac{c_3^2 + c_4^2}{c_3c_4} \right). \]

Next we turn to the March 2004 number of the Corner and readers' solutions to problems of the Hungary-Israel Binational Mathematical Competition 2001, Individual Competition [2004 : 82].

1. Find positive integers $x, y, z$ such that $x > z > 1999 \cdot 2000 \cdot 2001 > y$ and $2000x^2 + y^2 = 2001z^2$.

Solution by Michel Bataille, Rouen, France.

If $(x, y, z)$ is such a triple of positive integers, then
\[ \frac{2000(x - z)}{z - y} = \frac{z + y}{x + z}. \]
Denoting this rational by \( \frac{m}{n} \) (with \( m, n \in \mathbb{N} \)), we have
\[
n(y + z) = m(x + z) \quad \text{and} \quad 2000n(x - z) = m(z - y).
\]
These equations are satisfied by the integers
\[
x = d(2000n^2 + 2mn - m^2), \quad y = d(m^2 + 4000mn - 2000n^2),
\]
\[
z = d(m^2 + 2000n^2)
\]
for any integer \( d \). Taking \( m = 1, n = 2, \) and \( d = 1999 \times 2000 \), we find that
\[
x = 1999 \times 2000 \times 8003, \quad y = 1999 \times 2000, \quad z = 1999 \times 2000 \times 8001.
\]
These integers \( x, y, \) and \( z \) satisfy all the given conditions.

2. Points \( A, B, C, D \) lie on the line \( \ell \), in that order. Find the locus of points \( P \) in the plane for which \( \angle APB = \angle CPD \).

Solved by Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya’s version.

Let \( P \) be a point such that \( \angle APB = \angle CPD \). Let \( Q \) be the point such that \( QC \parallel PA \) and \( QD \parallel PB \). Then \( \triangle PAB \sim \triangle QCD \), implying that \( \angle CQD = \angle APB = \angle CPD \). Thus, \( P, C, D, \) and \( Q \) are concyclic.

Case 1. \( AB \neq CD \).

Then \( \triangle PAB \equiv \triangle QCD \). Thus, \( PB = QD \). Therefore, \( PBDQ \) is a parallelogram. Hence, \( PQ \parallel BD \); that is \( PQ \parallel CD \). Since \( P, C, D, Q \) are concyclic, we have \( PC = QD \). Hence, \( PB = PC \). Thus, the locus of \( P \) is the perpendicular bisector of \( BC \) (excluding the mid-point of \( BC \)).

Case 2. \( AB \neq CD \).

Let \( O \) be the point of intersection of \( PQ \) with \( AD \). Since \( PA \parallel QC \), we have \( OA : OC = PA : QC \). Also, since \( \triangle PAB \sim \triangle QCD \), we have \( PA : QC = AB : CD \). Therefore, \( OA : OC = AB : CD \), which is a constant ratio. Thus, \( O \) is a fixed point.

Since \( P, C, Q, D \) are concyclic, we have \( \angle QPD = \angle QCD = \angle PAB \); that is, \( \angle OPD = \angle PAO \), from which we get \( OP^2 = OA \cdot OD \). Thus, \( OP = \sqrt{OA \cdot OD} \), a constant. Therefore, the locus of \( P \) is the circle with centre \( O \) and radius \( \sqrt{OA \cdot OD} \) (excluding the points of intersection with the line \( \ell \)).
3. Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that, for all real \( x \),

\[
f(f(x)) = f(x) + x.
\]

Solution by Pierre Bornsztein, Maisons-Laffitte, France, modified by the editor.

Note that \( f(x) = \varphi x \) and \( f(x) = -x/\varphi \) are solutions of the problem, where \( \varphi = (1 + \sqrt{5})/2 \) is the golden ratio. We will prove that there is no other solution.

Let \( f \) be a solution. The function \( f \) must be injective, because if \( f(a) = f(b) \), then \( a = f(f(a)) - f(a) = f(f(b)) - f(b) = b \). Thus, \( f \) is a bijection from \( \mathbb{R} \) onto \( f(\mathbb{R}) \). Since \( f \) is continuous, it follows that \( f \) is strictly monotonic on \( \mathbb{R} \). Moreover, \( f(f(0)) = f(0) \); whence, \( f(0) = 0 \).

Since \( f \) is monotonic, it follows that \( \lim_{x \to \pm \infty} f(x) \) is either a real number or \( \pm \infty \). If \( \lim_{x \to \pm \infty} f(x) = L \in \mathbb{R} \), then, using the continuity of \( f \), we obtain

\[
f(L) - L = \lim_{x \to \pm \infty} (f(f(x)) - f(x)) = \lim_{x \to \pm \infty} x = \pm \infty,
\]

which is absurd. We have a similar result for \( \lim_{x \to -\infty} f(x) \). Therefore, \( f \) is unbounded above and unbounded below, and \( f(\mathbb{R}) = \mathbb{R} \).

Let \( f^0 \) denote the identity function on \( \mathbb{R} \). For each positive integer \( n \), let \( f^{n+1} = f \circ f^n \) and \( f^{-(n+1)} = f^{-1} \circ f^{-n} \). Since we are given that \( f^2 = f^1 + f^0 \), it follows that \( f^{n+2} = f^{n+1} + f^n \) for all \( n \in \mathbb{Z} \). Solving this difference equation, we get

\[
f^n = g + \frac{h}{\sqrt{5}} \varphi^n + \frac{h}{\sqrt{5}} \left( -\frac{1}{\varphi} \right)^n,
\]

where \( g = \frac{1}{\varphi} f^0 + f^1 \) and \( h = \varphi f^0 - f^1 \); that is, \( g(x) = \frac{1}{\varphi} x + f(x) \) and \( h(x) = \varphi x - f(x) \). Note that for all \( x \in \mathbb{R} \), since \( \varphi > 1 \), we have

\[
\lim_{n \to -\infty} \frac{g(x)}{\sqrt{5}} \varphi^n = 0 \quad \text{and} \quad \lim_{n \to +\infty} \frac{h(x)}{\sqrt{5}} \left( -\frac{1}{\varphi} \right)^n = 0.
\]

Case 1. \( f \) is increasing.

We will show that \( h(x) = 0 \) for all \( x \in \mathbb{R} \). Then we will have \( f(x) = \varphi x \) for all \( x \in \mathbb{R} \).

First consider any \( x > 0 \). Since \( f(0) = 0 \) and \( f \) is increasing, we deduce that \( f^n(x) > 0 \) for all \( n \in \mathbb{Z} \). If \( h(x) < 0 \), then

\[
\lim_{n \to -\infty} f^{2n}(x) = \lim_{n \to -\infty} \frac{h(x)}{\sqrt{5}} \left( -\frac{1}{\varphi} \right)^{2n} = -\infty,
\]

which is impossible, since \( f^n(x) > 0 \) for all \( n \in \mathbb{Z} \). Similarly, if \( h(x) > 0 \), then \( \lim_{n \to -\infty} f^{2n+1}(x) = -\infty \), and again we have a contradiction. Therefore, \( h(x) = 0 \).
Now consider any \( x < 0 \). In this case, we have \( f^n(x) < 0 \) for all \( n \in \mathbb{Z} \). In much the same manner as above, we see that if \( h(x) \neq 0 \), then either \( \lim_{n \to -\infty} f^{2n}(x) = \infty \) or \( \lim_{n \to -\infty} f^{2n+1}(x) = \infty \), giving a contradiction. Thus, \( h(x) = 0 \).

Finally, we note that \( h(0) = 0 - f(0) = 0 \).

**Case 2.** \( f \) is decreasing.

By an argument similar to the argument in Case 1, but with \( x \to +\infty \) instead of \( x \to -\infty \), we see that \( g(x) = 0 \) for all \( x \in \mathbb{R} \). Then \( f(x) = -x/\varphi \) for all \( x \in \mathbb{R} \).

**4.** Let \( P(x) = x^3 - 3x + 1 \). Find the polynomial \( Q \) whose roots are the fifth power of the roots of \( P \).

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein’s write-up.

Let \( a, b, c \) be the roots of \( P \). Then \( a + b + c = 0 \), \( ab + bc + ca = -3 \), and \( abc = -1 \). The required polynomial \( Q \) is given by

\[
Q(x) = (x - a^5)(x - b^5)(x - c^5) = x^3 - (a^5 + b^5 + c^5)x^2 + (a^5b^5 + b^5c^5 + c^5a^5)x - a^5b^5c^5
\]

where \( S_5 = a^5 + b^5 + c^5 \) and \( T_5 = a^5b^5 + b^5c^5 + c^5a^5 \).

Generalizing our definition of \( S_n \), we define \( S_n = a^n + b^n + c^n \), for each positive integer \( n \). Observe that \( T_5 = \frac{1}{2}(S_5^2 - S_{10}) \). Therefore, the determination of \( Q(x) \) will be complete if we can calculate \( S_5 \) and \( S_{10} \).

We have \( S_1 = a + b + c = 0 \) and \( S_2 = (a + b + c)^2 - 2(ab + bc + ca) = 6 \). Since \( a, b, \) and \( c \) are roots of \( P \), they each satisfy the equation \( x^3 = 3x - 1 \). It follows that \( S_{n+3} = 3S_{n+1} - S_n \) for all \( n \geq 0 \). Therefore,

\[
S_3 = 3 \times 0 - 3 = -3 \\
S_4 = 3 \times 6 - 0 = 18 \\
S_5 = 3 \times (-3) - 6 = -15 \\
S_6 = 3 \times 18 - (-3) = 57 \\
S_7 = 3 \times (-15) - 18 = -63 \\
S_8 = 3 \times 57 - (-15) = 186 \\
S_{10} = 3 \times 186 - (-63) = 621.
\]

Now we have \( T_5 = \frac{1}{2}(S_5^2 - S_{10}) = \frac{1}{2}(15^2 - 621) = -198 \). Thus,

\[
Q(x) = x^3 + 15x^2 - 198x + 1.
\]
We also give the method used by Díaz-Barrero.

We consider the companion matrix of the polynomial $P(x)$, namely,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 0 \end{pmatrix},$$

for which the characteristic polynomial is $\det(xI - A) = x^3 - 3x + 1$. For each integer $k \geq 1$, the zeroes of the characteristic polynomial of $A^k$ are the $k^{th}$ powers of the zeroes of $P(x)$. Taking this into account, we find that $Q(x) = \det(xI - A^5)$. It is an easy exercise to see that

$$A^5 = \begin{pmatrix} -3 & 9 & -1 \\ 1 & -6 & 9 \\ -9 & 28 & -6 \end{pmatrix}.$$

Then

$$Q(x) = \det(xI - A^5) = \begin{vmatrix} x + 3 & -9 & 1 \\ -1 & x + 6 & -9 \\ 9 & -28 & x + 6 \end{vmatrix} = x^3 + 15x^2 - 198x + 1.$$

**5.** A triangle $ABC$ is given. The mid-points of sides $AC$ and $AB$ are $B_1$ and $C_1$, respectively. The centre of the incircle of $\triangle ABC$ is $I$. The lines $B_1I$, $C_1I$ meet the sides $AB$, $AC$ at $B_2$, $C_2$, respectively. Given that the areas of $\triangle ABC$ and $\triangle AB_2C_2$ are equal, what is $\angle BAC$?

[Ed. The line $C_1I$ was originally stated in error as $B_2I$. It has been corrected above.]

Solved by Toshio Seimiya, Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya’s solution.

We set $BC = a$, $CA = b$, $AB = c$, $AC_2 = x$ and $AB_2 = y$. Since $[AB_2C_2] = [ABC]$ (where $[PQR]$ denotes the area of triangle $PQR$), we have $xy = bc$.

Let $D$ and $E$ be the intersections of $BI$ and $CI$ with $AC$ and $AB$, respectively. Since $AI$ and $CI$ are the bisectors of $\angle BAD$ and $\angle BCD$, we have

$$\frac{BI}{ID} = \frac{AB}{AD} = \frac{BC}{CD} = \frac{AB + BC}{AD + CD} = \frac{AB + BC}{AC} = \frac{a + c}{b}.$$

Similarly, $\frac{CI}{IE} = \frac{a + b}{c}$.

Since $BD$ is the bisector of $\angle ABC$, we have


From this, we get $AD = \frac{bc}{a + c}$. Similarly, $AE = \frac{bc}{a + b}$. 

Case 1. \( c = a \).

Then \( B_1 \) coincides with \( D \), and therefore \( C_2 \) coincides with \( B \). Thus, \( x = c \) and \( y = b \). Hence, \( B_2 \) coincides with \( C \), and \( C_1 \) coincides with \( E \). Thus, \( AC = BC \); that is \( b = a \). Thus, \( a = b = c \). Hence, \( \triangle ABC \) is equilateral, and we get \( \angle BAC = 60^\circ \).

Case 2. \( a < c \).

Then \( x > c \), and \( y < b \), from which we get \( a > b \). By Menelaus' Theorem for \( \triangle ABD \), we have

\[
\frac{AC_2}{C_2B} \cdot \frac{BI}{ID} \cdot \frac{DB_1}{B_1A} = 1.
\]

(1)

Since \( DB_1 = AD - AB_1 = \frac{bc}{a + c} - b = \frac{b(c - a)}{2(a + c)} \), we have \( \frac{DB_1}{B_1A} = \frac{c - a}{a + c} \).

Now (1) becomes

\[
\frac{x}{x - c} \cdot \frac{a + c}{b} \cdot \frac{c - a}{a + c} = 1;
\]

whence, \( \frac{x}{x - c} = \frac{b}{c - a} \). Thus, \( x = \frac{bc}{b - c + a} \). Similarly, \( y = \frac{bc}{c - b + a} \).

Since \( xy = bc \), we get

\[
bc = \frac{b^2c^2}{(b - c + a)(c - b + a)}.
\]

Then \( a^2 - (b - c)^2 = bc \), or equivalently, \( a^2 = b^2 - bc + c^2 \). Since \( a^2 = b^2 + c^2 - 2bc \cos \angle BAC \), we get \( \cos \angle BAC = \frac{1}{2} \). Thus, \( \angle BAC = 60^\circ \).

Case 3. \( a > c \).

As in case 2, we can prove that \( \angle BAC = 60^\circ \).
Now we present reader’s solutions to the Team Competition of the Hungary-Israel Binational Mathematical Competition given [2004:82–83].

In the following questions, $G_n$ is a simple undirected graph with $n$ vertices, $K_n$ is the complete graph with $n$ vertices, $K_{n,m}$ is the complete bipartite graph with $m$ vertices in one of the two partite sets and $n$ vertices in the other, and $C_n$ is a circuit with $n$ vertices. The number of edges in the graph $G_n$ is denoted $e(G_n)$.

1. The edges of $K_n$, $n \geq 3$, are coloured with $n$ colours, and every colour appears at least once. Prove that there is a triangle whose sides are coloured with 3 different colours.

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and the Samford University Problem Solving Group, Birmingham, AL, USA. We give the solution by the Samford Group.*

Since $K_3$ is a triangle, if it is coloured with 3 different colours, then it is a tri-coloured triangle.

The rest of the proof is by induction. Suppose that, for some $n$, if $K_n$ is coloured with at least $n$ colours, then $K_n$ contains a tri-coloured triangle.

Now, suppose $K_{n+1}$ is coloured with $n+1$ colours. Let $v$ be any vertex in $K_{n+1}$. Either $v$ is, or is not, incident on at least two edges $e_1$ and $e_2$ such that $e_1$ is the only edge coloured $c_1$ and $e_2$ is the only edge coloured $c_2$.

If so, then let $v_1$ and $v_2$ be the vertices adjacent to $v$ via $e_1$ and $e_2$, respectively. Then the triangle with vertices $v$, $v_1$ and $v_2$ is a tri-coloured triangle, as the colour of the edge connecting $v_1$ and $v_2$ cannot be either $c_1$ or $c_2$.

If not, then remove $v$ and all its incident edges from $K_{n+1}$, creating a copy of $K_n$. Since $v$ is incident on at most one edge $e$ in $K_{n+1}$ such that $e$ is the only edge of a particular colour, the resulting copy of $K_n$ is coloured with at least $n$ different colours. By the induction hypothesis, then, there is a tri-coloured triangle in this copy of $K_n$, and therefore in its supergraph $K_{n+1}$.

2. An integer $n \geq 5$ is given. If $e(G_n) \geq \frac{n^2}{4} + 2$, prove that there exist two triangles which have exactly one common vertex.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France.*

We will use induction on $n$.

For $n = 5$, let $A, B, C, D, E$ be the vertices. Since $\frac{5^2}{4} + 2 = 8.25$, we have at least 9 edges. Thus, at most one pair of vertices, say $D$ and $E$, is not connected by an edge. It follows that the triangles $ABD$ and $ACE$ exist and we are done.

Now assume that the result holds for a given $n \geq 5$. Now consider a graph $G_{n+1}$ with $e(G_{n+1}) \geq \frac{(n+1)^2}{4} + 2$.

If some vertex $M$ has degree $d(M) \leq (2n+1)/4$, then, by deleting $M$
and all the edges with endpoint $M$, we obtain a subgraph $G_n$ with

$$e(G_n) \geq e(G_{n+1}) = \frac{2n + 1}{4} \geq \frac{(n + 1)^2}{4} + 2 - \frac{2n + 1}{4} \geq \frac{n^2}{4} + 2.$$ 

The induction hypothesis ensures the existence of two triangles in $G_n$, and hence in $G_{n+1}$, which have exactly one common vertex, and we are done in that case.

Therefore, we may assume that every vertex $M$ has degree $d(M) > (2n + 1)/4$. Since $2n + 1$ is odd and $d(M)$ is an integer, it follows that $d(M) \geq (2n + 2)/4 = (n + 1)/2$. Let $n = 2p + e$, where $e = 0$ or $e = 1$, according to the parity of $n$. Then, for each vertex $M$, we have

$$d(M) \geq p + 1. \quad (1)$$

Let $A$ be a vertex of $G_{n+1}$ with maximal degree $d(A) = k \geq p + 1$. Let $A$ be adjacent to $A_1, \ldots, A_k$, and not adjacent to $B_1, \ldots, B_{n-k}$ (if any).

**Case 1.** $n$ is even ($n = 2p$).

Note that $k \geq p + 1 \geq 4$.

**Lemma 1.** If $MN$ is an edge, then there exists a vertex $P$ such that $MNP$ is a triangle.

**Proof:** From (1), $M$ is adjacent to at least $p$ vertices different from $N$, and $N$ is adjacent to at least $p$ vertices different from $M$. Since $2p = n > n - 1$, it follows that there exists a vertex $P$ which is adjacent to $M$ and $N$, which proves the lemma.

From the lemma, since $AA_1$ is an edge, we may assume that $AA_1A_2$ is a triangle.

Now $AA_2$ is an edge. Then if $AA_3A_i$ is a triangle for some $i \geq 4$, the triangles $AA_3A_4$ and $AA_1A_2$ lead to the desired conclusion. From now, we assume that $A_3A_i$ is not an edge for $i \geq 4$.

From the lemma, it follows that $A_3A_1$ or $A_3A_2$ is an edge. With no loss of generality, we may assume that $AA_1A_3$ is a triangle.

Let $i \geq 4$. Since $AA_i$ is an edge, then if $AA_iA_j$ is a triangle for some $j \geq 2$ and $j \neq 3$, we deduce that the triangles $AA_3A_j$ and $AA_1A_2$ give the desired conclusion. Otherwise, the only triangle with vertices $A$ and $A_i$ is $AA_1A_i$. In that case, $A_i$ is adjacent to $A$ and to all $A_j$ for $i \geq 2$. Thus, $d(A_1) \geq k = d(A)$.

From the maximality of $k$, we deduce that $d(A_1) = k$, which means that $A_1$ is adjacent to none of the $B_i$.

On the other hand, $d(A_2) \geq p + 1 \geq 4$ and $A_2$ is not adjacent to $A_i$ for $i \geq 3$. Then, $A_2$ must be adjacent to at least one of the $B_i$, say $B_1$. From the lemma, it follows that there exists a vertex $P$ such that $A_2B_1P$ is a triangle. This vertex $P$ is neither $A$ nor $A_1$, since they are not adjacent to $B_1$. Thus, $AA_1A_2$ and $A_2B_1P$ is a pair of triangles with exactly one common vertex, and we are done.
Case 2. \( n \) is odd \((n = 2p + 1)\).

Suppose first that \( k < p + 2 \). From the maximality of \( k \) and (1), it follows that \( d(M) = k + 1 \) for each vertex \( M \) of \( G_{n+1} \). Thus,

\[
e(G_{n+1}) = \frac{1}{2} \sum_{M} d(M) = \frac{1}{2} (n+1)(p+1) = \frac{(n+1)^2}{4} < \frac{(n+1)^2}{4} + 2,\]

a contradiction. Then,

\[
k \geq p + 2.\tag{2}
\]

Lemma 2. If \( AM \) is an edge, then there exists a vertex \( P \) such that \( AMP \) is a triangle.

Lemma 2 can be proved in exactly the same way as Lemma 1.

Now assume, for a contradiction, that there is no pair of triangles which have exactly one common vertex.

Exactly as above, we deduce that \( d(A_1) = d(A) \) and \( A_1A_j \) is not an edge for all \( i, j \geq 2 \). And that \( A_1 \) is adjacent to none of the \( B_j \)'s.

For \( i = 2, \ldots, k \), since \( d(A_i) \geq p+1 \geq 3 \), it follows that \( A_i \) is adjacent to at least \( p-1 \) of the \( B_j \)'s. In particular, we must have \( p-1 \leq n-k \) which leads to \( k \leq p + 2 \). From (2), it follows that \( k = p + 2 \) and \( n-k = p-1 \). Thus, \( A_iB_j \) is an edge for \( i \geq 2 \) and \( j \geq 1 \). That leads to

\[
d(A_i) = p + 1 \quad \text{for} \quad i \geq 2.\tag{3}
\]

From our hypothesis, no pair \((B_i, B_j)\) is an edge; otherwise, the triangles \( B_iB_jA_2 \) and \( AA_1A_2 \) would have exactly one common vertex. Thus,

\[
d(B_i) = k - 1 = p + 1 \quad \text{for} \quad i \geq 1.\tag{4}
\]

Now, from (2), (3) and (4), we deduce that

\[
e(G_{n+1}) = \frac{1}{2} \left( d(A) + d(A_1) + \sum_{i=2}^{k} d(A_i) + \sum_{i=1}^{n-k} d(B_i) \right)
\]

\[
= \frac{1}{2} \left( (p + 2) + (p + 2) \sum_{i=2}^{p+2} (p + 1) + \sum_{i=1}^{p-1} (p + 1) \right)
\]

\[
= \frac{(n + 1)^2}{4} + 1,
\]

a contradiction. Thus, there are two triangles which have exactly one common vertex, and we are again done.

This ends the induction step and the proof.

3. If \( e(G_n) \geq \frac{n \sqrt{n}}{2} + \frac{n}{4} \), prove that \( G_n \) contains \( C_4 \).

Solution by Pierre Bornszein. Maisons-Laffitte, France.

Let \( A_1, \ldots, A_n \) be the vertices, and for each \( i \) let \( d_i \) be the degree of \( A_i \). Let \( e = e(G_n) \). It is well known that \( e = \frac{1}{2} \sum_{i=1}^{n} d_i \).
For each \( i \), the number of undirected paths of the form \( MA_iP \), where \( M \) and \( P \) are two distinct vertices, is \( \binom{d_i}{2} \). On the other hand, there are \( \binom{n}{2} \) pairs of distinct vertices \( M \) and \( P \). Thus, from the Pigeonhole Principle, if \( \sum_{i=1}^{n} \binom{d_i}{2} > \binom{n}{2} \), there is a pair \((M, P)\) of vertices which belongs to at least two of the above paths of length 2, and in that case we have a \( C_4 \).

Therefore, if \( G_n \) does not contain a \( C_4 \), we must have \( \sum_{i=1}^{n} \binom{d_i}{2} \leq \binom{n}{2} \); that is,

\[
\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i \leq n(n - 1) .
\]

From the inequality between arithmetic and quadratic means, we have

\[
\sum_{i=1}^{n} d_i^2 \geq \frac{1}{n} \left( \sum_{i=1}^{n} d_i \right)^2 = \frac{4e^2}{n}.
\]

Thus, if \( G_n \) does not contain a \( C_4 \), we must have \( \frac{4e^2}{n} - 2e \leq n(n - 1) \); that is, \( 4e^2 - 2ne - n^2(n - 1) \leq 0 \). Treating the left side as a quadratic in \( e \), we deduce that, if \( G_n \) does not contain a \( C_4 \), then \( e \leq \frac{n + n\sqrt{4n - 3}}{4} \).

This means that, if \( e > \frac{n + n\sqrt{4n - 3}}{4} \), then \( G_n \) contains a \( C_4 \). Since \( \frac{n + n\sqrt{4n - 3}}{4} < \frac{n\sqrt{n}}{2} + \frac{n}{4} \), we have a better result than desired.

4. (a) If \( G_n \) does not contain \( K_{2,3} \), prove that \( e(G_n) \leq \frac{n\sqrt{n}}{\sqrt{2}} + n \).

(b) Given \( n \geq 16 \) distinct points \( P_1, P_2, \ldots, P_n \) in the plane, prove that at most \( n\sqrt{n} \) of the segments \( P_iP_j \) have unit length.

**Solution by Pierre Bornschein, Maisons-Laffitte, France.**

(a) Let \( A_1, \ldots, A_n \) be the vertices, and for each \( i \) let \( d_i \) be the degree of \( A_i \). Let \( e = e(G_n) \). It is well known that \( e = \frac{1}{2} \sum_{i=1}^{n} d_i \).

For each \( i \), the number of undirected paths of the form \( MA_iP \), where \( M \) and \( P \) are two distinct vertices, is \( \binom{d_i}{2} \). On the other hand, there are \( \binom{n}{2} \) pairs of distinct vertices \((M, P)\). Thus, from the Pigeonhole Principle, if \( \sum_{i=1}^{n} \binom{d_i}{2} > 2\binom{n}{2} \) there is a pair \((M, P)\) of vertices which belongs to at least three of the above paths of length 2, and in that case we have a \( K_{2,3} \).

Thus, if \( G_n \) does not contain a \( K_{2,3} \), we must have \( \sum_{i=1}^{n} \binom{d_i}{2} \leq 2\binom{n}{2} \); that is,
\[
\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i \leq 2n(n - 1).
\]

From the inequality between arithmetic and quadratic means, we have

\[
\sum_{i=1}^{n} d_i^2 \geq \frac{1}{n} \left( \sum_{i=1}^{n} d_i \right)^2 = \frac{4e^2}{n}.
\]

Thus, if \( G_n \) does not contain a \( K_{2,3} \), we must have \( \frac{4e^2}{n} - 2e \leq 2n(n - 1) \); that is, \( 2e^2 - ne - n^2(n - 1) \leq 0 \). Solving it as a quadratic expression in \( e \), we deduce that, if \( G_n \) does not contain a \( K_{2,3} \), then \( e \leq \frac{n + n\sqrt{8n} - 7}{4} \).

Since \( \frac{n + n\sqrt{8n} - 7}{4} < \frac{4n}{4} + \frac{n\sqrt{8n}}{4} = \frac{n\sqrt{n}}{\sqrt{2}} + n \), we have a better result than desired.

(b) Let us consider the graph \( G_n \) whose vertices are the \( n \) given points, any two connected by an edge if and only if the segment they form has unit length.

Suppose that \( G_n \) contains a \( K_{2,3} \). Then there are two points, say \( A \) and \( B \), which are connected to three other common points, say \( C, D \) and \( E \). It follows that the circles with radii 1 and respective centres \( A \) and \( B \) intersect in at least three points, which is impossible. Thus, \( G_n \) does not contain a \( K_{2,3} \).

From (a), we have \( e \leq \frac{n + n\sqrt{8n} - 7}{4} \). Since \( \frac{n + n\sqrt{8n} - 7}{4} \leq n\sqrt{n} \), we have a better result than desired, which holds even if \( n \leq 16 \).

5. (a) Let \( p \) be a prime. Consider the graph whose vertices are the ordered pairs \( (x, y) \) with \( x, y \in \{0, 1, 2, \ldots, p - 1\} \), and whose edges join vertices \( (x, y) \) and \( (x', y') \) if and only if \( xx' + yy' \equiv 1 \pmod{p} \). Prove that this graph does not contain \( C_4 \).

(b) Prove that for infinitely many values of \( n \), there is a graph \( G_n \) that does not contain \( C_4 \) and satisfies \( e(G_n) \geq \frac{n\sqrt{n}}{2} - n \).

Solution by Pierre Bornstein, Maisons-Laffitte, France.

(a) First note that if \( x, y \in \{0, 1, \ldots, p - 1\} \) and \( x \equiv y \pmod{p} \), then \( x = y \).

Let \( A_1(a_1, b_1), A_2(a_2, b_2), A_3(a_3, b_3), \) and \( A_4(a_4, b_4) \) be four pairwise distinct vertices from the graph. Suppose, for a contradiction, that \( A_1A_2A_3A_4A_1 \) is a \( C_4 \). Thus,

\[
\begin{align*}
a_1a_2 + b_1b_2 & \equiv 1 \pmod{p} \quad (1) \\
a_2a_3 + b_2b_3 & \equiv 1 \pmod{p} \quad (2) \\
a_3a_4 + b_3b_4 & \equiv 1 \pmod{p} \quad (3) \\
a_4a_1 + b_4b_1 & \equiv 1 \pmod{p} \quad (4)
\end{align*}
\]
From (1) and (2), we deduce that
\[ a_2(a_3 - a_1) + b_2(b_3 - b_1) \equiv 0 \pmod{p} . \]  
\hspace{1cm} (5)

From (3) and (4), we deduce that
\[ a_4(a_3 - a_1) + b_4(b_3 - b_1) \equiv 0 \pmod{p} . \]  
\hspace{1cm} (6)

**Case 1.** \( p \) divides \( a_3 - a_1 \).

Then \( a_3 = a_1 \), and \( b_3 \neq b_1 \) (since \( A_1 \neq A_3 \)). From (5) and (6), it follows that \( p \) divides \( b_2 \) and \( b_4 \); that is, \( b_2 = b_4 = 0 \). Thus, from (1), we have \( a_1 \neq 0 \pmod{p} \). From (1) and (4), we then have \( a_1a_2 \equiv a_1a_4 \pmod{p} \), which leads to \( a_2 \equiv a_4 \pmod{p} \) and to \( a_2 = a_4 \). Hence, \( A_2 = A_4 \), a contradiction.

Similarly, we prove that \( p \) divides none of the numbers \( b_3 - b_1, a_2 - a_4, \) and \( b_2 - b_4 \) (for the two last cases, we may start from similar relations, rather than (5) and (6)).

**Case 2.** \( p \) divides \( a_2 \).

Then from above and from (5), \( p \) divides \( b_2 \) and \( a_2 = b_2 = 0 \). In that case, \( a_1a_2 + b_1b_2 \equiv 0 \pmod{p} \), which contradicts (1). Similarly, we prove that \( p \) divides none of the numbers \( a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \).

Now, it follows from (5) and (6) that
\[ a_2b_2^{-1} \equiv (b_3 - b_1)(a_1 - a_3)^{-1} \equiv a_4b_4^{-1} \pmod{p} ; \]
therefore,
\[ a_2b_4 \equiv a_4b_2 \pmod{p} . \]  
\hspace{1cm} (7)

Similarly,
\[ a_1b_3 \equiv a_3b_1 \pmod{p} . \]  
\hspace{1cm} (8)

On the other hand, from (1) and (3), we deduce that
\[ a_1a_2 - a_3a_4 + b_1b_2 - b_3b_4 \equiv 0 \pmod{p} . \]

Then, using (7) and (8), we have
\[ 0 \equiv b_3b_4(a_1a_2 - a_3a_4 + b_1b_2 - b_3b_4) \]
\[ \equiv a_3a_4b_1b_2 - b_3b_4a_3a_4 + b_3b_4(b_1b_2 - b_3b_4) \]
\[ \equiv (b_1b_2 - b_3b_4)(a_3a_4 + b_3b_4) \]
\[ \equiv b_1b_2 - b_3b_4 \pmod{p} \quad \text{from (3)} . \]

Thus, \( b_1b_2 \equiv b_3b_4 \pmod{p} \). Similarly, we have \( b_3b_2 \equiv b_1b_4 \pmod{p} \), and \( a_1a_2 \equiv a_3a_4 \pmod{p} \) and \( a_3a_2 \equiv a_1a_4 \pmod{p} \). It follows that \( b_3b_2b_4^{-1} \equiv b_1 \equiv b_3b_4b_2^{-1} \pmod{p} \), which implies that \( b_2 \equiv b_4 \pmod{p} \); that is, \( b_2 = b_4 \). Similarly, \( a_2 = a_4 \), and \( A_2 = A_4 \), a contradiction. Then, the graph contains no \( C_4 \).
(b) Let \( a, b \in \{0, 1, 2, \ldots, p - 1\} \) with \((a, b) \neq (0, 0)\).

If \( a \neq 0 \), then \( a \) is invertible modulo \( p \) and, for any given \( y \) in \( \{0, 1, 2, \ldots, p - 1\} \), the equation \( ax + by \equiv 1 \pmod{p} \) has a unique solution, namely \( x = (1 - by)a^{-1} \).

Thus, \( A(a, b) \) is connected to each of the points \( M((1 - by)a^{-1}, y) \) for \( y = 0, 1, \ldots, p - 1 \). It follows that \( d(A) \geq p \).

If \( a = 0 \), then \( b \neq 0 \), and \( A(0, b) \) is connected to each of the points \( M(x, b^{-1}) \) for \( x = 0, 1, \ldots, p - 1 \). Thus, \( d(A) \geq p \). Then \( d(A) \geq p \) for each point \( A \neq 0 \). It follows that the number \( e \) of edges of the graph defined in (a) satisfies

\[
e = \frac{1}{2} \sum_{A \neq 0} d(A) \geq \frac{1}{2} (p^2 - 1)p.
\]

Since the number \( n \) of vertices is \( n = p^2 \), we deduce that

\[
e \geq \frac{n \sqrt{n} - \sqrt{n}}{2} \geq \frac{n \sqrt{n}}{2} - n.
\]

Since we have proved that the graph contains no \( C_4 \), we are done.

---

Next we turn to solutions for problems of the Second Hong Kong (China) Mathematical Olympiad 1999 appearing [2004: 83-84].

1. [5 marks] Determine all positive rational numbers \( r \neq 1 \) such that \( r^{1/(r-1)} \) is rational.

Comment by Michel Bataille, Rouen, France. Solved by Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's comment.

This problem is known: a solution and references can be found in Mathematìcs Magazine, Vol. 69, No. 1, February 1996, p. 68.

2. [10 marks] Let \( I \) and \( O \) be the incentre and circumcentre, respectively, of \( \triangle ABC \). Assume \( \triangle ABC \) is not equilateral (so that \( I \neq O \)). Prove that \( \angle AIO \leq 90^\circ \) if and only if \( 2BC \leq AB + CA \).

Solved by Michel Bataille, Rouen, France; and Toshio Seimiyu, Kawasaki, Japan. We give Bataille's solution, adapted by the editor.

We will prove that \( \angle AIO \leq 90^\circ \) if and only if \( 2BC \leq AB + CA \) (as required), and we will also prove that equality holds on one side of this equivalence if and only if it holds on the other side. We will use standard notation for the elements of the triangle \( ABC \).

By the Cosine Law in \( \triangle AIO \), we have

\[ AO^2 = AI^2 + IO^2 - 2(AI)(IO) \cos \angle AIO. \]
Since $\angle AIO \leq 90^\circ$ if and only if $\cos \angle AIO \geq 0$, we deduce that

$$\angle AIO \leq 90^\circ \iff AO^2 \leq AI^2 + IO^2.$$  

(1)

Furthermore, equality occurs on one side of this equivalence if and only if it occurs on the other side.

We have

$$AI^2 = \frac{(s - a)^2}{\cos^2(A/2)} = \frac{(b + c - a)^2}{2(1 + \cos A)} = \frac{bc(b + c - a)}{2s},$$

where the last step makes use of the Cosine Law in $\triangle ABC$. Now $rs = \frac{abc}{4R}$ (since both expressions are equal to the area of $\triangle ABC$), and therefore,

$$AI^2 = \frac{2Rr}{a}(b + c - a).$$

We also have $AO^2 = R^2$ and $IO^2 = R^2 - 2Rr$. Hence,

$$AI^2 + IO^2 = \frac{2Rr}{a}(b + c - a) + (R^2 - 2Rr)
= AO^2 + \frac{2Rr}{a}(b + c - 2a).$$

Thus, $AI^2 + IO^2 \geq AO^2$ if and only if $b + c \geq 2a$, and equality in either of these inequalities implies equality in the other. Recalling (1), we obtain the desired conclusion.

3. [10 marks] Students have taken a test in each of $n$ subjects ($n \geq 3$). It is known that, for any subject, exactly three students got the best score in the subject, and for any two subjects, exactly one student got the best score in both of the subjects. Determine the smallest $n$ so that the above conditions imply that exactly one student got the best score in all $n$ subjects.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that $n = 8$ is the desired minimum.

Consider the graph $\mathcal{G}$ whose vertices are the students, two of them being connected by an edge if and only if they share the best score in a subject. Since, for any subject, exactly three students got the best score in the subject, it follows that to each subject corresponds a unique triangle in the graph, and that each edge belongs to a triangle. (Whenever we refer to a “triangle”, we will mean one of these triangles which correspond to the subjects.) Since, for any two subjects, exactly one student got the best score in both subjects, it follows that the graph is simple (no multiple edges). Thus, each edge belongs to exactly one triangle. Moreover, two triangles share exactly one vertex.

Now, we remark that if 4 triangles have a common vertex, then all the triangles have this vertex. Otherwise, a fifth triangle would have a common vertex with each of the 4 triangles, and these vertices would have to be distinct, which forces this fifth triangle to have more than 3 vertices.
Assume first that $n > 8$. Let $T$ be an arbitrary triangle. It shares a vertex with at least 7 other triangles. The Pigeonhole Principle ensures that it shares the same vertex with at least three other triangles, so that we are in the situation described above.

To complete the proof, we now give an example with 7 subjects for which there is no student who got the best score in all the subjects. (In the table, a number denotes a subject, a letter denotes a student, and a cross denotes the best score.)

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4. [10 marks] Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$,

$$f(x + yf(x)) = f(x) + xf(y).$$

**Solution by Michel Bataille, Rouen, France.**

The solutions are the zero function and the identity function. Clearly, these functions satisfy the given condition (which we will denote by $C$). Conversely, we show that any non-zero function $f$ satisfying $C$ is given by $f(x) = x$ for all real $x$. The proof goes through eleven steps.

(1) $f(0) = 0$.

This follows from $C$ with $x = 1$ and $y = 0$.

(2) $f(x) = 0 \implies x = 0$.

If $f(x) = 0$, then $0 = f(x) = f(x + yf(x)) = xf(y)$, which implies that $x = 0$, since we can choose $y$ such that $f(y) \neq 0$.

(3) $f(-1) = -1$.

This follows from step (2) and $C$ with $x = y = -1$.

(4) $f(x - f(x)) = f(x) - x$ for all $x$.

Apply $C$ with $y = -1$ and use (3).

(5) $f(1) = 1$.

Applying $C$ with $x = 1 - f(1)$ and $y = 1$, and using (4), we have $f(0) = -(1 - f(1))^2$; then use (1).

(6) $f(t - 1) = f(t) - 1$ for all $t$.

Apply $C$ with $x = 1$ and $y = t - 1$, and use (5).
(7) $f(u) = u \implies f(tu) = uf(t)$ for all $t$.
Suppose that $f(u) = u$. Then
\[
f(tu) = f(u + (t-1)f(u)) = f(u) + uf(t-1),
\]
using $C$ with $x = u$ and $y = t - 1$. Using (6), we get
\[
f(tu) = f(u) + u(f(t) - 1) = uf(t).
\]

(8) $f$ is odd.
Apply (7) with $u = -1$, noting (3).

(9) $f(v) = -v \implies f(tv) = -vf(t)$ for all $t$.
This is analogous to (7). Apply $C$ with $x = v$ and $y = 1 - t$, and use (8)
and (6) to get $f(1 - t) = -f(t - 1) = 1 - f(t)$.

(10) $f(x^2) = (f(x))^2$ for all $x$.
If $f(x) = x$, the result follows from (7) with $u = t = x$. Otherwise, let
$v = x - f(x)$. Then $v \neq 0$ and by (4) we have $f(v) = -v$. Now, (9) and $C$
with $y = \frac{v}{x}$ together yield
\[
f(x^2) = f(vx + xf(x)) = -vf\left(x + \frac{x}{v}f(x)\right)
= -vf(x) + x(-v)f\left(\frac{x}{v}\right) = -vf(x) + xf(x) = (f(x))^2.
\]

(11) $f(r) \geq 0$ if $r \geq 0$.
Use $x = \sqrt{r}$ in (10).

Now, let $x$ be any real number. If $x \geq f(x)$, then $f(x - f(x)) \geq 0$ by
(11); hence, by (4), we have $f(x) \geq x$, and eventually $f(x) = x$. If $x \leq f(x)$,
then $f(f(x) - x) \geq 0$ and, since $f$ is odd, we have $-f(x - f(x)) \geq 0$.
Therefore, $x - f(x) \geq 0$ and eventually $f(x) = x$. In either case, $f(x) = x$.

For an alternative solution, see Mathematics Magazine, Vol. 72, No. 3,

Next we turn to problems of the 17th Balkan Mathematical Olympiad,
appearing [2004 : 84].

1. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ with the property that
\[
f(xf(x) + f(y)) = (f(x))^2 + y,
\]
for any real numbers $x$ and $y$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-
Laftite, France; and José Luis Díaz-Barrero, Universitat Politècnica de
Catalunya, Barcelona, Spain. We give the solution by Díaz-Barrero.

The functions $f(x) = x$ and $f(x) = -x$ are solutions. We claim that
these are the only solutions.
Let $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers $x$ and $y$,
\[ f(xf(x) + f(y)) = (f(x))^2 + y. \]  
(1)

Let $f(0) = a$. Setting $x = 0$ in (1) yields
\[ f(f(y)) = a^2 + y \]
(2)
for all $y \in \mathbb{R}$. This equation shows that $f$ is a bijection. As a consequence, there exists $b$ such that $f(b) = 0$. Setting $x = b$ in (1), we get, for all $y \in \mathbb{R}$,
\[ f(f(y)) = y. \]
(3)
Comparing (2) and (3), we see that $a = 0$ (and hence $b = 0$). Then, substituting $y = 0$ into (1), we get
\[ f(xf(x)) = (f(x))^2, \]
(4)
for all $x \in \mathbb{R}$. Now, setting $x = f(t)$ in (4) gives
\[ f(tf(t)) = t^2, \]
(5)
for all $t \in \mathbb{R}$. Comparing (4) and (5), we get
\[ (f(x))^2 = x^2 \]
(6)
for all $x \in \mathbb{R}$. Thus, for each $x \in \mathbb{R}$, we have either $f(x) = x$ or $f(x) = -x$.

Suppose there exist non-zero numbers $\alpha$ and $\beta$ such that $f(\alpha) = -\alpha$ and $f(\beta) = \beta$. Then, taking $x = \alpha$ and $y = \beta$ in (1), we get
\[ f(-\alpha^2 + \beta) = \alpha^2 + \beta, \]
which contradicts (6). We conclude that $f(x) = x$ for all $x \in \mathbb{R}$ or $f(x) = -x$ for all $x \in \mathbb{R}$.

2. Let $ABC$ be a non-isosceles acute triangle, and let $E$ be an interior point of the median $AD$, with $D$ on $BC$. Let $F$ be the orthogonal projection of $E$ onto the line $BC$. Let $M$ be an interior point of the segment $EF$, and let $N$ and $P$ be the orthogonal projections of $M$ onto the lines $AC$ and $AB$, respectively. Prove that the bisectors of angles $PMN$ and $PEN$ are parallel.

Solved by Yoshio Seimiya, Kawasaki, Japan, and D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's write-up.

Since $\triangle ABC$ is not isosceles, we may assume that $AB < AC$.

As usual, we set $\alpha = \angle CAB, \beta = \angle ABC$, and $\gamma = \angle BCA$. We further set $\alpha_1 = \angle BAD$ and $\alpha_2 = \angle CAD$. Then
\[ \sin \alpha_1 : \sin \alpha_2 = \sin \beta : \sin \gamma = b : c. \]  
(1)

Let $K$ and $L$ be the points on $AC$ and $AB$, respectively, such that $EK \perp AC$ and $EL \perp AB$. Then $EK = AE \sin \alpha_2$ and $EL = AE \sin \alpha_1$. Hence, using (1), we get
\[ EK : EL = \sin \gamma : \sin \beta. \]  
(2)
Since quadrilateral $BFMP$ is cyclic, we have $\angle PME = \beta$. Similarly, $\angle NME = \gamma$ and $\angle NMQ = \alpha$, where $Q$ is the point where the production of $PM$ meets the line $AC$. Then $KN = EM \sin \gamma$ and $LP = EM \sin \beta$. This, together with (2), implies that $EK : EL = KN : LP$. Since we also have $\angle NKE = \angle PLE = 90^\circ$, it follows that

$$\triangle EKN \sim \triangle ELP.$$  

Setting $\varphi = \angle ENK = \angle EPL$, we see that

$$\angle PEN = 360^\circ - (180^\circ - \alpha) - 2(90^\circ - \varphi) = 2\varphi + \alpha.$$  

Let the bisector of $\angle PEN$ meet $BC$ at $U$. Then $\angle UEN = \varphi + \frac{1}{2}\alpha$. Let the production of $UE$ intersect $AC$ at $V$. In $\triangle NEV$, we see that $\angle EVN = \angle VNE = \varphi + \frac{1}{2}\alpha$. Since $\angle VNE = \varphi$, we have $\angle EVN = \frac{1}{2}\alpha$. This implies that $EU$ is parallel to the bisector of $\angle BAC$.

Let the bisector of $\angle NMQ$ intersect $AC$ at $S$, and the production of $SM$ intersect $AB$ at $T$. It is easy to see that $\triangle ATS$ is isosceles, since both $\angle ATS$ and $\angle AST$ have the value $90^\circ - \frac{1}{2}\alpha$. Thus, the bisector of $\angle BAC$ is perpendicular to $ST$. The bisector of $\angle PMN$ is also perpendicular to $ST$, since it is perpendicular to the bisector of $\angle NMQ$.

Thus, the bisectors of angles $PMN$ and $PEN$ are both parallel to that of $\angle BAC$. 


4. We say that a positive integer \( r \) is a power if it has the form \( r = t^s \), for some integers \( t \geq 2 \) and \( s \geq 2 \). Show that, for any positive integer \( n \), there exists a set \( A \) of \( n \) positive integers which satisfies the following conditions:

(i) Every element of \( A \) is a power.

(ii) For any \( k \) elements \( r_1, r_2, \ldots, r_k \) from \( A \) (where \( 2 \leq k \leq n \)), the number \( \frac{r_1 + r_2 + \cdots + r_k}{k} \) is a power.

**Solution by Pierre Bornszein, Maisons-Laffitte, France.**

First we prove the following lemma:

**Lemma.** For each integer \( m \geq 1 \) there exists an integer \( a > 0 \) such that each of the numbers \( a, 2a, \ldots, ma \) is a power.

**Proof:** We use induction on \( m \). For \( m = 1 \), simply choose \( a = 4 \). Let \( m \geq 1 \) be given, and assume that the result holds for this value of \( m \). Then, there exists an integer \( a > 0 \) such that, for each \( i = 1, 2, \ldots, m \), there exist two integers \( x_i \geq 2 \) and \( \alpha_i \geq 2 \) such that \( ia = x_i^{\alpha_i} \).

Let \( \alpha = \text{lcm}\{\alpha_i \mid i = 1, 2, \ldots, m\} \) and let \( b = a((m+1)a)^\alpha \). Let \( i \in \{1, 2, \ldots, m\} \). Then there exists \( \beta_i \in \mathbb{N}^+ \) such that \( \alpha = \alpha_i \beta_i \). Thus,

\[
b = \left( x_i^{\alpha_i} \right) ((m+1)a)^{\alpha \beta_i} = \left( x_i \left((m+1)a\right)^{\beta_i}\right)^{\alpha_i}.\]

Moreover, \( (m+1)b = ((m+1)a)^{\alpha+1} \). This proves the induction step. \( \blacksquare \)

Now consider a positive integer \( n \). Using the lemma with \( m = n \times n! \), it follows that there exists a positive integer \( a \) such that each of the numbers \( a, 2a, \ldots, n \times n!a \) is a power. Now let \( A = \{n!a, 2n!a, \ldots, n \times n!a\} \). Clearly \( |A| = n \) and each of the elements of \( A \) is a power. Now, let us give \( r_1, r_2, \ldots, r_k \) from \( A \), where \( r_i = ax_i/n! \) with \( x_i \in \{1, 2, \ldots, n\} \). Then

\[
r_1 + r_2 + \cdots + r_k = an!(x_1 + x_2 + \cdots + x_k).
\]

But \( 1 \leq x_1 + x_2 + \cdots + x_k \leq nk \). Thus, \( \frac{r_1 + r_2 + \cdots + r_k}{k} = ra \), where

\[
r = \frac{n!(x_1 + x_2 + \cdots + x_k)}{k}
\]

is a positive integer (since \( k \) divides \( n! \)) and \( r \leq n \times n! \). From the construction, it follows that \( \frac{r_1 + r_2 + \cdots + r_k}{k} = ra \) is a power, and we are done.

Next we turn to solutions from our readers to problems given in the April 2004 number of the Corner starting with the Israel Mathematical Olympiad 2001 given [2004 : 140–141].

1. Find all solutions of

\[
\begin{align*}
x_1 + x_2 + \cdots + x_{2000} &= 2000, \\
x_1^4 + x_2^4 + \cdots + x_{2000}^4 &= x_1^3 + x_2^3 + \cdots + x_{2000}^3.
\end{align*}
\]
Comment by Pierre Bornsztein. Maisons-Laffitte, France.

This is the same problem (with 2000 instead of 1997) as one of the Ukrainian Mathematical Olympiad problems from 1997. A solution appears in [2003: 89–90].

2. Given 2001 real numbers \( x_1, x_2, \ldots, x_{2001} \) such that \( 0 \leq x_n \leq 1 \) for each \( n = 1, 2, \ldots, 2001 \), find the maximum value of

\[
\left( \frac{1}{2001} \sum_{n=1}^{2001} x_n^2 \right) - \left( \frac{1}{2001} \sum_{n=1}^{2001} x_n \right)^2.
\]

Where is this maximum attained?

Solution by Pierre Bornsztein. Maisons-Laffitte, France.

The given expression is convex in each of the variables. Thus, it reaches its maximum when all the \( x_n \)s are end-points of the interval of study; that is, \( x_n \in \{0, 1\} \) for each \( n \).

Let \( k \) be the number values \( n \) such that \( x_n = 1 \). Then the expression is

\[
S = \frac{k}{2001} - \frac{k^2}{2001^2} = \frac{1}{2001^2} \left( \frac{2001^2}{4} - \left( k - \frac{2001}{2} \right)^2 \right).
\]

Since \( k \) is an integer, we deduce that \( S \leq \frac{1}{2001^2} \left( \frac{2001^2}{4} - \frac{1}{4} \right) = \frac{1001000}{4004001} \)

and equality occurs if and only if \( k = 1000 \) or \( k = 1001 \).

3. We are given 2001 lines in the plane, no two of which are parallel and no three of which pass through a common point. These lines partition the plane into some regions (not necessarily finite) bounded by segments of these lines. These segments are called sides, and the collection of the regions is called a map. Two regions on the map are called neighbours if they share a side.

The set of intersection points of the lines is called the set of vertices. Two vertices are called neighbours if they are found on the same side.

A legal colouring of the map is a colouring of the regions (one colour per region) such that neighbouring regions have different colours.

A legal colouring of the vertices is a colouring of the vertices (one colour per vertex) such that neighbouring vertices have different colours.

(i) What is the minimum number of colours required for a legal colouring of the map?

(ii) What is the minimum number of colours required for a legal colouring of the vertices?

Solution by Pierre Bornsztein. Maisons-Laffitte, France.

Though detailed, the statement is not really clear. I assume that in the graph theoretic wording, the regions are the faces of the representation
of the planar graph defined by the set of lines (the ‘elementary’ regions),
and that two intersection points are connected by an edge if and only if they
are end-points of some segment on a line, but with no other point between
them.
(i) We will answer the problem in the case of \( n \geq 1 \) lines, and prove that the
minimum is \( 2 \).
Clearly, we need at least two colours, since there are at least two neigh-
brouring regions.
We prove by induction on \( n \) that a legal 2-colouring does exist. It is
obvious for \( n = 1 \).
Next, suppose that, for some given \( n \), any map defined by \( n \) lines (as
in the statement of the problem) has a legal 2-colouring.
Now consider a map defined by \( n + 1 \) lines, say \( \ell_1, \ldots, \ell_{n+1} \). First,
delete \( \ell_{n+1} \), and use a legal 2-colouring for the map defined by \( \ell_1, \ldots, \ell_n \) as
given by the induction hypothesis. Now restore \( \ell_{n+1} \). This line separates
the plane into two half-planes, say \( \Pi_1 \) and \( \Pi_2 \). Some of the “old” regions
may be separated by \( \ell_{n+1} \) into two regions, one in each of the two half-planes.
Then, keep the colours of all the regions belonging to \( \Pi_1 \). And give to each
of the regions belonging to \( \Pi_2 \) the opposite colour to what it had before.
Since the initial colouring was legal, we do not have two neighbouring
regions which belong to \( \Pi_1 \) (respectively, \( \Pi_2 \)) with the same colour. And,
with the interchanging of colours made in \( \Pi_2 \), we do not have two neigh-
brouring regions (which formed an old region divided by \( \ell_{n+1} \)), one in each
of the half-planes, with the same colour.
Thus, we have found a legal 2-colouring for the map defined by the
\( n + 1 \) lines. This ends the induction step and the proof.

(ii) We will answer the problem in the case of \( n \geq 3 \) lines.
First note that it is easy to see that there always will be at least one
region which is a triangle. In fact, we may prove (see [1]) that there are at
least \( n - 2 \) regions which are triangles. It follows that we need at least three
colours for a legal colouring of the vertices.
Now we prove that a legal 3-colouring of the vertices does exist: Since
there are a finite number of lines, there are a finite number, say \( k \), of inter-
section points (namely, \( k = n(n - 1)/2 \) since no two lines are parallel, and
no three pass through the same point).
Thus, we may choose an orthogonal system of coordinates such that the
intersection points are \( M_1, \ldots, M_k \), with \( M_i = (x_i, y_i) \) and \( x_1 < \cdots < x_k \).
Now we colour the \( M_i \)'s in the increasing order of their respective subscripts,
as follows: Colour \( M_1 \) with colour \( c_1 \) and \( M_2 \) with colour \( c_2 \).
If, for \( 1 \leq i \leq k - 1 \), the points \( M_1, \ldots, M_i \) have been coloured
with only three colours, and that 3-colouring is legal for the set of already
coloured vertices, then we note that \( M_{i+1} \) has at most 4 neighbours, and
among them at most two can have an \( x \)-coordinate smaller than \( x_{i+1} \). Then,
at least one colour is not used for the coloured neighbours of \( M_{i+1} \). Thus,
give any of these unused colours to \( M_{i+1} \). This produces a legal 3-colouring
for the set of vertices $M_1, \ldots, M_{i+1}$. Then, we construct by induction a legal 3-colouring of the set of vertices, and we are done.

Reference.


4. The lengths of the sides of triangle $ABC$ are 4, 5, 6. For any point $D$ on one of the sides, drop the perpendiculars $DP$, $DQ$ onto the other two sides ($P$, $Q$ are on the sides). What is the minimal value of $PQ$?

Solved by Toshio Seimiya. Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya’s solution.

First we consider an arbitrary acute triangle $ABC$. We set $a = BC$, $b = CA$, $c = AB$, $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$, and we denote by $R$ the circumradius of $\triangle ABC$.

Suppose that $D$ is on the side $BC$. Let $H$ be the foot of the perpendicular from $A$ to $BC$, and let $P$ and $Q$ be the feet of the perpendiculars from $D$ to $AC$ and $AB$, respectively. Since $AQDP$ is a cyclic quadrilateral, Ptolemy’s Theorem implies that

$$PQ = AD \sin \angle PAQ = AD \sin \alpha \geq AH \sin \alpha.$$

Since

$$AH = AB \sin \beta = c \sin \beta = 2R \sin \beta \sin \gamma,$$

we see that $AH \sin \alpha = 2R \sin \alpha \sin \beta \sin \gamma$. Thus, the minimal value of $PQ$ is $2R \sin \alpha \sin \beta \sin \gamma$.

If $D$ is a point on either $AB$ or $AC$, the minimal value of $PQ$ is also $2R \sin \alpha \sin \beta \sin \gamma$. Thus, the minimal value of $PQ$ for any point $D$ on the perimeter of $\triangle ABC$ is $2R \sin \alpha \sin \beta \sin \gamma$; that is, $AH \sin \alpha$.

Now let $a = 6$, $b = 5$, and $c = 4$. Since $a > b > c$, we have $\alpha > \beta > \gamma$. By the Law of Cosines, $a^2 = b^2 + c^2 - 2bc \cos \alpha$, which means that $a^2 = 5^2 + 4^2 - 2 \cdot 5 \cdot 4 \cos \alpha$. Hence, $40 \cos \alpha = 5^2 + 4^2 - 6^2 = 5$, or $\cos \alpha = \frac{1}{8}$. Therefore, $\alpha$ is acute.

Since $\alpha > \beta > \gamma$, we see that $\triangle ABC$ is an acute triangle. Thus, the minimal value of $PQ$ is $AH \sin \alpha$.

Since $\cos \alpha = \frac{1}{8}$, we have $\sin^2 \alpha = 1 - \cos^2 \alpha = 1 - \frac{64}{63} = \frac{1}{62}$.

Since $AH \cdot BC = AB \cdot AC \sin \alpha$, we also have $AH \cdot a = bc \sin \alpha$; that is, $AH = \frac{bc \sin \alpha}{a}$. Hence,

$$AH \sin \alpha = \frac{bc \sin^2 \alpha}{a} = \frac{5 \cdot 4 \cdot 63}{64} = \frac{105}{32}.$$

Therefore, the minimal value of $PQ$ is $\frac{105}{32}$. 

\[ \text{Fig. 4} \]
Next we turn to readers' solutions to problems of the 21st Brazilian Mathematical Olympiad 2001 given at [2004 : 141-142].

1. Let $ABCDE$ be a regular pentagon such that the star $ACEBD$ has area 1. Let $P$ be the point of intersection of $AC$ and $BE$, and let $Q$ be the point of intersection of $BD$ and $CE$. Find the area of $APQD$.

**Solution by Geoffrey A. Kandall, Hamden, CT, USA.**

Let $R$ be the point of intersection of $AD$ and $BE$, and $S$ that of $AD$ and $CE$. Each corner angle of a regular pentagon is trisected by the two diagonals passing through its vertex into three $36^\circ$ angles. It follows that $PQ \parallel AR$ and $AP \parallel RQ$.

Let $\alpha = [PQR]$ and $\beta = [QRS]$. Then

$$[APQD] = 3\alpha + \beta = \frac{1}{2}(6\alpha + 2\beta) = \frac{1}{2}[ACEBD] = \frac{1}{2}.$$

5. There are $n$ football teams in Tumbolia. A championship is to be organized in which each team plays against every other exactly once. Every match must take place on a Sunday, and no team can play more than once on the same day.

Find the least positive integer $m$ for which it is possible to set up a championship lasting $m$ Sundays.

**Comment by Pierre Bornsztein, Maisons-Laffitte, France.**

This problem is similar to one given at the Romanian Olympiad 1978, 10th class, final round. The answer is $m = n$ if $n$ is odd, and $m = n - 1$ if $n$ is even. A solution appears in: R. Honsberger, *More Mathematical Morsels*, pp. 80-82, Mathematical Association of America.

To complete this month’s *Corner*, we give solutions to problems of the 49th Mathematical Olympiad of Lithuania 2000 given in [2004 : 142-143].

1. In a family there are four children of different ages, each age being a positive integer not less than 2 and not greater than 16. A year ago the square of the age of the eldest child was equal to the sum of the squares of the ages of the remaining children. One year from now the sum of the squares of the youngest and the oldest will be equal to the sum of the squares of the other two. How old is each child?

**Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.**

Let $x, y, z, t$ be the ages of the four children, where

$$2 \leq x < y < z < t \leq 16.$$
We have
\[(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = (t - 1)^2 \tag{1}\]
and \[(x + 1)^2 + (t + 1)^2 = (y + 1)^2 + (z + 1)^2. \tag{2}\]

By taking the sum and difference of these equations and simplifying, we get
\[2(y + z - t) = x^2 + 1 \tag{3}\]
and \[y^2 + z^2 = t^2 + 2x - 1. \tag{4}\]

Since \(z < t\) and \(y \leq 14\), it follows that \(2(y + z - t) \leq 28\). Hence, \(x^2 + 1 \leq 28\) from (3). It also follows from (3) that \(x\) is odd. Thus, we have \(x \in \{3, 5\}\).

If \(x = 3\), then (3) and (4) become \[y + z - t = 5\]
and \[y^2 + z^2 = t^2 + 5.\]

Then \(y^2 + z^2 = (y + z - 5)^2 + 5\), which gives \(yz - 5y - 5z + 15 = 0\), or \((y - 5)(z - 5) = 10\). Since \(y < z\), we must have either \(y - 5 = 2\) and \(z - 5 = 5\), or \(y - 5 = 5\) and \(z - 5 = 10\). The first case gives \(x = 3, y = 7, z = 10,\) and \(t = 12\), and the second gives \(x = 3, y = 6, z = 15,\) and \(t = 16\).

If \(x = 5\), we get \[y + z - t = 13\]
and \[y^2 + z^2 = t^2 + 9.\]

Then \(y^2 + z^2 = (y + z - 13)^2 + 9\), which gives \(yz - 13y - 13z + 89 = 0\), or \((y - 13)(z - 13) = 120\). But this equation has no solutions, since \(y - 13 \leq 1\) and \(z - 13 \leq 2\).

Therefore, the given problem has exactly two solutions, namely \[(x, y, z, t) = (3, 6, 15, 16)\] and \[(x, y, z, t) = (3, 7, 10, 12).\]

2. A sequence \(a_1, a_2, a_3, \ldots\) is defined such that \(a_n = n^2 + n + 1\) for all \(n \geq 1\). Prove that the product of any two consecutive members of the sequence is itself a member of the given sequence.

Solved by Michel Bataille, Rouen, France; Pierre Bornszein, Maisons-Lafitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's write-up.

It suffices to remark that \[a_n a_{n+1} = (n^2 + n + 1)(n^2 + 3n + 3) = n^4 + 4n^3 + 7n^2 + 6n + 3 = (n + 1)^4 + (n + 1)^2 + 1 = a_{(n+1)^2}.\]

3. In the triangle \(ABC\), the point \(D\) is the mid-point of the side \(AB\). Point \(E\) divides \(BC\) in the ratio \(BE : EC = 2 : 1\). Given that \(\angle ADC = \angle BAE\), determine \(\angle BAC\).
Let the lines $CD$ and $AE$ intersect at $O$. Let $F$ be the intersection of $BO$ and $CA$ and let $P$ be the intersection of $EF$ and $CD$.

Then, by Ceva's Theorem, \[ \frac{CF}{FA} \cdot \frac{AD}{DB} \cdot \frac{BE}{EC} = 1. \]

But we are assuming that $AD = DB$. Hence, \[ \frac{CF}{FA} = \frac{CE}{EB}, \]
which implies that $EF$ is parallel to $AB$ and \[ \angle EFC = \angle BAC. \] (1)

Triangles $OAD$ and $OEP$ have equal corresponding angles and therefore are similar; and since $\triangle OAD$ is isosceles, so is $\triangle OEP$ with $OE = OP$.

Similarly, $\triangle CFE$ is similar to $\triangle CAB$. It follows that \[ \frac{EF}{AD} = \frac{EF}{AB} = 2 \cdot \frac{EF}{AB} = 2 \cdot \frac{CE}{BC} \]

and \[ \frac{AE}{CD} = \frac{PD}{CD} = \frac{BE}{BC} \]
because $AE = AO + OE = DO + OP = PD$.

Since $BE : EC = 2 : 1$, we have $2 \cdot \frac{CE}{BC} = \frac{BE}{BC} \left(= \frac{2}{3}\right)$. Thus, \[ \frac{EF}{AD} = \frac{AE}{CD} \] (from above). Therefore, triangles $EFA$ and $DAC$ are similar (SAS) with \[ \angle EFA = \angle DAC = \angle BAC. \] (2)

Now $\angle EFA$ and $\angle EFC$ are supplementary angles and thus sum to $180^\circ$. It follows from (1) and (2) that $\angle BAC = 90^\circ$.

4. Find all the triples of positive integers $x, y, z$ with $x \leq y \leq z$ such that \[ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \]
is a positive integer.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Lafitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Bornsztein's solution.

Let $x, y, z$ satisfy the conditions in the problem.

If $x \geq 4$, then $y, z \geq 4$ and \[ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{4} < 1. \]
Thus, $x \in \{1, 2, 3\}$. 
Case 1. $x = 1$.

Then $\frac{1}{y} + \frac{1}{z}$ is a positive integer. If $y \geq 3$, then $z \geq 3$ and $\frac{1}{y} + \frac{1}{z} < 1$.

It follows that $y \in \{1, 2\}$. If $y = 1$, then $\frac{1}{z}$ is an integer, which forces $z = 1$.

If $y = 2$, then $\frac{1}{z} = k - \frac{1}{2} = \frac{2k - 1}{2}$, where $k$ is a positive integer. Thus, $z = \frac{2}{2k - 1}$, which forces $k = 1$ and $z = 2$.

Conversely, the triples $(1, 1, 1)$ and $(1, 2, 2)$ are solutions of the problem.

Case 2. $x = 2$.

Then $\frac{1}{y} + \frac{1}{z} = \frac{2k - 1}{2}$, where $k$ is a positive integer. Thus, $\frac{1}{y} + \frac{1}{z} \geq \frac{1}{2}$.

If $y \geq 5$, then $z \geq 5$ and $\frac{1}{y} + \frac{1}{z} \leq \frac{2}{5} < \frac{1}{2}$, a contradiction. It follows that $y \in \{2, 3, 4\}$.

If $y = 4$, then, as above, $z = \frac{4}{4k - 3}$, and hence, $k = 1$ and $z = 4$.

If $y = 3$, then $z = \frac{6}{6k - 5}$, implying that $k = 1$ and $z = 6$. If $y = 2$, then $\frac{1}{z} = k - 1$ is an integer, even though $z \geq 2$, which is absurd.

Conversely, the triples $(2, 4, 4)$ and $(2, 3, 6)$ are solutions of the problem.

Case 3. $x = 3$.

Then $\frac{1}{y} + \frac{1}{z} = k - \frac{1}{3} \geq \frac{2}{3}$, where $k$ is a positive integer. If $y \geq 4$, then $z \geq 4$ and $\frac{1}{y} + \frac{1}{z} \leq \frac{1}{2} < \frac{2}{3}$, a contradiction. It follows that $y = 3$. Then $\frac{1}{2} = k - \frac{2}{3} = \frac{3k - 2}{3}$. Thus, $z = \frac{3}{3k - 2}$, which forces $k = 1$ and $z = 3$.

Conversely, the triple $(3, 3, 3)$ is a solution of the problem.

In conclusion, the solutions are $(1, 1, 1)$, $(1, 2, 2)$, $(2, 4, 4)$, $(2, 3, 6)$, and $(3, 3, 3)$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following equation for all real $x$ and $y$:

$$(x + y)(f(x) - f(y)) = f(x^2) - f(y^2).$$

Find: (a) one such function; (b) all such functions.

Solved by Michel Bataille, Rouen, France; and Pierre Borsztein, Maisons-Laffitte, France. We give Bataille’s solution.

Any affine function $x \mapsto ax + b$ (for some real numbers $a$, $b$) clearly satisfies the functional equation. Conversely, we show that any solution is an affine function.

Let $f$ be any solution. Set $b = f(0)$ and $g(x) = f(x) - b$ ($x \in \mathbb{R}$). It is readily seen that $g$ is a solution as well and satisfies $g(0) = 0$. Taking $y = 0$
in the given equation (written for $g$) yields
\[(x + 0)(g(x) - g(0)) = g(x^2) - g(0) ;\]
that is, $xg(x) = g(x^2)$ for all real $x$. Substituting $-x$ for $x$ gives us
\[-xg(-x) = g(x^2),\]
and it follows that $g$ is an odd function. Thus,
\[(x + y)(g(x) - g(y)) = g(x^2) - g(y^2) = (x - y)(g(x) + g(y)).\]
Thus, $xg(y) = yg(x)$ for all $x$ and $y$. As a result, $g(x)/x$ is constant on $\mathbb{R}\setminus\{0\}$
and $g(x) = xg(1)$ for all real numbers $x$. It follows that $f(x) = xg(1) + b$
and $f$ is an affine function.

7. A line divides both the area and the perimeter of a triangle into two equal parts. Prove that this line passes through the incentre of the triangle. Does the converse statement always hold?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Toshio Seimiya, Kawasaki, Japan; D.J. Smeenk, Zalkommel, the Netherlands; and Tiku Zvonaru, Comănesti, Romania. First, we give Seimiya’s solution.

Let $ABC$ be a given triangle with incentre $I$. We may assume without loss of generality that the line in the problem intersects the sides $AB$ and $AC$ at $P$ and $Q$, respectively.

We set $a = BC$, $b = CA$, and $c = AB$. Let $D$ be the intersection of $AI$ with $BC$. Since $BI$ and $CI$ are the bisectors of $\angle{ABD}$ and $\angle{ACD}$, respectively, we have

\[
\frac{AI}{ID} = \frac{AB}{BD} = \frac{AC}{CD} = \frac{AB + AC}{BD + CD} = \frac{AB + AC}{BC} = \frac{b + c}{a}.
\]

Let $\ell$ be a line through $I$ which intersects sides $AB$ and $AC$ at $P'$ and $Q'$, respectively. Let $X$ and $Y$ be points on the line $P'Q'$ such that $BX \parallel AD$ and $CY \parallel AD$. Then

\[
\frac{P'B}{AP'} = \frac{BX}{AI}, \quad \text{and} \quad \frac{Q'C}{AQ'} = \frac{CY}{AI}.
\]

Since $BD : DC = AB : AC = c : b$, and $BX \parallel DI \parallel CY$, we get

\[
b \cdot BX + c \cdot CY = (b + c) \cdot DI.
\]

Hence,

\[
b \frac{BX}{AI} + c \frac{CY}{AI} = (b + c) \frac{DI}{AI};
\]
that is,
\[
b \frac{P'B}{AP'} + c \frac{Q'C}{AQ'} = (b + c) \frac{DI}{AI}.
\]

Conversely, if \( P' \) and \( Q' \) are points on the sides \( AB \) and \( AC \), respectively, and if
\[
b \frac{P'B}{AP'} + c \frac{Q'C}{AQ'} = (b + c) \frac{ID}{AI},
\]
then \( P' \), \( Q' \), and \( I \) are collinear. (Proof: Let \( I' \) be the intersection of \( P'Q' \) with \( AD \). Then
\[
b \frac{P'B}{AP'} + c \frac{Q'C}{AQ'} = (b + c) \frac{I'D}{AI'},
\]
from which we have \( \frac{I'D}{AI'} = \frac{ID}{AI} \). Then \( I' \) coincides with \( I \).)

Therefore, \( P \), \( Q \), and \( I \) are collinear if and only if (1) holds.

We set \( x = AP \) and \( y = AQ \). Then (1) becomes
\[
b \frac{c - x}{x} + c \frac{b - y}{y} = (b + c) \frac{a}{b + c} = a;
\]
that is,
\[
bc \left( \frac{1}{x} + \frac{1}{y} \right) = a + b + c. \tag{2}
\]
If \( PQ \) divides both the area and the perimeter of \( \triangle ABC \) into two equal parts, then \( xy = \frac{1}{2}bc \) and \( x + y = \frac{1}{2}(a + b + c) \). Thus,
\[
bc \left( \frac{1}{x} + \frac{1}{y} \right) = bc \frac{x + y}{x} = a + b + c,
\]
and (2) holds. Therefore, \( PQ \) passes through \( I \).

Next we consider converses.

I. If \( PQ \) passes through \( I \) and divides the area of \( \triangle ABC \) into two equal parts, then \( PQ \) divides the perimeter of \( \triangle ABC \) into two equal parts.

Since \( bc \left( \frac{1}{x} + \frac{1}{y} \right) = a + b + c \) and \( xy = \frac{1}{2}bc \), we obtain
\[
x + y = \frac{a + b + c}{2}. \quad \text{Thus, } PQ \text{ divides the perimeter of } \triangle ABC \text{ into two equal parts.}
\]

II. If \( PQ \) passes through \( I \) and divides the perimeter of \( \triangle ABC \) into two equal parts, then \( PQ \) divides the area of \( \triangle ABC \) into two equal parts.

Since \( bc \left( \frac{1}{x} + \frac{1}{y} \right) = a + b + c \) and \( x + y = \frac{a + b + c}{2} \), we have
\[
xy = \frac{1}{2}bc. \quad \text{Thus, } PQ \text{ divides the area of } \triangle ABC \text{ into two equal parts.}
\]

III. If \( PQ \) passes through \( I \), then \( PQ \) divides both the area and the perimeter of \( \triangle ABC \) into two equal parts.

This converse is not correct.
Next we give Bornsztein's generalization.

More generally, we will prove that the result holds for any convex polygon $\mathcal{P}$ into which we can inscribe a circle:

Let $\ell$ be a line which divides $\mathcal{P}$ into two sub-polygons, say $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $p(\mathcal{P}_1) = p(\mathcal{P}_2)$ and $|\mathcal{P}_1| = |\mathcal{P}_2|$ (where $p(\cdot)$ denotes the perimeter, and $|\cdot|$ denotes area). Let $A$ and $B$ be the points where $\ell$ meets $\mathcal{P}$.

Let $I$ and $r$ be the centre and the radius, respectively, of the incircle of $\mathcal{P}$ and let $H$ be the orthogonal projection of $I$ onto $\ell$. With no loss of generality, we may assume that $I$ belongs to the boundary of $\mathcal{P}_1$ (and $\mathcal{P}_2$) or is an interior point of $\mathcal{P}_1$ (and an exterior point of $\mathcal{P}_2$).

Then

$$[\mathcal{P}_1] = \frac{1}{2} r (p(\mathcal{P}_1) - AB) + \frac{1}{2} IH \cdot AB$$

and

$$[\mathcal{P}_2] = \frac{1}{2} r (p(\mathcal{P}_2) - AB) - \frac{1}{2} IH \cdot AB$$

Since $p(\mathcal{P}_1) = p(\mathcal{P}_2)$ and $|\mathcal{P}_1| = |\mathcal{P}_2|$, we get $IH \cdot AB = 0$; that is, $I = H$. Then $I \in \ell$, as desired.

The converse does not hold: Consider an equilateral triangle $ABC$ and the line through $I$ parallel to the line $BC$, which meets the sides $AB$ and $AC$ in $M$ and $P$, respectively.

Then $[AMP] = \frac{1}{4} [ABC]$, and $[BMP] = \frac{5}{16} [ABC] \neq [AMP]$.

8. The equation $x^2 + y^2 + z^2 + u^2 = xyz + 6$ is given. Find:

(a) at least one solution in positive integers;

(b) at least 33 such solutions;

(c) at least 100 such solutions.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's solution.

For all $n \in \mathbb{N}$, $(x, y, z, u) = (1, 2, n, n+1)$ satisfies the given equation since $x^2 + y^2 + z^2 + u^2 = xyz + 6 = 2n^2 + 2n + 6$. Hence, there are infinitely many solutions in positive integers.

That completes the Corner for this issue and this volume of CRUX with MAYHEM. Remember to send in your solutions promptly so that we can aim to keep the time between giving problems and looking at the solutions reasonably short. Also send me your Olympiad contest materials for the Corner.