SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2951. [2004 : 296, 298] *Proposed by Nevena Sybeva, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

Let M and N be interior points of $\triangle ABC$. Define the point T_A to be the point on BC such that light travelling from M to T_A, and undergoing perfect reflection at T_A, will pass through N. Define T_B and T_C similarly.

Prove that if the three possible light paths MT_AN, MT_BN, MT_CN have equal length, then the lines AT_A, BT_B, and CT_C are concurrent.

Combination of similar solutions by Michel Bataille, Rouen, France; John G. Heuver, Grande Prairie, AB; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $2a$ be the common length of the three light paths. Since $T_AM + T_AN = 2a$, the point T_A lies on the ellipse $\mathcal{E}$ with foci M and N and major axis $2a$. Moreover, from the hypothesis of perfect reflection at T_A, we have $\angle MT_AB = \angle NT_AC$, which implies that BC is the tangent to $\mathcal{E}$ at T_A. Similar results hold for T_B and T_C. Thus, $\mathcal{E}$ is inscribed in $\triangle ABC$. The desired concurrency of AT_A, BT_B, CT_C follows immediately from Brianchon's Theorem applied to the (degenerate) hexagon with sides AT_C, T_CB, BT_A, T_AC, CT_B, T_BA. For more about Brianchon's Theorem applied to triangles see N. Sato, "Ellipses in Polygons" [2000 : 361–371].

Also solved by Toshio Seimiya, Kawasaki, Japan; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

The argument used in the three other solutions came down to an application of Ceva's Theorem. Zhao added the observation, which he easily verified, that M is the isogonal conjugate of N with respect to $\triangle ABC$.

2952. [2004 : 296, 299] *Proposed by C.R. Pranesachar and Prithu Bharti, Indian Institute of Science, Bangalore, India.*

Find a closed form for the real series

$$\sum_{\substack{r \geq 0 \\ r \geq -n}} \binom{n+2r}{r} x^r \,, \qquad x \in \left(-\tfrac{1}{4}, \tfrac{1}{4}\right) \,,$$

where n is an integer (positive, negative, or zero).

I. *Solution by Michel Bataille, Rouen, France, modified by the editor.*

First, suppose that $n \geq 0$. The given sum is then $\sum\limits_{r \geq 0} \binom{n+2r}{r} x^r$. We will use the following Taylor series expansion:

$$\frac{1}{(1-z)^{m+1}} = \sum_{k=0}^{\infty} \binom{m+k}{k} z^k .$$

Here m may be any non-negative integer, and the expansion is valid for all complex numbers z such that $|z| < 1$.

Let an integer r be fixed, and let $f(z) = \frac{1}{z^{r+1}(1-z)^{n+r+1}}$. Then, for $0 < |z| < 1$, we have

$$f(z) = \sum_{k=0}^{\infty} \binom{n+r+k}{k} z^{k-r-1} .$$

In this Laurent series for $f(z)$, the coefficient of z^{-1} is $\binom{n+2r}{r}$. This is the residue of $f(z)$ at 0. By the Residue Theorem,

$$\binom{n+2r}{r} = \frac{1}{2\pi i} \oint_C f(z)\, dz ,$$

where C denotes the circle $|z| = \frac{1}{2}$ traversed once counterclockwise.

Let $x \in \left(-\frac{1}{4}, \frac{1}{4}\right)$ be fixed. For all z on C, we have $|1-z| \geq \frac{1}{2}$, and hence,

$$\left|\frac{x}{z(1-z)}\right| \leq \frac{|x|}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = 4|x| < 1 .$$

Then, for all z on C,

$$\sum_{r=0}^{\infty} \left(\frac{x}{z(1-z)}\right)^r = \frac{1}{1 - \frac{x}{z(1-z)}} = \frac{-z(1-z)}{z^2 - z + x} ,$$

and the convergence is uniform with respect to z on C. The uniform convergence allows us to interchange the integration with the summation in the following calculation:

$$\begin{aligned}
\sum_{r=0}^{\infty} \binom{n+2r}{r} x^r &= \sum_{r=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C f(z)\, dz\right) x^r \\
&= \frac{1}{2\pi i} \oint_C \frac{1}{z(1-z)^{n+1}} \sum_{r=0}^{\infty} \left(\frac{x}{z(1-z)}\right)^r dz \\
&= \frac{1}{2\pi i} \oint_C \frac{-1}{(1-z)^n (z^2 - z + x)}\, dz .
\end{aligned}$$

The polynomial $p(z) = z^2 - z + x$ has roots $\zeta_+ = \dfrac{1+\sqrt{1-4x}}{2}$ and $\zeta_- = \dfrac{1-\sqrt{1-4x}}{2}$. Note that $\zeta_+ + \zeta_- = 1$ and $\zeta_+ - \zeta_- = \sqrt{1-4x}$. The integrand above has simple poles at ζ_+ and ζ_-, but only ζ_- is inside C. By the Residue Theorem,

$$\begin{aligned}
\sum_{r=0}^{\infty} \binom{n+2r}{r} x^r &= \operatorname{Res}\left(\frac{-1}{(1-z)^n p(z)}, \zeta_-\right) = \lim_{z\to\zeta_-} \frac{-(z-\zeta_-)}{(1-z)^n p(z)} \\
&= \left.\frac{-1}{(1-z)^n (z-\zeta_+)}\right|_{z=\zeta_-} = \frac{1}{\zeta_+^n(\zeta_+ - \zeta_-)} \\
&= \frac{1}{\sqrt{1-4x}} \left(\frac{2}{1+\sqrt{1-4x}}\right)^n . \qquad (1)
\end{aligned}$$

This is the required closed form for the case $n \geq 0$.

Now suppose that $n < 0$. Letting $m = -n$ and $s = r - m$, we have, from (1),

$$\begin{aligned}
\sum_{r\geq -n} \binom{n+2r}{r} x^r &= x^m \sum_{r=m}^{\infty} \binom{-m+2r}{r} x^{r-m} = x^m \sum_{s=0}^{\infty} \binom{m+2s}{m+s} x^s \\
&= x^m \sum_{s=0}^{\infty} \binom{m+2s}{s} x^s = \frac{(2x)^m}{\left(1+\sqrt{1-4x}\right)^m \sqrt{1-4x}} \\
&= \frac{(2x)^{-n}}{\left(1+\sqrt{1-4x}\right)^{-n} \sqrt{1-4x}} .
\end{aligned}$$

II. *Solution by the proposers, modified slightly by the editor.*

Observe that

$$\lim_{r\to\infty} \frac{\binom{n+2r+2}{r+1}}{\binom{n+2r}{r}} = \lim_{r\to\infty} \frac{(n+2r+2)(n+2r+1)}{(r+1)(n+r+1)} = 4 ;$$

whence the radius of convergence of the given series is $1/4$, by the Ratio Test.

Let y_n denote the sum of the series. First, we consider the case $n \geq 0$. For $n = 0$, we have

$$y_0 = \sum_{r\geq 0} \binom{2r}{r} x^r = \sum_{r=0}^{\infty} \binom{-\frac{1}{2}}{r} (-4x)^r = \frac{1}{\sqrt{1-4x}} .$$

For $n = 1$,

$$\begin{aligned}
y_1 &= \sum_{r\geq 0} \binom{2r+1}{r} x^r = \sum_{r\geq 0} \left(\frac{2r+1}{r+1}\right) \binom{2r}{r} x^r \\
&= \sum_{r\geq 0} \left(2 - \frac{1}{r+1}\right) \binom{2r}{r} x^r = 2\sum_{r\geq 0} \binom{2r}{r} x^r - \sum_{r\geq 0} \frac{1}{r+1} \binom{2r}{r} x^r .
\end{aligned}$$

The last summation above corresponds to the generating function of the sequence of Catalan numbers and is known to equal $(1-\sqrt{1-4x})/(2x)$. Hence,

$$\begin{aligned} y_1 &= \frac{2}{\sqrt{1-4x}} - \frac{1-\sqrt{1-4x}}{2x} \;=\; \frac{4x-\sqrt{1-4x}+(1-4x)}{2x\sqrt{1-4x}} \\ &= \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right). \end{aligned}$$

Furthermore, for $n \geq 1$, we have

$$\begin{aligned} y_n - y_{n-1} &= \sum_{r\geq 0}\left[\binom{n+2r}{r} - \binom{n-1+2r}{r}\right]x^r \\ &= \sum_{r\geq 1}\binom{n-1+2r}{r-1}x^r \;=\; \sum_{r\geq 0}\binom{n+1+2r}{r}x^{r+1} \;=\; xy_{n+1}\,. \end{aligned}$$

Changing n to $n-1$, we obtain, for $n \geq 2$, the recurrence relation

$$xy_n - y_{n-1} + y_{n-2} \;=\; 0\,.$$

The characteristic equation is $x\lambda^2 - \lambda + 1 = 0$ and the characteristic roots are $\lambda = (1 \pm \sqrt{1-4x})/(2x)$. Hence,

$$y_n \;=\; A\left(\frac{1+\sqrt{1-4x}}{2x}\right)^n + B\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n,$$

where A and B are functions of x yet to be determined.

Setting $n=0$, we obtain

$$A+B \;=\; y_0 \;=\; \frac{1}{\sqrt{1-4x}}\,,$$

and setting $n=1$, we get

$$A\left(\frac{1+\sqrt{1-4x}}{2x}\right) + B\left(\frac{1-\sqrt{1-4x}}{2x}\right) \;=\; y_1 \;=\; \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right).$$

Solving (or simply by inspection), we find that $A=0$ and $B=\dfrac{1}{\sqrt{1-4x}}$. Hence, for all $n \geq 0$,

$$y_n \;=\; \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n.$$

To make the right side meaningful at $x=0$, we rewrite y_n as

$$y_n \;=\; \frac{1}{\sqrt{1-4x}}\left(\frac{2}{1+\sqrt{1-4x}}\right)^n.$$

[*Ed.:* The case where $n<0$ was handled as in Solution I above.]

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2953. *Proposed by Titu Zvonaru, Bucharest, Romania.*

Let m, n be positive integers with $n > 1$, and let a, b, c be positive real numbers satisfying $a^{m+1} + b^{m+1} + c^{m+1} = 1$. Prove that

$$\frac{a}{1-ma^n} + \frac{b}{1-mb^n} + \frac{c}{1-mc^n} \geq \frac{(m+n)^{1+\frac{m}{n}}}{n}.$$

Comments by the editor: The given inequality, which is an obvious attempt to generalize ***CRUX with MAYHEM*** Problem 2935 [2004 : 174; 2005 : 188] by the same proposer, is clearly false as stated. This was pointed out by several readers. For one thing, some of the denominators of the terms on the left side could easily be 0. OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA noted that if $m = n = 2$, $a = b = 1/\sqrt{2}$, and $c = (1 - 1/\sqrt{2})^{1/3}$, then $1 - ma^n = 0$. WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria and D. KIPP JOHNSON, Beaverton, OR, USA both gave examples showing that the left side could be negative; while LI ZHOU, Polk Community College, Winter Haven, FL, USA gave a counterexample in which the left side is positive.

There were four incorrect solutions, all of which proved the "validity" of the given inequality under the "implicit" assumption that the quantities $1 - ma^n$, $1 - mb^n$, and $1 - mc^n$ are all positive. This was apparently the intention of the proposer. The published proof for Problem 2935 [2005 : 188] can be modified easily with little change to yield a proof for the revised version of the current inequality.

2954. [2004 : 297, 299] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let Γ be the circumcircle of $\triangle ABC$. The tangents to Γ at B and C intersect at M. The line through M parallel to AB intersects Γ at D and E, and intersects AC at F.

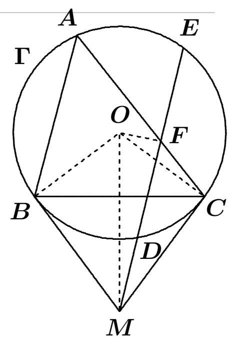
Prove that F is the mid-point of DE.

I. *Solution by Michel Bataille, Rouen, France.*

We use the notation $[\cdot, \cdot, \cdot, \cdot]$ for the cross ratio (of four points on a line or a circle, or four concurrent lines).

Let K be the point of intersection of ME and BC. Then K is on the polar of M with respect to Γ. Therefore, we have $[K, M, D, E] = -1$, which implies that $[CB, CM, CD, CE] = -1$. Thus, $[B, C, D, E] = -1$, and hence $[AB, AC, AD, AE] = -1$. Cutting by a transversal DE, we get $[\infty, F, D, E] = -1$. This means that F is the mid-point of DE.

II. *Composite of similar solutions by Toshio Seimiya, Kawasaki, Japan; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.*



Let O be the centre of Γ. Since MB and MC are tangent to Γ at B and at C, respectively, we have $\angle OBM = \angle OCM = 90^\circ$. Therefore, the points O, B, M and C lie on a circle with diameter OM.

We have $\angle MBC = \angle BAC$, because MB is tangent to Γ at B. Also, $\angle BAC = \angle MFC$, because $MF \parallel AB$. Thus, $\angle MBC = \angle MFC$, which implies that B, M, C, and F are cyclic. But we already know that B, M, and C lie on the circle with diameter OM. Therefore, F lies on this same circle. Hence, $\angle OFM = 90^\circ$. Then F is the mid-point of the chord DE.

III. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Seydewitz's Theorem states: If a triangle is inscribed in a conic section, then any line conjugate to one side meets the other two sides in conjugate points (see [1]). Here we have DM conjugate to BC with respect to Γ, since the pole of BC is M. Thus, DM intersects AB and AC at points which are harmonic conjugates with respect to the segment DE. Since DM meets AB at ∞, F must be the mid-point of DE.

Reference:

[1] H.S.M. Coxeter, *Projective Geometry*, 2nd edition, Springer, 1987.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; BOB SERKEY, Leonia, NJ, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; BABIS STERGIOU, Chalkida, Greece; MIHAELA VÂJIAC, Chapman University, Orange, CA, USA, and BOGDAN SUCEAVĂ, California State University, Fullerton, CA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania.

2955. [2003 : 297, 299] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer. For each positive integer k, let f_k be the k^{th} Fibonacci number; that is, $f_1 = 1$, $f_2 = 1$, and $f_{k+2} = f_{k+1} + f_k$ for all $k \geq 1$. Prove that

$$\left(\sum_{k=1}^{n} f_{k+1}^2\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) \;\geq\; n^2\,.$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC, USA.

The result follows immediately from the observations that $f_{n+1} \geq n$ for every positive integer n and $\sum\limits_{k=1}^{n} \frac{1}{f_{2k}} \geq \frac{1}{f_2} = 1$.

The following claim gives a sharper inequality for $n \neq 3$:

Claim. For every positive integer n except $n = 3$,

$$\left(\sum_{k=1}^{n} f_{k+1}\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) \;\geq\; n^2\,.$$

Proof: It is straightforward to check that the claim holds for $n = 1$, $n = 2$, and $n = 4$ (but, alas, not $n = 3$).

To apply induction, we assume that the claim holds for some integer $n \geq 4$. We have $f_{n+2} \geq 2n$ and $\sum\limits_{k=1}^{n} \frac{1}{f_{2k}} \geq \frac{4}{3}$ (since $n \geq 4$), and hence,

$$\begin{aligned}
&\left(\sum_{k=1}^{n+1} f_{k+1}\right)\left(\sum_{k=1}^{n+1} \frac{1}{f_{2k}}\right)\\
&\quad= \left(\sum_{k=1}^{n} f_{k+1}\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) + f_{n+2}\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) + \left(\sum_{k=1}^{n+1} f_{k+1}\right)\left(\frac{1}{f_{2n+2}}\right)\\
&\quad\geq n^2 + 2n(4/3) \;\geq\; (n+1)^2\,.
\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2956. [2004 : 297, 299] *Proposed by David Loeffler, student, Trinity College, Cambridge, UK.*

Let A, B, C be the angles of a triangle. Prove that

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \;<\; 2$$

if and only if

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) \;<\; 2\,.$$

Solution by YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON.

Since $A+B+C=\pi$, we have

$$\tan\left(\frac{A}{2}\right) = \cot\left(\frac{B+C}{2}\right) = \frac{1-\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}{\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)},$$

and thus,

$$\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)+\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)+\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right) = 1.$$

Hence, $\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right) < 2$ is equivalent to

$$\left(\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)\right)^2 < 4,$$

which can be expressed as

$$\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right) < 2,$$

since the left side of the last inequality is clearly positive.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; VEDULA N. MURTY, Dover, PA, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2957. [2004 : 297] *Proposed by K.R.S. Sastry, Bangalore, India.*

Let ABC and $A'B'C'$ be two triangles having $BC=a$, $B'C'=s-a$, etc., where $s=\frac{1}{2}(a+b+c)$. Prove that the triangles are isosceles if and only if $\tan\left(\frac{B}{2}\right)$ is the geometric mean of $\tan\left(\frac{A'}{2}\right)$ and $\tan\left(\frac{B'}{2}\right)$.

Solution by D. Kipp Johnson, Beaverton, OR, USA.

The statement of the problem is not sufficiently precise. We shall prove that the given tangent condition is true if and only if the triangles are isosceles with respective apex angles at A and A'.

By the Triangle Inequality in $\triangle A'B'C'$, we must have

$$(s-a)+(s-b) \;>\; s-c\,,$$

with similar inequalities obtained by permuting a, b, c. The above inequality simplifies to $c > s/2$. Thus, each side of $\triangle ABC$ must exceed one quarter of the perimeter $2s$. Equivalently, three times the shortest side must exceed the sum of the other two sides.

Note that the semiperimeter of $\triangle A'B'C'$ is half that of the $\triangle ABC$:

$$s' \;=\; \frac{(s-a)+(s-b)+(s-c)}{2} \;=\; \frac{3s-(a+b+c)}{2} \;=\; \frac{s}{2}\,.$$

We will need the following known identity: $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$. [Briefly, we have $\tan\left(\frac{B}{2}\right) = \frac{r}{s-b} = \frac{[ABC]}{s(s-b)} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$.] Using this identity, we get

$$\begin{aligned}
\tan\left(\frac{A'}{2}\right)&\tan\left(\frac{B'}{2}\right)\\
&= \sqrt{\frac{(s'-(s-b))(s'-(s-c))}{s'(s'-(s-a))}}\cdot\sqrt{\frac{(s'-(s-a))(s'-(s-c))}{s'(s'-(s-b))}}\\
&= \sqrt{\frac{(s'-s+c)^2}{(s')^2}} \;=\; \sqrt{\frac{(c\,-\,s/2)^2}{(s')^2}} \;=\; \frac{c\,-\,s/2}{s'} \qquad \text{since } c > s/2\\
&= \frac{2c}{s}-1\,,
\end{aligned}$$

and

$$\begin{aligned}
\tan^2\left(\frac{B}{2}\right) &= \frac{(s-a)(s-c)}{s(s-b)} \;=\; \frac{s^2-(a+b+c)s+bs+ac}{s(s-b)}\\
&= \frac{-s^2+bs+ac}{s(s-b)} \;=\; \frac{ac}{s(s-b)}-1\,.
\end{aligned}$$

The condition that $\tan\left(\frac{B}{2}\right)$ is the geometric mean of $\tan\left(\frac{A'}{2}\right)$ and $\tan\left(\frac{B'}{2}\right)$ can therefore be written in the following equivalent forms:

$$\begin{aligned}
\frac{ac}{s(s-b)}-1 &= \frac{2c}{s}-1\,,\\
\frac{c(b-c)}{s(s-b)} &= 0\,,\\
b &= c\,,\\
b' &= c'\,.
\end{aligned}$$

Comment. It is clear that when we interchange A and B, we will have $\tan^2\left(\frac{A}{2}\right) = \tan\left(\frac{A'}{2}\right)\tan\left(\frac{B'}{2}\right)$ if and only if $a = c$ (equivalently, $a' = c'$). Interestingly, replacing B by C produces a somewhat different conclusion. In this case, a similar argument shows that $\tan^2\left(\frac{C}{2}\right) = \tan\left(\frac{A'}{2}\right)\tan\left(\frac{B'}{2}\right)$ if and only if $a = c$ *or* $b = c$ (or equivalently, $a' = c'$ *or* $b' = c'$).

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2960. [2004 : 298, 300] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let m_a, w_a, and h_a be the lengths of the median, the angle-bisector, and the altitude, respectively, from the right-angled vertex A of triangle ABC to the hypotenuse. Suppose that the sides a and c are fixed in length, while the length of side b varies subject to $a > b \geq c$.

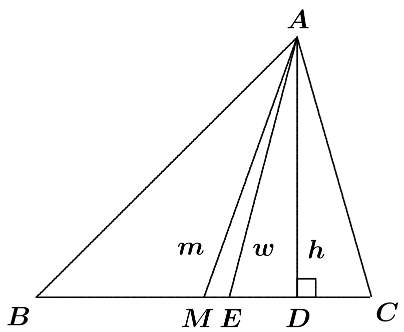
Evaluate $\lim\limits_{b\to c} \dfrac{m_a - h_a}{w_a - h_a}$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

First we note that the problem as stated is incorrect. If sides a and c are fixed in length, then so is side b, by the Pythagorean Theorem. The problem will be correct if we assume that only c is fixed. We will drop the condition $\angle A = 90°$ and solve the problem for any triangle with $a > b > c$, assuming that c is fixed.

There will be no confusion if we drop the subscripts and write m, w, and h instead of m_a, w_a, and h_a. Let the median, the angle-bisector, and the altitude from vertex A intersect the side BC at points M, E, and D, respectively. By the Pythagorean Theorem, we have $m^2 - h^2 = MD^2$. Thus,

$$m - h = \frac{MD^2}{m+h}.$$



Similarly, $w^2 - h^2 = ED^2$, which implies that $w - h = \dfrac{ED^2}{w+h}$.

We also have

$$MB = \frac{a}{2}, \quad EB = \frac{ac}{b+c},$$

and

$$BD = c\cos B = \frac{a^2+c^2-b^2}{2a},$$

where the last expression follows from the Law of Cosines. Using these results, we obtain

$$\begin{aligned} MD &= MB - BD = \frac{(b-c)(b+c)}{2a} \\ \text{and} \quad ED &= EB - BD = \frac{(b-c)\big((b+c)^2-a^2\big)}{2a(b+c)}. \end{aligned}$$

Therefore,

$$\frac{m-h}{w-h} = \frac{MD^2(w+h)}{ED^2(m+h)} = \frac{(b+c)^4(w+h)}{\big((b+c)^2-a^2\big)^2(m+h)}.$$

Now, $\lim\limits_{b\to c} m = \lim\limits_{b\to c} w = h$. Hence,

$$\lim_{b\to c}\frac{m-h}{w-h} = \frac{(2c)^4}{\big((2c)^2-a^2\big)^2}.$$

In the case when $\angle A = 90^\circ$, we have $\lim\limits_{b\to c} a = c\sqrt{2}$; whence,

$$\lim_{b\to c}\frac{m-h}{w-h} = 4.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect solution submitted.

Many solvers pointed out the error in the condition and then proceeded to assume that c is fixed or that a is fixed. Both approaches produce the same result. Zvonaru was the only other solver who initially solved the problem for any triangle and then deduced the answer for the case $\angle A = 90^\circ$, as in the featured solution above.

2961. [2004 : 298, 300] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC and $A'B'C'$ be two right triangles with right angles at A and A'. If w_a and $w_{a'}$ are the interior angle bisectors of angles A and A', respectively, prove that $aw_a a'w_{a'} \geq bcb'c'$, with equality if and only if both ABC and $A'B'C'$ are isosceles.

Solution by Titu Zvonaru, Bucharest, Romania.

Let h_a be the altitude from A. Because the shortest distance from A to BC is along the perpendicular, we have

$$aw_a \;\geq\; ah_a \;=\; 2[ABC] \;=\; bc\,.$$

Equality holds if and only if $h_a = w_a$; that is, if and only if $\triangle ABC$ is isosceles. This is true for as many triangles as we wish—we can have one or many triangles instead of two. Since the quantities involved are all positive, one can multiply the respective sides of each inequality together while maintaining the inequality.

Comment. The conclusion continues to hold if the bisectors are replaced by any other cevians.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2962. [2003 : 298, 300] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC and $A'B'C'$ be two triangles satisfying $a \geq b \geq c$ and $a' \geq b' \geq c'$. If h_a, $h_{a'}$ are the altitudes from the vertices A, A', respectively, to the opposite sides, prove that

(i) $bb' + cc' \geq ah_{a'} + a'h_a$, (ii) $bc' + b'c \geq ah_{a'} + a'h_a$.

Comment by Michel Bataille, Rouen, France.

This is the same problem as 2860 [2003 : 318; 2004 : 315]. [The editor apologizes for this embarassment: the solution to 2860 appeared in the same issue in which problem 2962 was posed.]

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2963. [2004 : 367, 370] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any acute-angled triangle. Let r and R be the inradius and circumradius, respectively, and let s be the semiperimeter; that is, $s = \frac{1}{2}(a+b+c)$. Let m_a be the length of the median from A to BC, and let w_a be the length of the internal bisector of $\angle A$ from A to the side BC. We define m_b, m_c, w_b and w_c similarly. Prove that

(a) $\dfrac{3s^2 - r^2 - 4Rr}{8sRr} \le \sum_{\text{cyclic}} \dfrac{m_a}{aw_a} \le \dfrac{s^2 - r^2 - 4Rr}{7sRr}$;

(b) $\dfrac{3}{4} \le \sum_{\text{cyclic}} \dfrac{m_a^2}{b^2+c^2} \le \dfrac{4R+r}{4R}$.

Solution to part (a) by Arkady Alt, San Jose, CA, USA.

First we note that the right inequality is incorrect. For example, if ABC is equilateral, then

$$\sum_{\text{cyclic}} \frac{m_a}{aw_a} = \frac{3}{a} > \frac{6}{7a} = \frac{s^2 - r^2 - 4Rr}{7sRr}.$$

We will instead prove that

$$\frac{3s^2 - r^2 - 4Rr}{8sRr} \le \sum_{\text{cyclic}} \frac{m_a}{aw_a} \le \frac{s^2 - r^2 - 4Rr}{2sRr}.$$

We use the following well-known identities:

$$\begin{aligned} 4m_a^2 &= 2(b^2+c^2) - a^2\,, \\ w_a^2 &= \frac{bc\big((b+c)^2 - a^2\big)}{(b+c)^2}\,, \\ abc &= 4sRr\,, \\ a^2+b^2+c^2 &= 2s^2 - 2r^2 - 8Rr\,, \\ ab+bc+ca &= s^2 + r^2 + 4Rr\,. \end{aligned}$$

From the first two identities above, we get

$$\frac{m_a^2}{w_a^2} = \frac{(b+c)^2}{4bc} \cdot \frac{2(b^2+c^2) - a^2}{(b+c)^2 - a^2}\,. \tag{1}$$

Now we observe that

$$2bc \le (b+c)^2 - a^2 \le 4bc\,. \tag{2}$$

The left inequality is true because it is equivalent to $b^2 + c^2 \ge a^2$, which is true for any acute triangle, and the right inequality is true because it is equivalent to $|b - c| \le a$, which is true for any triangle. From (2), we get

$$\frac{(b-c)^2}{4bc} + 1 \;\le\; \frac{(b-c)^2}{(b+c)^2 - a^2} + 1 \;\le\; \frac{(b-c)^2}{2bc} + 1\,;$$

that is,

$$\frac{(b+c)^2}{4bc} \;\le\; \frac{2(b^2+c^2) - a^2}{(b+c)^2 - a^2} \;\le\; \frac{b^2+c^2}{2bc}\,.$$

Recalling (1), we get

$$\frac{(b+c)^4}{16b^2c^2} \;\le\; \frac{m_a^2}{w_a^2} \;\le\; \frac{(b+c)^2(b^2+c^2)}{8b^2c^2}\,.$$

Now, using the easy-to-prove inequality $(b+c)^2 \le 2(b^2+c^2)$, we obtain

$$\frac{(b+c)^4}{16b^2c^2} \;\le\; \frac{m_a^2}{w_a^2} \;\le\; \frac{(b^2+c^2)^2}{4b^2c^2}\,.$$

Taking square roots throughout and dividing by a gives

$$\frac{(b+c)^2}{4abc} \;\le\; \frac{m_a}{aw_a} \;\le\; \frac{b^2+c^2}{2abc}\,,$$

where equality occurs if and only if $b = c$.

Using similar inequalities for $\frac{m_b}{bw_b}$ and $\frac{m_c}{cw_c}$, we obtain

$$\sum_{\text{cyclic}} \frac{m_a}{aw_a} \;\le\; \sum_{\text{cyclic}} \frac{b^2+c^2}{2abc} \;=\; \frac{a^2+b^2+c^2}{abc} \;=\; \frac{s^2-r^2-4Rr}{2sRr}$$

and

$$\begin{aligned}\sum_{\text{cyclic}} \frac{m_a}{aw_a} \;&\ge\; \sum_{\text{cyclic}} \frac{(b+c)^2}{4abc} \;=\; \frac{1}{2abc}\left(\sum_{\text{cyclic}} a^2 + \sum_{\text{cyclic}} bc\right)\\ &=\; \frac{1}{8sRr}(2s^2 - 2r^2 - 8Rr + s^2 + r^2 + 4Rr)\\ &=\; \frac{3s^2 - r^2 - 4Rr}{8sRr}\,,\end{aligned}$$

as claimed. Equality occurs in both inequalities if and only if $a = b = c$.

Solution to part (b) by Michel Bataille, Rouen, France.

We prove that

$$\frac{3}{4} \;<\; \sum_{\text{cyclic}} \frac{m_a^2}{b^2+c^2} \;\le\; \frac{4R+r}{4R}\,.$$

Since $4m_a^2 = 2(b^2+c^2) - a^2$, it is easily seen that our inequality is equivalent to

$$2 - \frac{r}{R} \leq \sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} < 3\,.$$

Since the triangle is acute, the cosines of all angles are positive. Using the Cosine Law, we obtain

$$\sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} = \sum_{\text{cyclic}} \frac{b^2+c^2-2bc\cos A}{b^2+c^2} = 3 - \sum_{\text{cyclic}} \frac{2bc\cos A}{b^2+c^2} < 3\,.$$

On the other hand, using the well-known identity

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

and the easy inequality $2bc \leq b^2 + c^2$, we obtain

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} &= 3 - \sum_{\text{cyclic}} \frac{2bc\cos A}{b^2+c^2} \geq 3 - \sum_{\text{cyclic}} \cos A \\ &= 3 - \left(1 + \frac{r}{R}\right) = 2 - \frac{r}{R}\,.\end{aligned}$$

Equality holds if and only if $a = b = c$.

Also solved by ARKADY ALT, San Jose, CA, USA (part (b)); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (part (b)); MICHEL BATAILLE, Rouen, France (part (a)); JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (b)); VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Janous believes that the lower bound of $3/4$ in inequality (b) can be increased to 1, but he does not have a proof. We encourage our readers to try to find a bound better than $3/4$.

The editors apologize for the typo in the right side of the inequality of part (a). The proposer's version was the correct one (found also by Alt and Bataille). Several other solvers either gave a counterexample or suggested a correct version and solved it.