Two Generalizations of Popoviciu's Inequality

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Dedicated to the memory of Murray S. Klamkin

In 1965 the Romanian mathematician T. Popoviciu proved the following inequality [8]

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right) \geq 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right),$$

where f is a convex function on an interval I and x, y, $z \in I$. A. Lupas [7] generalized this in 1982 as follows (where p, q, and r are positive numbers)

$$\begin{aligned} & pf(x)+qf(y)+rf(z)+(p+q+r)f\left(\frac{px+qy+rz}{p+q+r}\right) \\ \geq \quad & (p+q)f\left(\frac{px+qy}{p+q}\right)+(q+r)f\left(\frac{qy+rz}{q+r}\right)+(r+p)f\left(\frac{rz+px}{r+p}\right). \end{aligned}$$

In this paper we present two generalizations of Popoviciu's Inequality to n variables with some applications. The first generalization was published in [1] with a more difficult proof than the one below. The second generalization (without solution) was posted on the Mathlinks Site – Inequalities Forum in 2004. It is possible that one or both generalizations might have been previously published elsewhere, but not as far as we know.

Our proof relies on Karamata's Inequality for convex functions, see [4] and [3], which we now recall. We say that a vector $\overrightarrow{A} = [a_1, a_2, \ldots, a_n]$ with $a_1 \geq a_2 \geq \ldots \geq a_n$ majorizes a vector $\overrightarrow{B} = [b_1, b_2, \ldots, b_n]$ with $b_1 \geq b_2 \geq \ldots \geq b_n$ and write this as $\overrightarrow{A} \succeq \overrightarrow{B}$, if

$$\begin{aligned} a_1 &\geq b_1 \\ a_1+a_2 &\geq b_1+b_2 \\ &\vdots \\ a_1+a_2+\cdots+a_{n-1} &\geq b_1+b_2+\cdots+b_{n-1} \\ a_1+a_2+\cdots+a_n &= b_1+b_2+\cdots+b_n\,. \end{aligned}$$

The Karamata Inequality states that if f is any convex function and $\overrightarrow{A} \succeq \overrightarrow{B}$, then one has the following

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq f(b_1)+f(b_2)+\cdots+f(b_n)\,.$$

The First Generalization.

Theorem 1. If f is a convex function on an interval I and $a_1, a_2, \ldots, a_n \in I$, then

$$f(a_1) + f(a_2) + \cdots + f(a_n) + n(n-2)f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geq (n-1)\big(f(b_1) + f(b_2) + \cdots + f(b_n)\big),$$

where $b_i = \dfrac{1}{n-1}\sum_{j \neq i} a_j$ for all i.

Proof: Without loss of generality, we may assume that $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. Then there is an integer m with $1 \leq m \leq n-1$ and $a_1 \leq \cdots \leq a_m \leq a \leq a_{m+1} \leq \cdots \leq a_n$, where $a = (a_1 + \cdots + a_n)/n$. We also have $b_1 \geq \cdots \geq b_m \geq a \geq b_{m+1} \geq \cdots \geq b_n$. It is clear that the inequality that we are trying to prove is the sum of the following two inequalities:

$$f(a_1) + f(a_2) + \cdots + f(a_m) + n(n-m-1)f(a) \geq (n-1)\big(f(b_{m+1}) + f(b_{m+2}) + \cdots + f(b_n)\big), \quad (1)$$

$$f(a_{m+1}) + f(a_{m+2}) + \cdots + f(a_n) + n(m-1)f(a) \geq (n-1)\big(f(b_1) + f(b_2) + \cdots + f(b_m)\big). \quad (2)$$

In order to prove (1), we apply Jensen's Inequality to get

$$f(a_1) + f(a_2) + \cdots + f(a_m) + (n-m-1)f(a) \geq (n-1)f(b),$$

where $b = \dfrac{a_1 + a_2 + \cdots + a_m + (n-m-1)a}{n-1}$. Thus, we still have to show

$$(n-m-1)f(a) + f(b) \geq f(b_{m+1}) + f(b_{m+2}) + \cdots + f(b_n).$$

Since $a \geq b_{m+1} \geq b_{m+2} \geq \cdots \geq b_n$ and

$$(n-m-1)a + b = b_{m+1} + b_{m+2} + \cdots + b_n,$$

we see that $\overrightarrow{A}_{n-m} = [a, \ldots, a, b]$ majorizes $\overrightarrow{B}_{n-m} = [b_{m+1}, b_{m+2}, \ldots, b_n]$. The inequality follows by Karamata's Inequality for convex functions.

The inequality (2) can be proved similarly by adding Jensen's Inequality

$$\frac{f(a_{m+1}) + f(a_{m+2}) + \cdots + f(a_n) + (m-1)f(a)}{n-1} \geq f(c)$$

to the inequality

$$f(c) + (m-1)f(a) \geq f(b_1) + f(b_2) + \cdots + f(b_m),$$

where $c = (a_{m+1} + a_{m+2} + \cdots + a_n + (m-1)a)/(n-1)$. The last inequality follows from Karamata's Inequality, because $b_1 \geq \cdots \geq b_m \geq a$ and $c + (m-1)a = b_1 + b_2 + \cdots + b_m$, and therefore $\overrightarrow{C}_m = [c, a, \ldots, a]$ majorizes $\overrightarrow{D}_m = [b_1, b_2, \ldots, b_m]$.

The Second Generalization.

Theorem 2. If f is a convex function on an interval I and $a_1, a_2, \ldots, a_n \in I$, then

$$(n-2)\big(f(a_1)+f(a_2)+\cdots+f(a_n)\big)+nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right) \geq 2\sum_{1\leq i<j\leq n} f\left(\frac{a_i+a_j}{2}\right).$$

Proof: We will prove this using induction. For $n=2$, one has equality. Suppose now that $n \geq 3$ and that the inequality is valid for $n-1$. We will show that it holds for n. Let $a=(a_1+a_2+\cdots+a_n)/n$ and let $x=(a_1+a_2+\cdots+a_{n-1})/(n-1)$. By the induction hypothesis, we have

$$(n-3)\big(f(a_1)+f(a_2)+\cdots+f(a_{n-1})\big)+(n-1)f(x) \geq 2\sum_{1\leq i<j\leq n-1} f\left(\frac{a_i+a_j}{2}\right).$$

Thus, it suffices to show that

$$f(a_1)+f(a_2)+\cdots+f(a_{n-1})+(n-2)f(a_n)+nf(a) \geq (n-1)f(x)+2\sum_{i=1}^{n-1} f\left(\frac{a_i+a_n}{2}\right).$$

From Jensen's Inequality, we have

$$f(a_1)+f(a_2)+\cdots+f(a_{n-1}) \geq (n-1)f(x).$$

Hence, we just have to show that

$$(n-2)f(a_n)+nf(a) \geq 2\sum_{i=1}^{n-1} f\left(\frac{a_i+a_n}{2}\right).$$

Since $(n-2)a_n+na = 2\sum_{i=1}^{n-1}\frac{a_i+a_n}{2}$, we will again use Karamata's Inequality for two cases.

Case 1. $2a \geq \min\{a_1, a_2, \ldots, a_n\}+\max\{a_1, a_2, \ldots, a_n\}$.

Without loss of generality, we may assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. Then $a \geq (a_1+a_n)/2$. According to Karamata's Inequality, it is enough to show that $a_n \leq \min\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$ and $a \geq \max\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$. The first condition is clearly true and the second condition reduces to $a \geq (a_1+a_n)/2$.

Case 2. $2a < \min\{a_1, a_2, \ldots, a_n\}+\max\{a_1, a_2, \ldots, a_n\}$.

Without loss of generality, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then $a \leq (a_1+a_n)/2$. According to Karamata's Inequality, it is enough to show that $a \leq \min\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$ and $a_n \geq \max\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$. The second condition is clearly true and the first condition reduces to $a \leq (a_1+a_n)/2$.

Some Applications.

Proposition 1. Let $a_1, a_2, \ldots, a_n$ be positive numbers with $a_1a_2\cdots a_n = 1$. Then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \;\geq\; (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Proof: This inequality follows from Theorem 1 if we consider the convex function $f(x) = e^x$ and replace $a_1, a_2, \ldots, a_n$ with $(n-1)\ln a_1$, $(n-1)\ln a_2, \ldots, (n-1)\ln a_n$, respectively.

Remark. For $n = 3$ and $a_1 = \dfrac{x^2}{yz}$, $a_2 = \dfrac{y^2}{zx}$, $a_3 = \dfrac{z^2}{xy}$, this inequality reduces to

$$x^6 + y^6 + z^6 + 3(xyz)^2 \;\geq\; 2(y^3z^3 + z^3x^3 + x^3y^3).$$

This inequality was proposed by Murray Klamkin in [5].

Proposition 2. If $a_1, a_2, \ldots, a_n$ are positive numbers satisfying $a_1 + a_2 + \cdots + a_n = n$, then

$$(n-a_1)(n-a_2)\cdots(n-a_n) \;\geq\; (n-1)^n(a_1a_2\cdots a_n)^{\frac{1}{n-1}}.$$

Proof: We apply Theorem 1 to the convex function $f(x) = -\ln x$ for $x > 0$.

Remark. Since $a_1 + a_2 + \cdots + a_n = n$ implies that $a_1a_2\cdots a_n \leq 1$ (the AM–GM Inequality), the above inequality is sharper than the inequality

$$(n-a_1)(n-a_2)\cdots(n-a_n) \;\geq\; (n-1)^n a_1a_2\cdots a_n,$$

which easily follows by multiplying the inequalities

$$\begin{aligned}
n - a_1 &= a_2 + a_3 + \cdots + a_n \;\geq\; (n-1)(a_2a_3\cdots a_n)^{\frac{1}{n-1}},\\
n - a_2 &= a_1 + a_3 + \cdots + a_n \;\geq\; (n-1)(a_1a_3\cdots a_n)^{\frac{1}{n-1}},\\
&\;\;\vdots\\
n - a_n &= a_1 + a_2 + \cdots + a_{n-1} \;\geq\; (n-1)(a_1a_2\cdots a_{n-1})^{\frac{1}{n-1}}.
\end{aligned}$$

Proposition 3. If $a_1, a_2, \ldots, a_n$ are positive numbers and $b_i = \dfrac{1}{n-1}\sum\limits_{j\neq i} a_j$ for all i, then

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \;\geq\; \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}. \tag{3}$$

Proof: We can prove this well-known inequality by applying Theorem 1. Let $a = (a_1 + a_2 + \cdots + a_n)/n$. Using the relations

$$\frac{(n-1)b_i}{a_i} = \frac{na}{a_i} - 1 \qquad \text{and} \qquad \frac{a_i}{b_i} = \frac{na}{b_i} - n + 1$$

for $i = 1, 2, \ldots, n$, we see that (3) is equivalent to the inequality

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{n(n-2)}{a} \;\geq\; (n-1)\left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}\right).$$

But this easily follows from Theorem 1 by using the convex function $f(x) = 1/x$ for $x > 0$.

Proposition 4. Suppose that $x_1, x_2, \ldots, x_n$ are positive numbers such that

$$x_1 + x_2 + \cdots + x_n \;=\; \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\,. \tag{4}$$

Then

$$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \;\geq\; 1\,. \tag{5}$$

Proof: This inequality may be derived from (3) in the following way. Suppose that

$$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \;<\; 1\,; \tag{6}$$

in other words, we suppose that (5) is false. Then we will show that (4) also does not hold. To this end, let $a_i = \dfrac{1}{1+(n-1)x_i}$ for $i = 1, 2, \ldots, n$. Note that $a_i > 0$ and that $x_i = \dfrac{1-a_i}{(n-1)a_i}$ for all $i = 1, 2, \ldots, n$. We also have $\sum a_i < 1$ by (6). Hence,

$$1 - a_i \;>\; \sum_{j \neq i} a_j \;=\; (n-1)b_i \tag{7}$$

for all $i = 1, 2, \ldots, n$. Thus,

$$\begin{aligned} x_1 + \cdots + x_n \;=\; \sum_{i=1}^{n} \frac{1-a_i}{(n-1)a_i} \;&>\; \sum_{i=1}^{n} \frac{b_i}{a_i} && \text{by (7)} \\ &\geq\; \sum_{i=1}^{n} \frac{a_i}{b_i} && \text{by (3)} \\ &>\; \sum_{i=1}^{n} \frac{(n-1)a_i}{1-a_i} && \text{by (7)} \\ &=\; \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\,. \end{aligned}$$

Hence, we have proved that if (5) does not hold, then (4) does not hold. Therefore, (4) implies (5), by the contrapositive.

Remark. Substituting $1/x_i$ for x_i in (5) and noting that (4) is still satisfied gives us

$$\frac{x_1}{n-1+x_1} + \frac{x_2}{n-1+x_2} + \cdots + \frac{x_n}{n-1+x_n} \;\geq\; 1\,.$$

Since

$$\frac{x_i}{n-1+x_i} = 1 - \frac{n-1}{n-1+x_i}$$

for each $i = 1, 2, \ldots, n$, the inequality can be rewritten in the form

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1\,.$$

This is an inequality in [2] that was derived using Lagrange Multipliers.

Proposition 5. If $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$(n-1)(x_1^2 + x_2^2 + \cdots + x_n^2) + n(x_1^2 x_2^2 \cdots x_n^2)^{\frac{1}{n}} \geq (x_1 + x_2 + \cdots + x_n)^2\,.$$

Proof: This is a known inequality [6] which follows from Theorem 2 using the convex function $f(x) = e^x$ and replacing $a_1, a_2, \ldots, a_n$ with $2\ln x_1$, $2\ln x_2, \ldots, 2\ln x_n$, respectively.

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