Maximum Area of a Triangle
Subject to Side Constraints

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An old Monthly problem by C.N. Schmall [3] was to show that the
maximum area $F$ of a triangle of sides $a$, $b$, $c$, where $a^3 + b^3 + c^3 = 3k^3$,
is $\sqrt{3}k^2/4$ and occurs when $a = b = c$. The solution by E. Swift was via
calculus and it was noted that the same method shows that the result also
holds if the constraint condition is $a^n + b^n + c^n = 3k^n$, $n > 1$.

Here we give generalizations without using calculus (which should not
be needed for elementary triangle inequalities). Firstly (see [2]),

$$F^3 \leq 3\sqrt{3}(abc)^2/64,$$

since $abc = 4FR$ and $R^2 \geq 4F/(3\sqrt{3})$, which geometrically corresponds to
the fact that the triangle of largest area which can be inscribed in a given circle
is the equilateral one. We now use the following variants of the
AM–GM Inequality which hold for all positive $a$, $b$, $c$:

$$(abc)^{\frac{2}{3}} \leq \sqrt[3]{((bc)^n + (ca)^n + (ab)^n)/3} \leq (a^n + b^n + c^n)/3$$

for $n > 0$. It follows that if any one of the latter three expressions is given,
all the other expressions to the left including the area $F$ are maximized when
$a = b = c$.

The above results can be generalized to $n$-dimensional simplexes by
starting with the known inequality [2]

$$V^{n-1} \leq n! \sqrt[3]{\frac{(n+1)^{n-1}}{n^3n} \left( \prod_{k=1}^{n+1} F_k \right)^{\frac{n}{n+1}}}, \quad (1)$$

where $V$ is the volume and $F_k$ are the volumes of the $(n-1)$-dimensional
facets. From this, we will now show that

$$V^{\frac{n+2}{2}} \leq c(n) \prod_{k=1}^{N} e_k, \quad (2)$$

where $N = n(n+1)/2$, $e_k$ are the edges of the simplex, and $c(n)$ is the
$(n+1)/2$ power of the volume of a regular simplex of unit edge length. Our
proof is inductive.

Write inequality (1) in the form $V^{n-1} \leq \lambda(n) \left( \prod_{k=1}^{n+1} F_k \right)^{\frac{n}{n+1}}$, and
assume the validity of (2) for $(n-1)$ dimensions. Then $F_i^{\frac{n}{2}} \leq c(n-1) \prod_{j} e_{ij}$,
where $e_{ij}$, for $j = 1, 2, \ldots, n(n-1)/2$, are the edge lengths of the $i^{th}$ facet. This implies that

$$\left(\prod_{i=1}^{n+1} F_i\right)^{\frac{n}{2}} \leq c(n-1)^{n+1} \left(\prod_{k=1}^{N} e_k\right)^{n-1},$$

since each edge of the given simplex belongs to $n - 1$ facets. Therefore,

$$V^{n-1} \leq \lambda(n)c(n-1)^{2} \left(\prod_{k=1}^{N} e_k\right)^{\frac{2(n-1)}{n+1}}.$$ 

Hence,

$$V^{\frac{n+1}{2}} \leq \lambda(n)^{\frac{n+1}{2(n-1)}}c(n-1)^{\frac{n+1}{n-1}} \left(\prod_{k=1}^{N} e_k\right) = c(n) \prod_{k=1}^{N} e_k.$$ 

Noting that $V^{\frac{n+1}{2}} / \prod_{k=1}^{N} e_k$ is invariant under similarity gives us the fact that $c(n)$ is the constant described above. Thus, the validity for the $(n - 1)$ simplices implies the validity for dimension $n$.

We now use the Maclaurin Inequalities [1]. Letting

$$(x + e_1^m) \cdots (x + e_{n+1}^m)$$

$$= x^{n+1} + \binom{n+1}{1} p_1 x^n + \binom{n+1}{2} p_2 x^{n-1} + \cdots + p_{n+1},$$

where $m > 0$, we have $p_1 \geq p_2 \geq \cdots \geq p_n \geq p_{n+1}$, with equality if and only if all the $e_j$s are equal. This implies that if any one of the $p_j$s is given, then all the $p_j$s where $j > i$ and $V$ achieve their maxima if and only if all the $e_j$s are equal.

References.


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