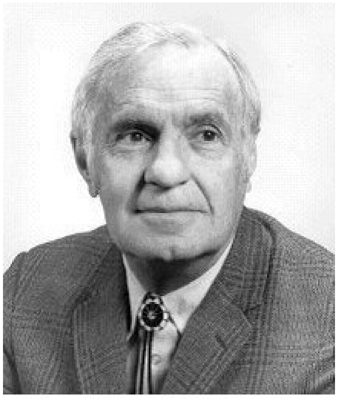
A Celebration of the Life of Murray Seymour Klamkin

Andy Liu



(1921–2004)

Let me first make it clear that this is not a eulogy. By my definition, a eulogy is an attempt to make the life of the departed sound much better than it was. In the present case, it is not only unnecessary, it is actually impossible. Murray Seymour Klamkin had a most productive and fulfilling life, divided between industry and academia.

Of the early part of his life, I knew little except that he was born in 1921 in Brooklyn, New York, where his father owned a bakery. This apparently induced in him his life-long fondness for bread. I read in his curriculum vitae that his undergraduate degree was in Chemical Engineering, obtained in 1942 from Cooper Union's School of Engineering. During the war, he was attached to a chemical warfare unit stationed in Maryland, as his younger sister, Mrs. Judith Horn, informed me.

In 1947, Murray obtained a Master of Science degree from the Polytechnic Institute of New York. He taught there until 1957, when he joined AVCO's Research and Advanced Development Division. In 1962, Murray returned briefly to academia as a professor at SUNY, Buffalo, and then became a visiting professor at the University of Minnesota. In 1965, he felt again the lure of industry and joined the Ford Motor Company as the Principal Research Scientist, staying there until 1976.

During all this time, Murray was extremely active in the field of mathematics problem solving. His main contribution was serving as the editor of the problem section of *SIAM Review*. He had a close working relationship with the Mathematical Association of America, partly arising from his involvement with the William Lowell Putnam Mathematics Competition.

In 1972, the MAA started the USA Mathematical Olympiad, paving the way for the country's entry into the International Mathematical Olympiad in 1974, hosted by what was then East Germany.

Murray was unable to obtain from Ford release time to coach the team. Disappointed, he began to look elsewhere for an alternative career. This brought him to Canada, at first as a Professor of Applied Mathematics at the University of Waterloo.

An abridged version of this article appeared in Math Horizons, November 2004, p. 23

However, it was not until the offer came from the University of Alberta that he made up his mind to leave Ford. I did not know if Murray had been to Banff before, but he must have visited this tourist spot during the negotiation period, fell in love with the place and closed the deal.

As Chair of the Mathematical and Statistical Sciences Department, Murray brought with him a management style from the private sector. Apparently not everyone was happy with that, but he did light some fires under several pairs of pants and rekindled the research programs of the wearers.

Murray was always interested in Euclidean Geometry. He often told me about his high school years when he and a friend would challenge each other to perform various Euclidean constructions. Although the Chair had no teaching duties at the time, Murray took on a geometry class himself.

At the same time, Murray began editing the Olympiad Corner in ***Crux Mathematicorum***, a magazine then published privately by Professor Leo Sauvé of Ottawa. It is now an official journal of the Canadian Mathematical Society. Murray also introduced the Freshmen and Undergraduate Mathematics Competitions in the Department.

Geometry, mathematics competitions, and ***Crux Mathematicorum*** were what brought me to Murray's attention. At the time, I was a post-doctoral fellow seeking employment, having just graduated from his Department. Thus, I was ready to do anything, and it happened that my interests coincided with those of Murray. I was maintaining office hours for his geometry class, helping to run the Department's competitions, and assisting him in his editorial duty.

I remember being called into his office one day. He had just received a problem proposal for ***Crux Mathematicorum***. "Here is a nice problem," he said, "but the proposer's solution is crappy. Come up with a nice solution, and I need it by Friday afternoon!"

As much as I liked problem-solving, I was not sure that I could produce results by an industrial schedule. Nevertheless, I found that I did respond to challenges, and although I was not able to satisfy him every time, I managed to do much better than if I was left on my own, especially after I had got over the initial culture shock.

The late seventies were hard times for academics, with few openings in post-secondary institutions. I was short-listed for every position offered by the Department, but always came just short. Eventually, I went elsewhere for a year as sabbatical replacement. Murray came over to interview me for a new position, pushed my appointment through the Hiring Committee, and brought me back in 1980.

Murray had been the Deputy Leader for the USA National Team in the IMO since 1975. In 1981, the USA became the host of the event, held outside Europe for the first time. Sam Greitzer, the usual Leader, became the chief organizer. Murray took over as the Leader, and secured my appointment as his Deputy Leader.

I stayed in that position for four years, and in 1982, made my first trip to Europe because the IMO was in Budapest. This was followed by IMO 1983 in Paris, and IMO 1984 in Prague. I was overawed by the international assembly, but found that they in turn were overawed by Murray's presence. He was arguably the most well-known mathematics problem-solver in the entire world.

We both retired from the IMO after 1984, even though I would later return to it. His term as Chair also expired in 1981. Thus, our relationship became collegial and personal. He and his wife Irene had no children, but they were very fond of company. I found myself a guest at their place at regular intervals, and they visited my humble abode a few times.

It was during this period that I saw a different side of Murray. Before, I found him very businesslike, his immense talent shining through his incisive insight and clinical efficiency.

Now I found him a warm person with many diverse interests, including classical music, ballroom dancing, adventure novels, kung-fu movies, and sports, in particular basketball.

Although Murray was highly successful in everything he attempted, he will probably be remembered most for his involvement in mathematics problem-solving and competitions. He authored or edited four problem books and left his mark in every major journal which had a problem section. He received an Honorary Doctorate from the University of Waterloo and was a Fellow of the Royal Society of Belgium. He won numerous prizes, and had some named after him.

Murray enjoyed remarkably good health during his long life. His health began to deteriorate in September 2000 when he underwent a by-pass operation. After his release from the hospital, he continued to exert himself, walking up to his office on the sixth floor, and skating in the West Edmonton Mall.

His heart valve gave way in November, fortunately while he was already in the hospital for physiotherapy. He was in a coma for some time. One day, when I visited him, he was bleeding profusely from his aorta. The doctor indicated to me that he did not expect Murray to last through the day.

Somehow, the inner strength of Murray came through, and on my next visit, he was fully conscious. He told me to make arrangements for his 80^{th} birthday party, stating simply that he would be out of the hospital by that time. It was a good thing that I took his words seriously, for he was out of the hospital by that time, ready to celebrate.

One of the last mathematical commitments he made was to edit the problem section in the MAA's new journal *Math Horizons*. During this difficult time, he asked me to serve with him as joint editor. Later, he passed the column on to me, but his finger-prints were still all over the pages.

Now I have to try to fill his shoes without the benefit of his wisdom. His passing marks the end of an era in the world of mathematics competitions and problem-solving. He will be deeply missed.

Klamkin Commemorative Issue

In the October 2004 issue of ***CRUX with MAYHEM***, having just learned that Murray Seymour Klamkin had passed away on August 6, 2004, we announced that we were planning a commemorative issue of ***CRUX with MAYHEM*** in his honour. Now we are pleased to present this issue to our readers.

Murray had been a long-time supporter of ***Crux Mathematicorum*** and ***CRUX with MAYHEM***, having created the Olympiad Corner and served as its first editor. He continued to contribute to ***CRUX with MAYHEM*** over the years through his ever-popular "Klamkin Quickies" in the Olympiad Corner, as well as many and varied problem proposals and solutions for that part of the journal, and an occasional article.

Accompanying our announcement of the Klamkin Commemorative Issue was a short biography prepared by Murray's long-time colleague and friend, Andy Liu. Andy later enlarged this article for *Math Horizons* and for ***CRUX with MAYHEM***. The enlarged article begins this issue.

We have received many contributions from our readers, including several problems and some articles dedicated to Murray's lasting memory, which you will find in this issue. As a bonus, we also have an article which Murray co-authored with G.D. Chakerian. This article was in the process of being refereed at the time that Murray passed away, which means that it is one of the last (if not the last) article that Murray wrote.

Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain enclosed a copy of a brief article he wrote for the on-line journal, Revista Escolar de la Olimoiada Iberoamericana de Matemática, for which he is the editor. Below is an excerpt (loosely translated from the Spanish) from that article, reprinted with permission.

". . . my first knowledge of Murray was as a reader and admirer of his articles and solutions, always brilliant. If my memory serves me well, my first exchange of letters with him was in 1983. He was widely known for many years, and at that time he was editing the Problem Corner of *The Mathematical Intelligencer*. I sent him a solution to one of his problem proposals, and he replied by mail with one of the most appreciated recommendations of my entire professional life. He said: 'You should get to know the journal ***Crux Mathematicorum***'. Of course, I followed his advice and I can tell you that today I have in my library the complete collection of the best journal of elementary mathematical problems in the whole world. For that, I will always be in debt to Murray."

The Canadian Mathematical Society, with the assistance of many of Murray's colleagues and in collaboration with the Mathematical Association of America, will be publishing an annotated collection of Murray's problem-solving work, which should begin to appear in 2006. Details will appear at `www.cms.math.ca/Publications/books`

SKOLIAD No. 87

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 January, 2006***. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our items this issue come from the 2005 BC Colleges Junior High School Mathematics Contest (Preliminary and Final rounds). Thanks go to Clint Lee of Okanagan College for supplying the material.

BC Colleges High School Mathematics Contest 2005
Junior Preliminary Round
Wednesday, March 2, 2005

1. A wire is cut into two parts in the ratio $3 : 2$. Each part is bent to form a square. The ratio of the perimeter of the larger square to the perimeter of the smaller square is:

(A) $3 : 2$ (B) $9 : 4$ (C) $5 : 3$ (D) $5 : 2$ (E) $12 : 5$

2. Given the following

I. even II. odd III. a perfect square IV. a multiple of 5

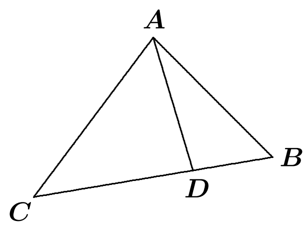
then it is true that the product $21 \times 35 \times 15$ is:

(A) II & IV (B) I & IV (C) II & III (D) III & I (E) II, III, & IV

3. The radius of the largest sphere that can fit entirely inside a rectangular box with dimensions 5 cm $\times 7$ cm $\times 11$ cm is:

(A) 2 cm (B) $\frac{5}{2}$ cm (C) 3 cm (D) $\frac{23}{6}$ cm (E) $\frac{9}{2}$ cm

4. In the diagram, the area of the triangle ABC is 60. If DB is one third of CB, then the area of triangle ACD is:



(A) 20 (B) 30 (C) 40

(D) 45 (E) 50

5. If $\dfrac{1}{n+5} = 4$, then $\dfrac{1}{n+6}$ equals:

(A) 3 (B) $\frac{1}{5}$ (C) $\frac{5}{4}$ (D) $\frac{4}{5}$ (E) None of these

6. A standard 6-sided die is tossed twice. The probability of obtaining a sum of 5 is:

(A) $\frac{1}{12}$ (B) $\frac{1}{9}$ (C) $\frac{5}{36}$ (D) $\frac{1}{6}$ (E) $\frac{2}{9}$

7. A rectangle has dimensions 20 cm $\times$ 50 cm. If the length is increased by 20% and the width is decreased by 20%, then the change in the area is:

(A) an 8% increase (B) a 4% increase (C) a 0% increase

(D) a 4% decrease (E) an 8% decrease

8. The greatest prime factor of 21831 is:

(A) 435 (B) 57 (C) 783 (D) 383 (E) 10917

9. The number of houses sold in Kamloops in 2004 is exactly 40% more than the number sold in 2003. Assuming that at least one house was sold in 2004, the smallest possible number of houses sold in Kamloops in 2004 is:

(A) 5 (B) 7 (C) 14 (D) 70 (E) 140

10. The minimum number of students that must be in a room to ensure that at least 10 are boys or at least 10 are girls is:

(A) 10 (B) 11 (C) 18 (D) 19 (E) 20

11. The number of integers that satisfy the inequality $\frac{3}{7} < \frac{n}{14} < \frac{2}{3}$ is:

(A) 0 (B) 2 (C) 3 (D) 4 (E) 5

12. Given that $20! = 20 \times 19 \times 18 \times \cdots \times 2 \times 1$ and 2^n is a factor of $20!$, then the largest possible value of n is:

(A) 10 (B) 12 (C) 18 (D) 20 (E) 24

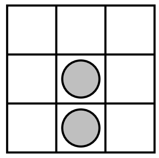
13. Terry has \$28.00 in nickels, dimes, and quarters. The value of the dimes is twice the value of the quarters, and it is half the value of the nickels. The total number of coins that Terry has is:

(A) 72 (B) 264 (C) 416 (D) 560 (E) 632

14. The number 2005 can be written in the form $a^2 - b^2$, where a and b are integers that are greater than one, in exactly one way. The value of $a^2 + b^2$ is:

(A) 160825 (B) 160801 (C) 80418 (D) 80413 (E) 80406

15. The game of Solitaire JumpIt is played on a 3×3 grid with two identical game discs. If the two discs are adjacent horizontally, vertically, or diagonally, one disc can jump the other by moving onto the open space opposite the other disc. The disc that is jumped is removed. (See the diagram). The number of ways to place two identical game discs on the grid so that no jump is possible is:



(A) 16 (B) 20 (C) 24 (D) 32 (E) 40

BC Colleges High School Mathematics Contest 2005
Junior Final Round, Part A
Friday, May 6, 2005

1. Two operations $\star$ and $\diamond$ are defined by the two tables below:

$\star$	1	2	3
1	1	3	2
2	1	3	1
3	3	3	1

$\diamond$	1	2	3
1	4	2	3
2	3	6	5
3	2	6	4

For example, $1 \diamond 2 = 2$. The value of $2 \diamond (3 \star 3)$ is:

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

2. Three people leave their coats in a check room. When they check out, three coats are distributed randomly among them. The probability that *none* of the three receives the correct coat is:

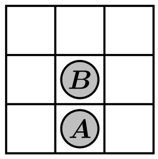
(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{5}{6}$

3. A grocer uses a pan balance on which weights can be placed on either of the pans together with the object being weighed. The grocer has three weights that will balance precisely any whole number of kilograms from 1 kg to 13 kg. The three weights are:

(A) 2, 5, 6 (B) 3, 4, 6 (C) 1, 5, 7 (D) 2, 4, 7 (E) 1, 3, 9

4. The number of cards that must be drawn from a standard deck of 52 playing cards to be sure that at least two are aces or three are of the same suit is:

(A) 9 (B) 13 (C) 27 (D) 49 (E) 50



5. The game of Solitaire JumpIt is played on a 3×3 grid. A single player places two or more game discs on the grid. If two discs, A and B, are adjacent horizontally, vertically, or diagonally and there is an open space on the side of B away from A, then A can jump B and disc B is removed. (See the diagram.) The player makes jumps as long as possible. The player wins if he or she can continue until only one disc remains. The maximum number of discs that can be placed on the grid in a way that the player still wins is:

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

6. The product $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n-1}\right)\left(1+\frac{1}{n}\right)$ is equal to:

(A) 1 (B) $\frac{1}{n}$ (C) $\frac{n+1}{n}$ (D) -1 (E) None of these

7. The number of integers between 500 and 600 which have 12 as the sum of their digits is:

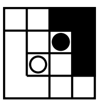
(A) 6 (B) 7 (C) 8 (D) 10 (E) 12

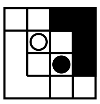
8. A grid point in the plane is a point (x, y) for which both x and y are integers. The number of grid points that lie within or on the boundary of the region bounded by the parabola $y = x^2$ and the line $y = 50$ is:

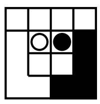
(A) 470 (B) 485 (C) 490 (D) 750 (E) 765

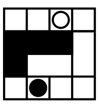
9. *Wot th'ell* is a game played on a 4×4 checker board. Both players have an *L*-shaped piece which covers four squares and a disc which covers one. The players alternate moves, one playing white pieces and the other playing black. A move consists of picking up the *L*-shaped piece, possibly turning it over, and placing it back on the board in a new position. Then the player removes his disk and puts it back on the board (the disk may be returned to where it came from). Neither the *L*-shaped piece nor the disc can be placed so that it covers any square that is already occupied by a disc or an *L*-shaped piece. A player who is unable to move loses.

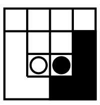
The one of the following boards on which white can play and win is:

(A) 

(B) 

(C) 

(D) 

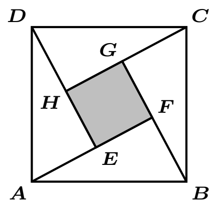
(E) 

10. There is a critical height (which is a whole number of floors above ground level), such that an egg dropped from that height (or higher) will break, but if dropped from a lower height (no matter how many times), it will not break. You are given two eggs and told that the critical height is between 1 floor and 37 floors (inclusive). You want to develop a plan that gives the most efficient way to determine this critical height. Obviously, starting at the first floor and going up one floor at a time could require only one drop, if the critical height is 1; but it could require as many as 37 drops, if the critical height is 37. The optimum plan will give the smallest value for the maximum possible number of drops. The maximum possible number of drops required by the optimum plan for determining the critical height is:

(A) 8 (B) 9 (C) 12 (D) 19 (E) 35

BC Colleges High School Mathematics Contest 2005
Junior Final Round, Part B
Friday, May 6, 2005

1. In the diagram, $ABCD$ is a square with side length 17 and the four triangles ABF, DAE, BCG, and CDH are congruent right triangles. Furthermore, $\overline{FB} = 8$. Find the area of the shaded quadrilateral $EFGH$.



2. A party went to a restaurant for dinner. At the end of the meal they decided to split the bill evenly among them. If each contributed \$16, they found that they were \$4 short, while if each put in \$19, they had enough to pay the bill, 15% for the tip, and \$2 left over. How much was the bill, and how many were in the party?

3. Find the number of solutions in integers (x, y) of the equation

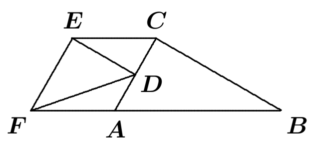
$$x^2y^3 = 6^{12}.$$

4. Nellie is 5 km south of a stream that flows due east. She is 8 km west and 6 km north of her cabin. She wishes to water her horse at the stream and then return to her cabin. What is the shortest distance that Nellie must travel?

5. In the diagram, triangle ABC is a 30°–60°–90° triangle with the right angle at vertex C, the 30° angle at vertex B, and side AB having length 20. Segment ED is perpendicular to side AC and D bisects AC. Segment EC is parallel to AB. Segment EF is perpendicular to ED and F is on the extension of AB.

(a) Find the length of segment ED.

(b) Find the length of segment DF.



Collèges de Colombie Britannique 2005
Concours Junior de Mathématiques du Secondaire
Ronde Préliminaire, Mercredi, 2 Mars 2005

1. Un fil est coupé en deux morceaux dans un ratio $3 : 2$. Chaque partie est tordue pour former un carré. Le ratio du périmètre du grand carré sur celui du petit est :

(A) $3 : 2$ (B) $9 : 4$ (C) $5 : 3$ (D) $5 : 2$ (E) $12 : 5$

2. Étant donné les descriptions :

I. pair II. impair III. un carré parfait IV. un multiple de 5

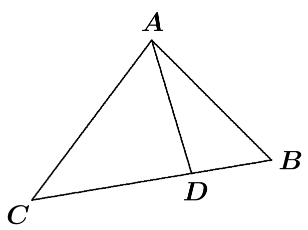
il est vrai que le produit $21 \times 35 \times 15$ est :

(A) II & IV (B) I & IV (C) II & III (D) III & I (E) II, III, & IV

3. Le rayon de la plus grande sphère qui rentre complètement dans une boîte rectangulaire de dimensions 5 cm $\times 7$ cm $\times 11$ cm est :

(A) 2 cm (B) $\frac{5}{2}$ cm (C) 3 cm (D) $\frac{23}{6}$ cm (E) $\frac{9}{2}$ cm

4. Dans le dessin, l'aire du triangle ABC est 60. Si DB est un tiers de CB, alors l'aire du triangle ACD est :



(A) 20 (B) 30 (C) 40

(D) 45 (E) 50

5. Si $\dfrac{1}{n+5} = 4$, alors $\dfrac{1}{n+6}$ égale :

(A) 3 (B) $\frac{1}{5}$ (C) $\frac{5}{4}$ (D) $\frac{4}{5}$ (E) Aucun de ces choix

6. Un dé normal à six faces est lancé deux fois. La probabilité d'obtenir une somme de 5 est :

(A) $\frac{1}{12}$ (B) $\frac{1}{9}$ (C) $\frac{5}{36}$ (D) $\frac{1}{6}$ (E) $\frac{2}{9}$

7. Un rectangle est de dimensions 20 cm $\times$ 50 cm. Si la longueur est augmentée de 20% et la largeur est réduite de 20%, alors la variation de l'aire est :

(A) 8% plus grande (B) 4% plus grande (C) 0% plus grande

(D) 4% plus petite (E) 8% plus petite

8. Le plus grand facteur premier de 21831 est :

(A) 435 (B) 57 (C) 783 (D) 383 (E) 10917

9. Le nombre de maisons vendues à Kamloops en 2004 est exactement 40% de plus que le nombre vendues en 2003. En supposant qu'au moins une maison a été vendue en 2004, le plus petit nombre possible de maisons vendues à Kamloops pour 2004 est :

(A) 5 (B) 7 (C) 14 (D) 70 (E) 140

10. Le nombre minimum d'étudiants qui doivent se trouver dans une salle pour être sûr d'avoir au moins 10 gars ou au moins 10 filles est :

(A) 10 (B) 11 (C) 18 (D) 19 (E) 20

11. Le nombre d'entiers qui satisfont à l'inégalité $\frac{3}{7} < \frac{n}{14} < \frac{2}{3}$ est :

(A) 0 (B) 2 (C) 3 (D) 4 (E) 5

12. Étant donné que $20! = 20 \times 19 \times 18 \times \cdots \times 2 \times 1$ et que 2^n est un facteur de 20!, alors la plus grande valeur possible de n est :

(A) 10 (B) 12 (C) 18 (D) 20 (E) 24

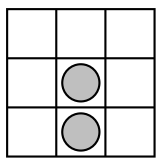
13. Thierry a $28,00 en cinq sous, dix sous et vingt-cinq sous. La valeur des dix sous est le double de la valeur des vingt-cinq sous et vaut la moitié des cinq sous. Le nombre total de pièces à Thierry est :

(A) 72 (B) 264 (C) 416 (D) 560 (E) 632

14. Le nombre 2005 peut s'écrire sous la forme $a^2 - b^2$, où a et b sont des entiers plus grands que un, d'une seule manière. La valeur de $a^2 + b^2$ est

(A) 160825 (B) 160801 (C) 80418 (D) 80413 (E) 80406

15. Le jeu de Solitaire Saute-Le est joué sur un grillage 3×3 avec deux jetons identiques. Si les deux jetons sont adjacents horizontalement ou diagonalement, alors un jeton peut sauter par dessus l'autre en bougeant sur l'espace libre de l'autre côté du jeton. Le jeton sauté est enlevé (voir le dessin). Le nombre de manières de placer 2 jetons identiques sur le grillage pour qu'aucun saut ne soit possible est :



(A) 16 (B) 20 (C) 24 (D) 32 (E) 40

Collèges de Colombie Britannique 2005
Concours Junior de Mathématiques du Secondaire
Ronde Finale, Partie A, Vendredi, 6 Mai 2005

1. Deux operations $\star$ and $\diamond$ sont définies par les deux tables suivantes :

$\star$	1	2	3
1	1	3	2
2	1	3	1
3	3	3	1

$\diamond$	1	2	3
1	4	2	3
2	3	6	5
3	2	6	4

Par exemple, $1 \diamond 2 = 2$. La valeur de $2 \diamond (3 \star 3)$ est :

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

2. Trois personnes laissent leurs manteaux au vestiaire. Lorsqu'ils sortent, trois manteaux sont distribués au hasard parmi eux. La probabilité qu'*aucun* d'eux ne reçoive son manteau est :

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{5}{6}$

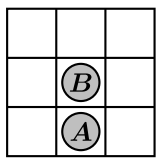
3. Un épicier utilise une balance à plateaux où des poids peuvent être placés sur n'importe lequel d'entre eux indépendamment de l'objet pesé. L'épicier a trois poids qui peuvent être utilisés pour peser précisement n'importe quel poids en kilogrammes allant de 1 kg à 13 kg. Les trois poids de l'épicier sont :

(A) $2, 5, 6$ (B) $3, 4, 6$ (C) $1, 5, 7$ (D) $2, 4, 7$ (E) $1, 3, 9$

4. Le nombre de cartes qui doivent être pigés d'un paquet normal de 52 cartes pour être sûr d'avoir au moins deux aces ou trois d'une même sorte est :

(A) 9 (B) 13 (C) 27 (D) 49 (E) 50

5. Le jeu de Solitaire Saute-Le est joué sur un grillage 3×3. Un joueur place deux jetons ou plus sur le grillage. Si deux jetons A et B sont adjacents horizontalement, verticalement ou diagonalement, et s'il y a un espace libre du côté à B opposé à A, alors A peut sauter par-dessus B et le jeton B est enlevé. (Voir le dessin.) Le joueur essai de sauter le plus longtemps possible. Le joueur gagne si il ou elle peut continuer jusqu'à ce qu'un seul jeton reste. Le nombre maximum de jetons que l'on peut mettre sur le grillage pour que le joueur gagne est :



(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

6. Le produit $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n-1}\right)\left(1+\frac{1}{n}\right)$ est égal à :

(A) 1 (B) $\frac{1}{n}$ (C) $\frac{n+1}{n}$ (D) -1 (E) Aucun de ces choix

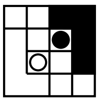
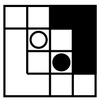
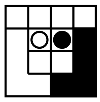
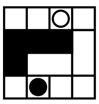
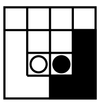
7. Le nombre d'entiers entre 500 et 600 dont la somme des chiffres est 12 est :

(A) 6 (B) 7 (C) 8 (D) 10 (E) 12

8. Un point entier du plan (x, y) est tel que x et y sont entiers. Le nombre de points entiers qui gisent à l'intérieur ou sur la frontière de la région limitées par la parabole $y = x^2$ et la ligne $y = 50$ est :

(A) 470 (B) 485 (C) 490 (D) 750 (E) 765

9. *Wot th'ell* est un jeu qui se joue sur un damier 4×4. Les deux joueurs ont chacun un morceau en forme de *L* qui couvre quatre cases et un jeton qui couvre une case. Les joueurs, dont un joue les blancs et l'autre les noirs, alternent leurs coups. Un coup consiste à ramasser une pièce en *L* pour la replacer sur le jeu dans une nouvelle position quitte à la retourner. Il doit ensuite prendre son jeton et le replacer sur le jeu (il peut revenir à la case qu'il a quitté). Le *L* ou le jeton ne peuvent pas chevaucher une case qui est déjà occupée. Le joueur qui ne peut plus jouer perd. Le jeu où le blanc peut jouer et gagner est :

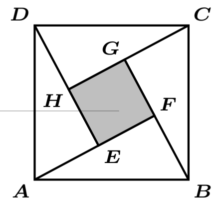
(A) 
(B) 
(C) 
(D) 
(E) 

10. Il y a une hauteur critique (qui est un nombre entier d'étages au-dessus du sol), telle qu'un oeuf largué de cette hauteur (ou au-dessus) va casser, mais que si on le largue d'une hauteur plus basse (quel qu'en soit le nombre de fois) ne cassera pas. On vous procure deux oeufs et on vous dit que cette hauteur est entre 1 et 37 étages (inclusivement). Vous voulez développer un plan qui permet de l'identifier de la manière la plus efficace. Évidement, on pourrait commencer au premier étage et monter un étage à la fois. Cette stratégie pourrait prendre un coup si la hauteur critique est un étage, mais cette technique prendrait jusqu'à 37 coups si celui-ci est la hauteur critique. La stratégie optimale va nécéssiter le nombre minimal de chutes. Dans le pire scénario, le nombre de chute nécessaire pour identifier la hauteur critique avec la stratégie optimale est :

(A) 8 (B) 9 (C) 12 (D) 19 (E) 35

Collèges de Colombie Britannique 2005
Concours Junior de Mathématiques du Secondaire
Ronde Finale, Partie B, Vendredi, 6 Mai 2005

1. Dans le diagramme, $ABCD$ est un carré dont les côtés mesurent 17 et que les quatre triangles ABF, DAE, BCG et CDH sont des triangles rectangles congrus. En plus, $\overline{FB} = 8$. Trouver l'aire du quadrilatère $EFGH$ ombragé.



2. Un groupe est allé diner au restaurant. À la fin du repas, ils décident de partager la note également entre eux. S'ils contribuent chacun \$16, ils se retrouvent à court de \$4, alors que s'ils mettent \$19, ils peuvent payer la facture, 15% de pourboire et il reste \$2. Combien de gens étaient-ils et la note s'élevait à combien ?

3. Trouver le nombre de solutions entières (x, y) de l'équation

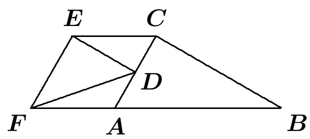
$$x^2y^3 = 6^{12}.$$

4. Nellie est 5 km au sud d'un cours d'eau qui va vers l'est. Elle est 8 km à l'ouest et 6 km au nord de sa cabine. Elle veut abreuver son cheval au cours d'eau puis retourner à sa cabine. Quelle est la plus courte distance que Nellie doit voyager ?

5. Dans le diagramme, le triangle ABC est un triangle $30°$–$60°$–$90°$ dont l'angle droit est à C, le $30°$ se trouve en B, et AB mesure 20. Le segment ED est perpendiculaire à AC et D est le milieu de AC. Le segment EC est parallèle à AB. Le segment EF est perpendiculaire à ED et F est sur le prolongement de AB.

(a) Trouver le longueur de ED.

(b) Trouver la longueur de DF.



Next we give the solutions to the Concours Montmorency 2002–2003 [2005 : 1–3].

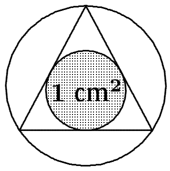
Concours Montmorency 2002–2003
Sec V, novembre 2002

1. ($*$) Show that the product of any two odd integers is always odd. (Warning: examples are not enough.)

Solution by Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

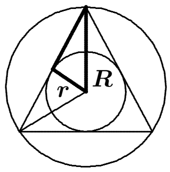
Because both numbers are odd, they can be written in the form $2k + 1$ where $k \in \mathbb{N}$. If $2a + 1$ and $2b + 1$ are the two odd numbers, then we have $(2a + 1) \times (2b + 1) = 2(2ab + b + a) + 1$, which is also in the form $2k + 1$ (in this case, $k = 2ab + b + a$ is clearly an integer). Thus, the product of two odd numbers is also odd.

2. $(*)$ L'aire d'un cercle inscrit dans un triangle équilatéral est 1 cm^2. Quelle est l'aire du cercle circonscrit au même triangle équilatéral ?



Solution officielle.

Soit r et R les rayons du cercle inscrit et du cercle circonscrit respectivement, et soit A l'aire du cercle circonscrit.



Dans le triangle équilatéral, tous les angles intérieurs sont de 60°. Le petit triangle rectangle (tracé en gras) possède donc un angle de 30°, car son hypoténuse coupe un angle du triangle équilatéral en son milieu.

Or, dans un triangle rectangle ayant un angle de 30°, la mesure du côté opposé à l'angle de 30° est la moitié de la mesure de l'hypoténuse. D'où $r = \frac{1}{2}R$. Donc le rapport des longueurs est $R/r = 2$.

Mais le rapport des aires est

$$\frac{A}{1 \text{ cm}^2} = \frac{\pi R^2}{\pi r^2} = \left(\frac{R}{r}\right)^2 = 4\,.$$

Donc, l'aire du grand cercle est 4 cm^2.

Solutioné aussi par Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

3. $(*)$ While descending a river, a boat goes 30 km/h; going up-river, the speed is only 22 km/h. To go between the cities of Bellerue and Beauparc on the river takes 4 hours less one way than the other. What is the distance between the cities?

Solution by Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

Let the distance in kilometres between the two cities be s. Let the time it takes to go down the river be t_1 hours and the time to go up the river be t_2 hours. Because the speed going down is faster than the speed going up, we have $t_1 < t_2$; thus, $t_2 = t_1 + 4$.

Since $t_1 = \dfrac{s}{\text{speed down the river}} = \dfrac{s}{30}$ and $t_2 = \dfrac{s}{\text{speed up the river}} = \dfrac{s}{22}$, we get

$$\begin{aligned} \frac{s}{22} - \frac{s}{30} &= 4\,, \\ \frac{4s}{330} &= 4\,, \\ s &= 330\,. \end{aligned}$$

Hence, the distance between the two cities is 330 km.

4. (∗) In algebra, the following simplification is not allowed: $\dfrac{2\cancel{x}}{\cancel{x}+y} = \dfrac{2}{y}$. Supposing that x and y are positive integers, show that the given equation is true only if $x = y = 2$.

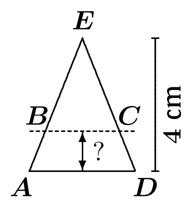
Solution by Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON, modified by the editor.

If the equation $\dfrac{2x}{x+y} = \dfrac{2}{y}$ is true, then we can simplify it to

$$\begin{aligned} 2xy &= 2(x+y)\,, \\ xy - x &= y\,, \\ x(y-1) &= y\,. \end{aligned}$$

We recall that x and y are positive integers. In fact, $y \geq 2$, since the above equation is not satisfied when $y = 1$. Thus, $y - 1$ is a positive integer. Now the equation tells us that $\dfrac{y}{y-1} = x$. Hence, $y - 1$ divides y evenly. But $y - 1$ and y are consecutive integers. Therefore, $y - 1$ can divide y evenly only when $y - 1 = 1$; that is, $y = 2$. This immediately yields $x = 2$.

5. (∗) On coupe un triangle isocèle AED par BC, une droite parallèle à sa base AD. À quelle hauteur au-dessus de cette base devons-nous couper le triangle pour que l'aire du trapèze $ABCD$ soit la moitié de l'aire du triangle AED sachant que la hauteur de ce dernier est de 4 cm ?



Solution officielle, modifiée légèrement par le rédacteur.

Si l'aire du trapèze $ABCD$ est la moitié de l'aire du triangle AED, alors l'aire du triangle BEC doit former l'autre moitié de AED. Donc,

$$\text{Aire } \triangle BEC = \frac{1}{2} \text{ Aire } \triangle AED\,.$$

Appelons x la hauteur de $\triangle BEC$ et y la hauteur du trapèze $ABCD$. Comme $\triangle AED$ se décompose en $\triangle BEC$ et le trapèze $ABCD$, on a $x + y = 4$.

Puisque l'aire du triangle BEC est la moitié de l'aire du triangle AED, nous avons que le rapport des aires de ces triangles est de $\frac{1}{2}$. Ceci nous permet de conclure que le rapport des hauteurs des triangles sera de $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Donc $\frac{x}{4} = \frac{\sqrt{2}}{2}$, donc $x = 2\sqrt{2}$. Puisque $x + y = 4$, on a

$$y \;=\; 4 - 2\sqrt{2} \;\approx\; 1,171572875\ldots$$

Solutioné aussi par Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

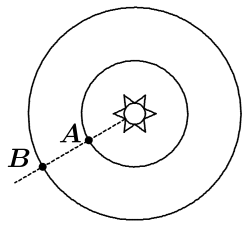
6. $(*)$ Decompose $x^3 - y^3 + x^2 - y^2 + x^2y - xy^2$ into 3 factors.

Solved by Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

Editor's comment: A few solutions are available, but all of them depend on double factoring, the shortest being:

$$\begin{aligned} &x^3 - y^3 + x^2 - y^2 + x^2y - xy^2 \\ &\qquad = x^3 - xy^2 + x^2y - y^3 + x^2 - y^2 \\ &\qquad = x(x^2 - y^2) + y(x^2 - y^2) + (x^2 - y^2) \\ &\qquad = (x^2 - y^2)(x + y + 1) \\ &\qquad = (x - y)(x + y)(x + y + 1)\,. \end{aligned}$$

7. $(*)$ Deux planètes A et B tournent sur des orbites circulaires autour d'un soleil central. La planète A est aujourd'hui exactement entre le soleil et la planète B. Sachant que A et B prennent respectivement 3 ans et 8 ans pour faire une révolution complète autour de leur soleil, évaluer le temps nécessaire pour qu'à nouveau, et pour la première fois, la planète A se retrouve exactement entre le soleil et la planète B.



Solution officielle, modifiée par le rédacteur.

Le plus petit commun multiple de 3 et 8 est 24. Ceci veut dire que dans 24 ans les 2 planètes seront exactement au même endroit qu'au départ. Cependant, on nous demande dans combien de temps les deux planètes seront alignées pour la première fois, mais pas nécessairement au même endroit. Au bout des 24 ans, la planète A aura fait 8 tours, car cela lui prend 3 ans pour faire 1 tour. La planète B aura fait 8 tours, car cela lui prend 8 ans pour faire 1 tour. Donc la planète A aura fait 5 tours de plus que la planète B. Et donc les planètes se seront alignées 5 fois. Donc le nombre d'années divisé par le nombre de fois qu'elle se sont alignées donnera le temps écoulé entre chaque alignement (et donc le temps du premier alignement), d'où :

$$\frac{\text{\# d'années}}{\text{\# d'alignements}} \;=\; \frac{24}{5} \;=\; 4,8\,\frac{\text{années}}{\text{alignement}}\,.$$

Ainsi, les deux planètes seront alignées après $4,8$ années, soit 4 ans 9 mois et environ 18 jours.

Solutioné aussi par Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

8. ($*$) Three faucets A, B, and C are placed above a pan. The table gives the time needed for the pan to fill up when only two of the three faucets are turned on simultaneously. How much time would be needed for the pan to fill up if all three faucets were on at the same time? (Note: Assume that the flow out of each faucet is constant when the faucet is on.)

A	B	C	Time
ON	ON	OFF	3 minutes
ON	OFF	ON	4 minutes
OFF	ON	ON	6 minutes

Solution by Alex Remorov, grade 9 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

Let a be the volume of water from faucet A in 1 minute; let b be the volume from faucet B in 1 minute; and let c be the volume from faucet C in 1 minute. Let the volume of the pan be V. Then the different conditions translate to:

$$\begin{aligned} 3(a+b) &= V\,, & (1)\\ 4(a+c) &= V\,, & (2)\\ 6(b+c) &= V\,. & (3) \end{aligned}$$

Expanding, and multiplying equation (1) by 4, (2) by 3, and (3) by 2, we get

$$\begin{aligned} 12a+12b &= 4V\,,\\ 12a+12c &= 3V\,,\\ 12b+12c &= 2V\,. \end{aligned}$$

Adding all three of these equations yields $24a+24b+24c = 9V$, which simplifies to $\frac{8}{3}(a+b+c) = V$.

Thus, it will take $\frac{8}{3}$ minutes, or 2 minutes and 40 seconds, to fill up the pan when all three faucets are turned on.

That brings us to the end of another issue.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). ~~The Assistant Mayhem Editor is John Grant McLoughlin~~ (University of New Brunswick). The other staff members are Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Editorial

Shawn Godin

Since I became Editor of the Mayhem section of ***CRUX with MAYHEM***, I have made a concerted effort to broaden its appeal to more than the elite high school and undergraduate student. I have also undertaken to promote it among high school teachers and students at every possible opportunity. While reader contribution to Mayhem is growing, it is growing very slowly.

Currently, many of the Mayhem problems are proposed by the editorial staff. We are fast running out of fresh problems at the high school level, and we appeal to you, our readers, to increase our supply of problems. We currently have sufficient problems to see us through to the end of 2005, but without an influx of Mayhem problem proposals, we will NOT be able to have a regular problems section in Mayhem, but only to have problems appear sporadically.

Please send us your favourite high school level problems. Thank you.

Mayhem Problems

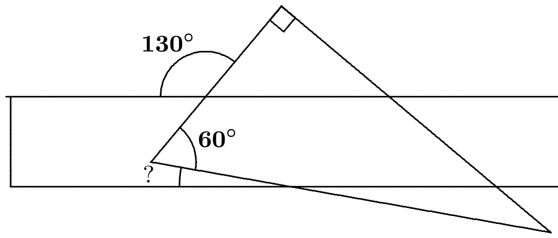
Please send your solutions to the problems in this edition by **1 February 2006**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M201. *Proposed by Robert Bilinski, Outremont, QC.*

A student drops his 30°–60°–90° triangle on his ruler so that a 130° angle appears as in the diagram below. What is the measure of the other marked angle?



M202. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

(a) By the end of the season, Samuel had made more than 50% of his foul shots, even though at the start of the season his average was below 50%. Show that there was a time during the season when his average was exactly 50%.

(b) For what other percentages p can you be certain that the average was exactly p at some time when you know only that the average was below p and then above p at a later time?

M203. *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

Chuck goes into the local 7-11 store and buys four items. The bill totals $\$7.11$. He notices that the product of the four prices is exactly 7.11. What are the prices of the four items?

[*Ed:* The proposer has indicated that the problem does not originate with him, nor does he know its origin.]

M204. *Proposed by Geneviève Lalonde, Massey, ON.*

Suppose that there is a line of 2005 buttons numbered 1 through 2005. Above each button is a counter initially set to 0. Each time a button is pushed, the corresponding counter advances by 1. A set of 2005 people now proceed down the line of buttons. The first person pushes every button, the second person pushes every second button starting at button #2, the third person pushes every third button starting at button #3, and so on, so that the 2005^{th} person pushes only button #2005. When everyone has gone, which buttons' counters will read 4?

Please provide a description of the set of buttons, rather than the actual list.

M205. *Proposed by John Katic, Ottawa, ON.*

Show that for every triangle ABC,

$$1 \;\leq\; \cos A + \cos B + \cos C \;\leq\; \frac{3}{2}\,.$$

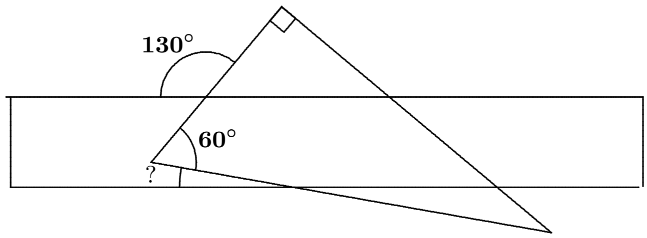
M206. *Proposed by Bill Arden, Rideau High School, Ottawa, ON.*

Let n be a composite number such that $a^{n-1} - 1$ is divisible by n for every number a that does not have a factor in common with n. Prove that at least 3 distinct primes divide n.

. . . .

M201. *Proposé par Robert Bilinski, Outremont, QC.*

Un étudiant laisse tomber son équerre 30°–60°–90° sur sa règlette de sorte qu'un angle de 130° apparaît comme dans la figure ci-dessous. Combien mesure l'autre angle noté ?



M202. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

(a) À la fin de la saison, Samuel avait une moyenne de plus de 50% sur ses coups manqués, bien qu'au début, sa moyenne était en-dessous de 50%. Montrer qu'il y avait un temps durant la saison où sa moyenne était exactement 50%.

(b) Pour quels autres pourcentages p pouvez-vous être certain que sa moyenne était exactement p si tout ce que vous savez est que sa moyenne était en-dessous de p, et en-dessus ultérieurement ?

M203. *Proposé par Richard K. Guy, Université de Calgary, Calgary, AB.*

Charles achète quatre articles dans une épicerie 7-11 (ouverte de 7 heures du matin à 11 heures du soir). La facture se monte à \$7.11. Il constate que le produit des quatre prix est exactement 7.11. Quel est le prix de chaque article ?

[*Ed :* Le proposeur a indiqué qu'il n'est pas l'auteur de ce problème et qu'il en ignore l'origine.]

M204. *Proposé par Geneviève Lalonde, Massey, ON.*

On suppose donné une rangée de 2005 boutons électriques numérotés de 1 à 2005. Au-dessus de chaque bouton se trouve un compteur indiquant 0. Chaque fois qu'on presse un bouton, son compteur augmente de 1. Une foule de 2005 personnes défile devant cette rangée de boutons. La première personne pousse chacun des boutons, la seconde personne pousse un bouton sur deux commençant au #2, la troisième personne pousse un bouton sur trois commençant au #3, et ainsi de suite, de sorte que la 2005-ième personne ne pousse que le bouton #2005. Quand tout le monde aura passé, à quels boutons correspondent les compteurs indiquant 4 ?

Donner s'il vous plaît une description de l'ensemble de tels boutons plutôt que la liste telle quelle.

M205. *Proposé par John Katic, Ottawa, ON.*

Montrer que pour tout triangle ABC,

$$1 \;\leq\; \cos A + \cos B + \cos C \;\leq\; \frac{3}{2}\,.$$

M206. *Proposé par Bill Arden, Rideau High School, Ottawa, ON.*

Soit n un nombre composé tel que $a^{n-1} - 1$ soit divisible par n pour tout nombre a sans diviseur commun avec n. Montrer qu'au moins 3 premiers distincts divisent n.

Mayhem Solutions

M133. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

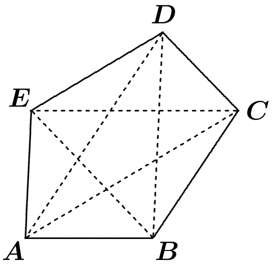
Dans un pentagone $ABCDE$, chaque côté est parallèle à une diagonale. Montrer que le rapport d'une diagonale au côté parallèle correspondant est constant. En fait, cette constante est le nombre d'or. (Un tel pentagone est appelé *un pentagone d'or*.)

Solution par Jacques Choné, Nancy, France.

Soit $ABCDE$ un pentagone d'or convexe dont les sommets ont respectivement pour affixe des nombres complexes 0, 1, c, d et e. Il existe cinq nombres réels positifs α, β, γ, δ et λ tels que

$$\overrightarrow{EC} \;=\; \alpha\overrightarrow{AB}\,, \quad \overrightarrow{AD} \;=\; \beta\overrightarrow{BC}\,, \quad \overrightarrow{BE} \;=\; \gamma\overrightarrow{CD}\,,$$
$$\overrightarrow{AC} \;=\; \delta\overrightarrow{ED}\,, \quad \overrightarrow{BD} \;=\; \lambda\overrightarrow{AE}\,;$$

c'est-à-dire tels que



$$\begin{aligned} c-e &= \alpha && (1)\\ d &= \beta(c-1) && (2)\\ e-1 &= \gamma(d-c) && (3)\\ c &= \delta(d-e) && (4)\\ d-1 &= \lambda e\,. && (5) \end{aligned}$$

Équations (1), (2), (3) donnent $c-\alpha-1=\gamma(\beta(c-1)-c)$; comme c n'est pas réel (les points ABC ne sont pas alignés), on a $1=\beta\gamma-\gamma$ et $-\alpha-1=-\beta\gamma$; c'est-à-dire $\alpha=\gamma$ et $\alpha\beta=\alpha+1$.

Équations (1), (2), (4) donnent $c=\delta\big(\beta(c-1)-c+\alpha\big)$; comme c n'est pas réel, on a $1=\beta\delta-\delta$ et $0=-\delta\beta+\delta\alpha$; c'est-à-dire (puisque $\delta\neq 0$) $\alpha=\beta$ et $\delta=1/(\alpha-1)$.

Équations (2) et (5) donnent $\lambda(c-\alpha)=\beta(c-1)-1$; toujours pour la même raison, on en déduit $\beta=\gamma$ et $\alpha=1+\dfrac{1}{\beta}$.

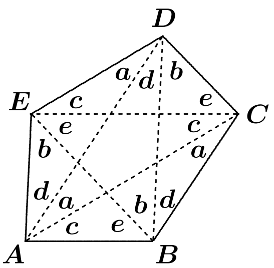
On obtient finalement $\alpha=\beta=\gamma=\delta=\lambda$ et $\alpha^2=\alpha+1$. Comme $\alpha>0$, le rapport des diagonales au côté correspondant est constant et égal au nombre d'or $\alpha=\frac{1}{2}(1+\sqrt{5})$.

Remarque : Dans le cas où le pentagone d'or est croisé, on obtient, par la même étude, $\alpha'=\frac{1}{2}(1-\sqrt{5})=-1/\alpha$ (voir le pentagone étoilé $ACEBD$ dont les côtés sont les diagonales du pentagone étudié ci-dessus).

M134. *Proposed by K.R.S. Sastry, Bangalore, India.*

In golden pentagon $ABCDE$ (see the preceding problem for the definition), we have $\angle EAB=\angle BCD$. Show that $\angle CDE=\angle DEA$.

Solution by Doug Newman, Lancaster, CA, USA, modified by the editor.



At each vertex of the pentagon, two diagonals meet. Let the measure of the angle between these two diagonals be denoted by the same (lower-case) letter as the vertex label (see the diagram). Since alternate interior angles are equal, the three angles formed by the diagonals at each vertex are now completely determined, as shown in the diagram.

The given relation can now be restated as $a+c+d=a+c+e$, from which we have $d=e$. This implies that $\triangle BAE$ and $\triangle BCD$ are similar; whence,

$$\frac{CD}{BD}=\frac{AE}{BE}\,. \qquad (1)$$

From problem M133 above, we know that $BD=k\,AE$ and $BE=k\,CD$, where $k=\frac{1}{2}(1+\sqrt{5})$. Equation (1) is then equivalent to

$$\frac{CD}{k\,AE} \;=\; \frac{AE}{k\,CD}\,,$$

from which we get $CD^2 = AE^2$; that is, $CD = AE$. Thus, $\triangle BAE$ is congruent to $\triangle BCD$, implying that $BA = BC$. Therefore, $\triangle BAC$ is isosceles, and it follows that $a = c$. Since $\angle CDE = a+b+d$ and $\angle DEA = b+c+e$, and since $d = e$ and $a = c$, we see that $\angle CDE = \angle DEA$.

M135. *Proposed by the Mayhem staff.*

Find all two-digit numbers with exactly 8 positive divisors.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_1, p_2, $\dots$, p_r are distinct primes and α_1, α_2, $\dots$, α_r are positive. The number of positive divisors of n is

$$d(n) \;=\; (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)\,.$$

Since $2^7 > 100$, we are looking for numbers which are products of at least two distinct primes. We have the following two cases:

Case (a). $n = p^\alpha q^\beta$.

For this case, $d(n) = 8$ if and only if $\alpha+1 = 2$ and $\beta+1 = 4$. The numbers n which also satisfy $n < 100$ are $2^1 3^3 = 54$, $2^3 3^1 = 24$, $2^3 5^1 = 40$, $2^3 7^1 = 56$ and $2^3 11^1 = 88$.

Case (b). $n = p^\alpha q^\beta r^\gamma$.

Then, $d(n) = 8$ if and only if $\alpha = \beta = \gamma = 1$. The two-digit numbers which are products of three distinct primes are $2\cdot3\cdot5 = 30$, $2\cdot3\cdot7 = 42$, $2\cdot3\cdot11 = 66$, $2\cdot3\cdot13 = 78$, and $2\cdot5\cdot7 = 70$.

From (a) and (b), all two-digit numbers with exactly 8 positive divisors are 24, 30, 40, 42, 54, 56, 66, 70, 78, and 88.

Also solved by Doug Newman, Lancaster, CA, USA.

M136. *Proposed by the Mayhem staff.*

The digits 1, 4, 5, 7, and 8 are each used once to form a five-digit number. Determine the sum of all such distinct five-digit numbers.

Combination of solutions by Doug Newman, Lancaster, CA, USA; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

There are 120 such distinct five-digit numbers. Of these, there are 24 which have 1 as their first digit. Each of these 24 gives rise to 4 others by cyclic permutation of the digits. (For example, the number 14 578 gives 81 457, 78 145, 57 814, and 45 781.) In this way, the 120 numbers can be arranged in 24 groups of 5, such that the 5 numbers in each group are cyclic permutations of each other. Since each digit occupies a given position exactly once within each group of 5, the sum of the numbers in each group of 5 is 277 775. Thus, the sum of all such distinct five-digit numbers is $277\,775 \times 24 = 6\,666\,600$.

M137. *Proposed by Babis Stergiou, Lycio Psachnon Evias, Greece.*

Suppose a, b, $c > 0$, $a + b + c = 3$, and $abc = 1$.

(a) Prove that $(a^2 + b)(a + b^2) \geq (a + a^2)(b + b^2)$.

(b) Hence, or otherwise, prove that

$$\frac{ab}{(a^2+b)(a+b^2)} + \frac{bc}{(b^2+c)(b+c^2)} + \frac{ca}{(c^2+a)(c+a^2)} \leq \frac{3}{4}.$$

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

(a) We can assume, without loss of generality, that $a \geq b$. Thus, since a, $b > 0$, we have

$$\begin{aligned} 0 &\leq (a-b)(a^2-b^2) = a^2(a-b) - b^2(a-b) \\ &= a^2(a-b) + b^2(b-a) = a^3 + b^3 - a^2b - ab^2 . \end{aligned}$$

Hence, $a^3+b^3 \geq a^2b+ab^2$, and $a^3+a^2b^2+ab+b^3 \geq ab+a^2b+ab^2+a^2b^2$, from which it follows that $(a^2 + b)(a + b^2) \geq (a + a^2)(b + b^2)$.

Thus, not only is the inequality established, but we only needed to use the fact that a and b are both positive.

(b) The inequality is true, because, for a, b, $c > 0$, the system

$$\begin{aligned} a+b+c &= 3, \\ abc &= 1, \end{aligned}$$

has the unique solution $a = b = c = 1$. Indeed, the system implies that $3 - a - b = 1/(ab)$; that is, $(3 - a)ab - ab^2 = 1$, or equivalently,

$$ab^2 + (a-3)ab + 1 = 0.$$

This quadratic equation in b has a solution if and only if $\big((a-3)a\big)^2 - 4a \geq 0$, or $(a-1)^2(a-4) \geq 0$. Since $a < 4$, we must have $a = 1$. By symmetry, $b = c = 1$. Then

$$\frac{ab}{(a^2+b)(a+b^2)} + \frac{bc}{(b^2+c)(b+c^2)} + \frac{ca}{(c^2+a)(c+a^2)} = \frac{3}{4}.$$

M138. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS and Sarah McCurdy, University of New Brunswick, Fredericton, NB.*

Five points are located on a line. When the ten distances between pairs of points are listed from smallest to largest, the list reads:

$$2,\quad 4,\quad 5,\quad 7,\quad 8,\quad k,\quad 13,\quad 15,\quad 17,\quad 19.$$

Determine the value of k.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let a_1, a_2, a_3, a_4, and a_5 be these five points. Let $d(x,y)$ be the distance between x and y. Since $d(a_1, a_5) = 19$, we have

$$19 = d(a_1, a_2) + d(a_2, a_5) = d(a_1, a_3) + d(a_3, a_5) = d(a_1, a_4) + d(a_4, a_5)\,;$$

that is, there exist three pairs of distances that add to give 19. Two of these pairs are $(2, 17)$ and $(4, 15)$. The third pair must be $(7, k)$ or $(8, k)$. Then $k = 12$ or $k = 11$.

On the other hand, since $d(a_2, a_5) = 17$ or $d(a_1, a_4) = 17$, there exist two pairs of distances that add to give 17. One of these pairs is $(4, 13)$. The other must be $(5, k)$, $(7, k)$, or $(8, k)$. Then $k = 12$, $k = 10$, or $k = 9$.

Now the only possibility is $k = 12$.

Also solved by Miguel Marañón, 4° ESO student, Instituto Práxedes Mateo Sagasta, Logroño, Spain; Doug Newman, Lancaster, CA, USA; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M139. *Proposed by the Mayhem Staff.*

The digits 1, 2, 3, 4, and 5 are each used once to compose a 5–digit number $abcde$, such that the 3–digit number abc is divisible by 4, bcd is divisible by 5, and cde is divisible by 3. Find the 5–digit number $abcde$.

Solution by Miguel Marañón, 4° ESO student, Instituto Práxedes Mateo Sagasta, Logroño, Spain.

Since $5 \mid bcd$, then $d = 5$. Also since $4 \mid abc$, then $c = 2$ or $c = 4$. Since $3 \mid cde$, we have $3 \mid (c + d + e)$.

If $c = 2$, then $3 \mid (2 + 5 + e)$, which implies that $e \equiv 2 \pmod 3$. This represents a contradiction, since e must be distinct from $d = 5$ and $c = 2$. Hence, $c = 2$ is false, and thus, $c = 4$. Then $3 \mid (4 + 5 + e)$. This implies that $e \equiv 0 \pmod 3$, from which we have $e = 3$.

Now a, $b \in \{1,\ 2\}$. Since 214 is not a multiple of 4, we must have $a = 1$ and $b = 2$.

Therefore, the problem has the unique solution: $abcde = 12453$.

Also solved by the Austrian 2004 IMO Team; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Doug Newman, Lancaster, CA, USA; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M140. *Proposed by the Mayhem Staff.*

Arthur, Bernie, and Charlie play a game in which the loser has to triple the money of each other player. Three games are played, in which the losers are Arthur, Bernie, and Charlie, in that order. Each player ends with \$27. How much money did each person have at the outset?

Solution by Miguel Marañón, 4° ESO student, Instituto Práxedes Mateo Sagasta, Logroño, Spain.

At the end, each player has \$27. To find out how much money each player had at the outset, we consider the process backwards, stage by stage, as shown in the following table.

	A	B	C
End	27	27	27
Game 2	$27 \div 3 = 9$	$27 \div 3 = 9$	$27 + 18 + 18 = 63$
Game 1	$9 \div 3 = 3$	$9 + 42 + 6 = 57$	$63 \div 3 = 21$
Start	$3 + 38 + 14 = 55$	$57 \div 3 = 19$	$21 \div 3 = 7$

Thus, Arthur started with $\$55$, Bernie with $\$19$, and Charlie with $\$7$.

Also solved by Doug Newman, Lancaster, CA, USA.

Problem of the Month

Ian VanderBurgh, University of Waterloo

In a problem which requires solving an equation, there are two different types of solutions that are often called for—all solutions (normally, all real solutions) or all integer solutions. Each of these types of problems presents its own difficulties and requires its own approaches. In this month's Problem of the Month, we will look at the problem of finding integer solutions.

Problem (1991 Canadian Invitational Mathematics Challenge, Grade 11) Find all positive integer values of a and b that satisfy the equation $\frac{1}{a}+\frac{a}{b}+\frac{1}{ab} = 1$.

We will look at two different approaches to solving this particular problem, each of which can be used for a lot of other problems also.

Solution 1. The first thing to recognize is that having fractions in equations can be a pain; hence, we multiply through by ab to clear out the fractions and obtain

$$b + a^2 + 1 \;=\; ab\,. \tag{1}$$

A clever thing to try at this stage is to solve for one variable in terms of the other. Let us solve for b in terms of a. Rearranging the equation, we get

$$\begin{aligned} a^2 + 1 &= ab - b \;=\; b(a-1)\,, \\ b &= \frac{a^2+1}{a-1}\,, \end{aligned} \tag{2}$$

since $a \neq 1$ (if $a = 1$, the equation (1) would simplify to $b + 2 = b$).

Why is this useful? We want both a and b to be positive integers. For b to be an integer, the right side must also be an integer. Thus, we need to determine for what positive integers a does $a - 1$ divide evenly into $a^2 + 1$.

At this stage, a trick comes in handy. In essence, the idea is to long-divide $a - 1$ into $a^2 + 1$ (or in other words, to write $a^2 + 1$ as a multiple of $a - 1$ plus a remainder):

$$\frac{a^2+1}{a-1} \;=\; \frac{(a^2-1)+2}{a-1} \;=\; \frac{a^2-1}{a-1}+\frac{2}{a-1} \;=\; a+1+\frac{2}{a-1}\,.$$

Why is this helpful? We want this expression to be an integer. Since we require a to be an integer, the expression $a+1+\dfrac{2}{a-1}$ will be an integer whenever $\dfrac{2}{a-1}$ is an integer. Thus, we need $a-1$ to divide evenly into 2.

We can now finish things quickly. Since a is a positive integer, $a-1$ is at least 0, and the only non-negative integer divisors of 2 are 1 and 2. Therefore, either $a-1=1$ (that is, $a=2$) or $a-1=2$ (that is, $a=3$).

Substituting back into equation (2) yields $b=5$ in both cases. Therefore, the solutions to the equation are $(a,b)=(2,5)$ and $(a,b)=(3,5)$.

It's never a bad idea at this stage to check our answers to make sure that they work in the original equation. Substituting the two possibilities, we get $\frac{1}{2}+\frac{2}{5}+\frac{1}{10}=1$ and $\frac{1}{3}+\frac{3}{5}+\frac{1}{15}=1$, which are both true.

This method of solution is a good one because it is very portable; that is, it works for a lot of different problems. Here is a similar but different approach to solving this same problem.

Solution 2. Again, we clear the fractions and obtain (1). This time, we try to solve for a in terms of b by rewriting equation (1) as a quadratic equation in a, namely

$$a^2+(-b)a+(b+1) \;=\; 0\,.$$

We can now solve for a in terms of b by using the quadratic formula:

$$a \;=\; \frac{b\pm\sqrt{(-b)^2-4(1)(b+1)}}{2} \;=\; \frac{b\pm\sqrt{b^2-4b-4}}{2}\,.$$

We require b to be an integer. In order for a to be an integer, it is necessary that b^2-4b-4 be a perfect square (otherwise $\sqrt{b^2-4b-4}$ is irrational). This does not guarantee that a is an integer, but it at least guarantees that a is rational.

When is b^2-4b-4 a perfect square? We need $b^2-4b+4-8=k^2$, for some integer k. We can rewrite this equation (by completing the square) as $(b-2)^2-8=k^2$. Thus, we are now looking for two perfect squares, k^2 and $(b-2)^2$, which differ by 8. The only such perfect squares are 1 and 9. (Why is this true? See if you can convince yourself of it.)

Consequently, we must have $(b-2)^2=9$, implying that $b=5$ (since b must be positive). Therefore, $a=\frac{1}{2}(5\pm 1)=2$ or 3 (both integers). Thus, the solutions are $(a,b)=(2,5)$ and $(a,b)=(3,5)$, as before.

We have seen here two quite different approaches to solving this particular problem, both of which can be useful in solving many other problems. For the sake of completeness, let us show that 1 and 9 are the only perfect squares which differ by 8. Suppose $x^2-y^2=8$ where $x>y>0$ are integers. Then $(x+y)(x-y)=8$. Both factors $x+y$ and $x-y$ are positive integers, and $x+y>x-y$. This means that either $x+y=8$ and $x-y=1$ or $x+y=4$ and $x-y=2$. The first possibility has no integer solutions, and the second gives $(x,y)=(3,1)$.

Pólya's Paragon

It Ain't So Complex (Part 1)

Shawn Godin

Numbers have been central in the development of mathematics over the centuries. The need to describe quantity led to the development of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$. As hard as it is to believe today, the idea of zero did not come easily; when it did come, the natural number system was enlarged, and the whole numbers $\mathbb{W} = \{0, 1, 2, 3, \dots\}$ came into being. The represention of numbers symbolically was then greatly streamlined with the introduction of place value systems like the Hindu-Arabic numbers that we use today.

At various other times, need and intellectual sophistication led to the integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, the rational numbers $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\}$, and the real numbers $\mathbb{R}$.

The equation $x + 1 = 0$ had no solutions before the development of negative numbers. Similarly, the equation $x^2 + 1 = 0$ has no solutions that are real numbers. A larger number system is needed for this equation to have solutions.

In the sixteenth century, Cardan developed methods for solving cubic equations, in which he would manipulate things like $\sqrt{-1}$ in his formulas. These "un-real" numbers gave him valid real results—the "imaginary" numbers were born!

If we define the unit imaginary number i to satisfy $i^2 = -1$ (so that our quadratic equation $x^2 + 1 = 0$ has two solutions, namely $\pm i$), we can define a *complex* number as a number of the form $a + bi$ where $a, b \in \mathbb{R}$. The collection of all such numbers is the set of *complex numbers*, which we denote by $\mathbb{C}$. Given a complex number $a + bi$, we call a the *real part* of the number and b the *imaginary part*. We will adopt the notation $\mathfrak{Re}(a+bi) = a$ for the real part, and $\mathfrak{Im}(a + bi) = b$ for the imaginary part. We will also adopt the convention of using z for a complex variable (as opposed to x, which we typically reserve for a real variable).

How do we define operations with complex numbers? Try to calculate each of the following before going on to see how they are done.

1. $(1 + 2i) + (-3 + 4i)$.
2. $(1 + 2i) - (-3 + 4i)$.
3. $(1 + 2i) \times (-3 + 4i)$.
4. $(1 + 2i) \div (-3 + 4i)$.

Operations with complex numbers can be defined intuitively using the laws of arithmetic and algebra. For instance, it seems logical to define addition by

$$(a+bi)+(c+di) \;=\; a+bi+c+di \;=\; (a+c)+(b+d)i\,;$$

that is, just add the real and imaginary parts to get the real and imaginary parts of the sum. Subtraction is similar. Thus, the answers to the first two exercises are:

1. $(1+2i)+(-3+4i)=-2+6i$. 2. $(1+2i)-(-3+4i)=4-2i$.

To define multiplication, we need to use the fact that $i^2=-1$. If we assume that we may treat the symbol i as we would any other symbol and use our well-known rules of algebra, we get

$$\begin{aligned}(a+bi)\times(c+di) \;&=\; ac+adi+bci+bdi^2\\ &=\; ac+adi+bci+bd(-1) \;=\; (ac-bd)+(ad+bc)i\,.\end{aligned}$$

Thus, the answer to the third exercise is

3. $(1+2i)\times(-3+4i)=-11-2i$.

To tackle division, we have to remember how division is related to multiplication. Thus, to find a complex number $x+yi$ such that

$$x+yi \;=\; \frac{1+2i}{-3+4i}\,,$$

we need to find x and y so that $(x+yi)\times(-3+4i)=1+2i$. Calculating the left side, we get

$$(-3x-4y)+(4x-3y)i \;=\; 1+2i\,.$$

This means that the left and right sides of the equation represent the same complex number; that is, the real parts of the left and right sides must be equal, and similarly for the imaginary parts. Thus, we get the equations:

$$\begin{aligned}-3x-4y \;&=\; 1\,,\\ 4x-3y \;&=\; 2\,.\end{aligned}$$

Solving, we get $x=\frac{1}{5}$, $y=-\frac{2}{5}$. Thus, the answer to the fourth exercise is:

4. $(1+2i)\div(-3+4i)=\frac{1}{5}-\frac{2}{5}i$.

Now, try to find a simpler method for doing the calculation in 4. (*Hint:* Calculate $1\div(-3+4i)$. How is the reciprocal of a complex number related to the original? How can you use this to help with division?) Then see if you can develop a general formula for division like we did for addition, subtraction, and multiplication.

Next month we will consider the complex number $z=a+bi$ as a point (a,b) in the (complex) plane. For homework, along with your exercise above, look at the each of the four examples and see what the answers look like when plotted as points (a,b).

THE OLYMPIAD CORNER

No. 247

R.E. Woodrow

We start this number with problems of the first and second rounds of the 2001-02 British Mathematical Olympiad. Thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for our use.

2001–2002 BRITISH MATHEMATICAL OLYMPIAD

Round 1

5 December, 2001

1. Find all positive integers m, n, where n is odd, that satisfy

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}\,.$$

2. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC and BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P.

Given that $CD = CP = DQ$, prove that $\angle CAD = 60^\circ$.

3. Find all positive real solutions to the equation

$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor ,$$

where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

4. Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross? (Nobody is allowed to shake hands with more than one person at once.)

5. Let f be a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers, which has the following properties:

(a) $f(n+1) > f(n)$ for each $n \in \mathbb{Z}^+$,

(b) $f(n + f(m)) = f(n) + m + 1$ for all $m, n \in \mathbb{Z}^+$.

Find all possible values of $f(2001)$.

Round 2
26 February, 2002

1. The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D, perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

2. A conference hall has a round table with n chairs. There are n delegates to the conference. The first delegate chooses his or her seat arbitrarily. Thereafter, the $(k+1)^{\text{st}}$ delegate sits k places to the right of the k^{th} delegate, for $1 \le k \le n-1$. (In particular, the second delegate sits next to the first.) No chair can be occupied by more than one delegate.

Find the set of values n for which this is possible.

3. Prove that the sequence defined by $y_0 = 1$ and

$$y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right) \qquad (\text{for } n \ge 0)$$

consists only of integers.

4. Suppose that $B_1, \ldots, B_N$ are N spheres of unit radius arranged in space so that each sphere touches exactly two others externally. Let P be a point outside all these spheres, and let the N points of contact be $C_1, \ldots, C_N$. The length of the tangent from P to the sphere B_i $(1 \le i \le N)$ is denoted by t_i. Prove the product of the quantities t_i is not more than the product of the distances PC_i.

Next we give the problems from the 15^{th} Korean Mathematical Olympiad written April 13–14, 2002. Again my thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining these problems for our use.

15^{th} KOREAN MATHEMATICAL OLYMPIAD
April 13–14, 2002
Day 1

1. For a prime p of the form $12k+1$ and $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$, let

$$\mathbb{E}_p = \{(a,b) \mid a\,, b \in \mathbb{Z}_p\,,\quad p \nmid (4a^3 + 27b^2)\}\,.$$

For (a,b), $(a',b') \in \mathbb{E}_p$, we say that (a,b) and (a',b') are equivalent if there is a non-zero element $c \in \mathbb{Z}_p$ such that $p \mid (a' - ac^4)$ and $p \mid (b' - bc^6)$. Find the maximal number of inequivalent elements in $\mathbb{E}_p$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x - y) = f(x) + xy + f(y)$ for every $x \in \mathbb{R}$ and every $y \in \{f(x) \mid x \in \mathbb{R}\}$, where $\mathbb{R}$ is the set of all real numbers.

3. The following facts are known in a mathematics contest:

(a) The number of problems tested was $n \geq 4$.

(b) Each problem was solved by exactly four contestants.

(c) For each pair of problems, there is exactly one contestant who solved both problems.

Assuming the number of contestants is greater than or equal to $4n$, find the minimum value of n for which there always exists a contestant who solved all the problems.

Day 2

4. For $n \geq 3$, let $S = a_1 + a_2 + \cdots + a_n$ and $T = b_1 b_2 \cdots b_n$ for positive real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, where the numbers b_i are pairwise distinct.

(a) Find the number of distinct real zeroes of the polynomial

$$f(x) = (x - b_1)(x - b_2) \cdots (x - b_n) \sum_{j=1}^{n} \frac{a_j}{x - b_j} .$$

(b) Prove the inequality

$$\frac{1}{n-1} \sum_{j=1}^{n} \left(1 - \frac{a_j}{S}\right) b_j > \left(\frac{T}{S} \sum_{j=1}^{n} \frac{a_j}{b_j}\right)^{\frac{1}{n-1}} .$$

5. Let ABC be an acute triangle, and let O be its circumcircle. Let the perpendicular line from A to BC meet O at D. Let P be a point on O, and let Q be the foot of the perpendicular line from P to the line AB. Prove that if Q is on the outside of O and $2\angle QPB = \angle PBC$, then D, P, Q are collinear.

6. Let p_n be the n^{th} prime counting from the smallest prime 2 in increasing order. For example, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$.

(a) For a given $n \geq 10$, let r be the smallest integer satisfying

$$2 \leq r \leq n - 2 , \quad n - 4 + 1 < p_r ,$$

and define $N_s = (s p_1 p_2 \cdots p_{r-1}) - 1$ for $s = 1, 2, \ldots, p_r$. Prove that there exists j, $1 \leq j \leq p_r$, such that none of $p_1, p_2, \ldots, p_n$ divides N_j.

(b) Using the result of (a), find all positive integers m for which

$$p_{m+1}^2 < p_1 p_2 \cdots p_m .$$

We have received a solution to Problem 5 of the 2000 Russian Mathematical Olympiad [2002 : 483] which avoids the calculus used in the featured solution [2005 : 95–96].

5. Prove that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}} \le \frac{2}{\sqrt{1+xy}} \qquad \text{for } 0<x,\,y\le 1.$$

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

We prove the inequality under the slightly weaker conditions that $x,\,y \ge 0$ and $xy \le 1$.

For $x,\,y\ge 0$, let

$$F(x,y) \;=\; \sqrt{1+xy}\left(\frac{2}{\sqrt{1+xy}}-\frac{1}{\sqrt{1+x^2}}-\frac{1}{\sqrt{1+y^2}}\right).$$

Then

$$\begin{aligned}
F(x,y) &= \frac{\sqrt{1+x^2}-\sqrt{1+xy}}{\sqrt{1+x^2}}+\frac{\sqrt{1+y^2}-\sqrt{1+xy}}{\sqrt{1+y^2}}\\
&= \frac{x(x-y)}{1+x^2+\sqrt{1+xy}\sqrt{1+x^2}}+\frac{y(y-x)}{1+y^2+\sqrt{1+xy}\sqrt{1+y^2}}\\
&= \frac{(x-y)G(x,y)}{\left(1+x^2+\sqrt{1+xy}\sqrt{1+x^2}\right)\left(1+y^2+\sqrt{1+xy}\sqrt{1+y^2}\right)},
\end{aligned}$$

where

$$\begin{aligned}
G(x,y) &= x\left(1+y^2+\sqrt{1+xy}\sqrt{1+y^2}\right)\\
&\qquad -y\left(1+x^2+\sqrt{1+xy}\sqrt{1+x^2}\right)\\
&= x-y-xy(x-y)+\sqrt{1+xy}\left(x\sqrt{1+y^2}-y\sqrt{1+x^2}\right)\\
&= x-y-xy(x-y)+\frac{\sqrt{1+xy}(x^2-y^2)}{x\sqrt{1+y^2}+y\sqrt{1+x^2}}\\
&= (x-y)\left(1-xy+\frac{\sqrt{1+xy}(x+y)}{x\sqrt{1+y^2}+y\sqrt{1+x^2}}\right).
\end{aligned}$$

Thus,

$$F(x,y) \;=\; \frac{(x-y)^2\left(1-xy+\dfrac{\sqrt{1+xy}(x+y)}{x\sqrt{1+y^2}+y\sqrt{1+x^2}}\right)}{\left(1+x^2+\sqrt{1+xy}\sqrt{1+x^2}\right)\left(1+y^2+\sqrt{1+xy}\sqrt{1+y^2}\right)}.$$

It is now evident that $F(x,y)\ge 0$ if $xy\le 1$. The desired result follows.

Next we return to readers' solutions to problems proposed and short-listed for the 2000 International Olympiad in Korea. These problems were given in [2003 : 215–216] and continued in [2003 : 279–280]. My thanks go to George Evagelopoulos, Athens, Greece for providing us with solutions.

8. (*Japan*) Determine all integers $n \geq 2$ such that for all integers a and b relatively prime to n, $a \equiv b \pmod{n}$ if and only if $ab \equiv 1 \pmod{n}$.

[*Ed.:* When this problem was given in [2003 : 216], the congruence $ab \equiv 1 \pmod{n}$ was stated incorrectly as $ab \equiv -1 \pmod{n}$.]

Solution supplied by George Evagelopoulos, Athens, Greece.

For any integer $n \geq 2$, the given condition is equivalent to

(I) For every integer a relatively prime to n, $a^2 \equiv 1 \pmod{n}$.

Indeed, the given condition implies (I), and conversely, if (I) holds, then for any integers a and b relatively prime to n, we have

$$\begin{array}{rrcl} & a \equiv b \pmod{n} & \Longrightarrow & ab \equiv a^2 \equiv 1 \pmod{n}\ , \\ \text{and} & ab \equiv 1 \pmod{n} & \Longrightarrow & a^2 \equiv ab \pmod{n} \\ & & \Longrightarrow & a \equiv b \pmod{n}\ . \end{array}$$

Let $n = p_1^{e_1} p_2^{e_2} \dots p_\ell^{e_\ell}$ ($e_i \geq 1$) be the factorization of n into primes.

Claim. Condition (I) is equivalent to

(II) For each i, for every integer a relatively prime to p_i, $a^2 \equiv 1 \pmod{p_i^{e_i}}$.

Proof of the Claim: Suppose that (I) holds. We prove (II). It is enough to consider $i = 1$. Suppose that a is relatively prime to p_1. If some of the primes p_2, p_3, $\dots$, p_ℓ divide a, say p_2, p_3, $\dots$, p_k divide a and $p_{k+1}, \dots, p_\ell$ do not divide a, then $a + p_1^{e_1} p_{k+1} p_{k+2} \cdots p_\ell$ is relatively prime to n. By (I), we have

$$(a + p_1^{e_1} p_{k+1} p_{k+2} \cdots p_\ell)^2 \equiv 1 \pmod{n}\ .$$

But we also have $(a + p_1^{e_1} p_{k+1} p_{k+2} \cdots p_\ell)^2 \equiv a^2 \pmod{p_1^{e_1}}$. Hence, $a^2 \equiv 1 \pmod{p_1^{e_1}}$.

Now suppose that (II) holds, and let a be relatively prime to n. Then a is relatively prime to each p_i. Thus, by (II), we have $a^2 \equiv 1 \pmod{p_i^{e_i}}$ for any i, implying that $a^2 \equiv 1 \pmod{p_1^{e_1} \cdots p_\ell^{e_\ell}}$. ■

Suppose now that $n = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ satisfies the given condition.

If $p_i = 2$, then, by (II), we have $2^2 \equiv 1 \pmod{p_i^{e_i}}$; that is, $e_i \leq 3$. Conversely, $a^2 \equiv 1 \pmod{8}$ for any odd integer a.

If $p_i > 2$, then, by (II), we have $2^2 \equiv 1 \pmod{p_i^{e_i}}$; that is, $p_i = 3$ and $e_i = 1$. Conversely, $a^2 \equiv 1 \pmod{3}$ for any integer a relatively prime to 3.

Thus, n satisfies the given condition if and only if n divides $2^3 \cdot 3$. The outcomes are

$$n \in \{2, 3, 4, 6, 8, 12, 24\}.$$

9. (*Romania*) Prove that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.

Solution supplied by George Evagelopoulos, Athens, Greece.

Suppose we have a positive integer N with the following properties

$$N = a_1^2 + a_2^2 + \cdots + a_m^2 \quad \text{and} \quad 2N = b_1^2 + b_2^2 + \cdots + b_n^2,$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n$ are positive integers such that none of the fractions a_α/a_β, a_α/b_δ, b_γ/a_β, b_γ/b_δ is a power of 2 (including $2^0 = 1$) for all α, β, γ, δ with $\alpha \neq \beta$, $\gamma \neq \delta$. We shall then prove that every integer $P > \sum\limits_{k=0}^{4N-2} (2kN+1)^2$ can be represented as a sum of distinct perfect squares.

Given any such integer P, we write P in the form $P = 4Nq + r$, where $0 \leq r \leq 4N - 1$. Since $r \equiv \sum\limits_{k=0}^{r-1} (2kN + 1)^2 \pmod{4N}$ and the latter sum is less than P, we may write

$$P = \sum_{k=0}^{r-1} (2kN + 1)^2 + 4Nt,$$

for some positive integer t, if $r \geq 1$. If $r = 0$, we just take $P = 4Nt$, where $t = q$.

Let $t = \sum\limits_i 2^{2u_i} + \sum\limits_j 2^{2v_j+1}$ be the binary expansion of t. Then

$$P = \begin{cases} \sum\limits_{k=0}^{r-1} (2kN + 1)^2 + \sum\limits_{i,\alpha} (2^{u_i+1} a_\alpha)^2 + \sum\limits_{j,\gamma} (2^{v_j+1} b_\gamma)^2 & \text{if } r \geq 1, \\ \sum\limits_{i,\alpha} (2^{u_i+1} a_\alpha)^2 + \sum\limits_{j,\gamma} (2^{v_j+1} b_\gamma)^2 & \text{if } r = 0. \end{cases}$$

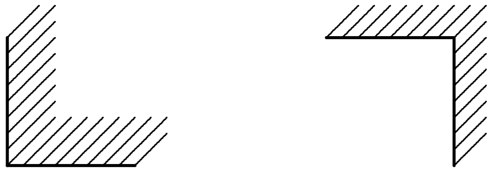
Thus, P can be represented as a sum of distinct perfect squares.

It remains only to show that there exists a positive integer N as above. But 29 is such a number, since $29 = 2^2 + 5^2$ and $58 = 3^2 + 7^2$.

10. (*Russia*) In the plane we have n rectangles with parallel sides. The sides of distinct rectangles lie on distinct lines. The boundaries of the rectangles cut the plane into connected regions. A region is said to be *nice* if it has at least one of the vertices of the n rectangles on its boundary. There can be non-convex regions, as well as regions with more than one boundary curve. Prove that the sum of the numbers of the vertices of all nice regions is less than $40n$.

Solution supplied by George Evagelopoulos, Athens, Greece, modified by the editors.

Each vertex of a nice region is either convex or concave, as in the figures below (where the shaded part is the interior of the region considered).



For a simple closed curve C contained in the boundary of a region R, the pair (R, C) will be called a *boundary pair*. A boundary pair (R, C) is said to be *outer* (respectively *inner*) if R is inside C (respectively outside C). Let B be the set of boundary pairs of nice regions. Let O be the set of boundary pairs in B which are outer, and let I be the set which are inner. Then B is the disjoint union of O and I. Note that the set I is not empty, since the only region with infinite area is a nice region and all of its boundary curves are inner boundaries.

For a boundary pair $b = (R, C) \in B$, let v_b and c_b be the numbers of convex vertices and concave vertices of R, respectively, which belong to the boundary curve C. Each vertex of the given rectangles creates exactly one convex vertex and one concave vertex. Conversely, each concave vertex of a region comes from a vertex of one of the original rectangles. Therefore,

$$\sum_{b \in B} c_b \;=\; 4n\,.$$

For a boundary pair $b = (R, C)$, both sides of the following equation express the sum of the angles at the vertices of C, where the angle at each vertex is chosen to be the angle interior to R:

$$v_b \cdot 90^\circ + c_b \cdot 270^\circ \;=\; \begin{cases} (v_b + c_b - 2) \cdot 180^\circ & \text{if } b \in O, \\ (v_b + c_b) \cdot 360^\circ - (v_b + c_b - 2) \cdot 180^\circ & \text{if } b \in I. \end{cases}$$

It follows that

$$v_b - c_b \;=\; \begin{cases} 4 & \text{if } b \in O, \\ -4 & \text{if } b \in I. \end{cases}$$

Since I is non-empty, we have

$$\sum_{b \in B} (v_b - c_b) \;\le\; 4(k-1) - 4 \;=\; 4k - 8\,,$$

where k is the number of boundary pairs in B.

For each boundary pair (R, C) of a nice region R, at least one of the vertices of C is a vertex of one of the given rectangles. Since every vertex of the given rectangles appears in exactly two boundary pairs, we see that $k \le 8n$.

Finally, the sum of the numbers of the vertices of all nice regions is

$$\sum_{b \in B} (v_b + c_b) \;=\; \sum_{b \in B} \big(2c_b + (v_b - c_b)\big) \;\le\; 2 \cdot 4n + 4k - 8 \;\le\; 40n - 8\,.$$

12. (*Belarus*) Find all pairs of functions f and g from the set of real numbers to itself such that $f\big(x+g(y)\big) = xf(y) - yf(x) + g(x)$ for all real numbers x and y.

Solution supplied by George Evagelopoulos, Athens, Greece, modified by the editors.

It is easy to see that if f is constant then $f(x) = g(x) \equiv 0$. Assume that f is non-constant. Replacing x by $g(x)$ in the given equation, we have

$$f\big(g(x)+g(y)\big) \;=\; g(x)f(y) - yf\big(g(x)\big) + g\big(g(x)\big)\,.$$

Since the left side is symmetric in x and y, the right side must be also, and we obtain

$$g(x)f(y) - yf\big(g(x)\big) + g\big(g(x)\big) \;=\; g(y)f(x) - xf\big(g(y)\big) + g\big(g(y)\big)\,. \quad (1)$$

Taking $x=0$ in the original equation we get

$$f\big(g(y)\big) \;=\; ay+b \quad (2)$$

where $a = -f(0)$ and $b = g(0)$. Setting $y = 0$ in (2) gives $f(b) = b$. Equation (1) with $y=0$ now gives

$$g\big(g(x)\big) \;=\; ag(x) + bf(x) - bx + g(b)\,. \quad (3)$$

Inserting (2) and (3) into (1), we obtain

$$g(x)f(y) + ag(x) + bf(x) \;=\; g(y)f(x) + ag(y) + bf(y)\,,$$

or equivalently,

$$\Big(g(x)-b\Big)\Big(f(y)+a\Big) \;=\; \Big(g(y)-b\Big)\Big(f(x)+a\Big)\,, \quad (4)$$

for all x and y.

Recalling that f is non-constant, we choose y_0 such that $f(y_0)+a \neq 0$. Then we see from (4) with $y=y_0$ that

$$g(x) - b \;=\; A\big(f(x)+a\big)\,,$$

where $A = \dfrac{g(y_0)-b}{f(y_0)+a}$. Letting $B = Aa + b$, we have

$$g(x) \;=\; Af(x) + B\,, \quad (5)$$

for all $x \in \mathbb{R}$. Inserting this into the original equation, we obtain

$$f\big(x+g(y)\big) \;=\; xf(y) - yf(x) + Af(x) + B\,.$$

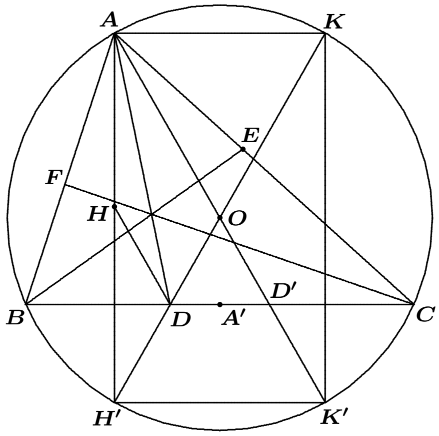
Putting $y = A$ gives $f\big(x+g(A)\big) = xf(A) + B$, which shows that f is a linear function. By (5), g is also linear. Now, a direct computation yields

$$f(x) \;=\; \frac{c}{c+1}(x-c)) \qquad \text{and} \qquad g(y) \;=\; c(y-c)\,,$$

for any constant $c \neq -1$.

13. (*India*) Let O be the circumcentre and H the orthocentre of an acute triangle ABC. Prove that there exist points D, E, and F on sides BC, CA, and AB, respectively, such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE, and CF are concurrent.

Solved by Michel Bataille, Rouen, France; George Evagelopoulos, Athens, Greece; and Toshio Seimiya, Kawasaki, Japan. We give Bataille's version.



Let Γ be the circumcircle of $\triangle ABC$. It is well known that the reflection H' of H in BC lies on Γ. Note that O and H' are on opposite sides of BC (because $\triangle ABC$ is acute). Let D be the point of intersection of BC and OH', and let E and F be similarly constructed on CA and AB, respectively. We show that the points D, E, F satisfy the conditions in the problem.

First, $OD + DH = OE + EH = OF + FH = R$, the radius of Γ. This follows from $OD + DH = OD + DH' = OH' = R$, for example. Now consider the points K and K' which are diametrically opposite to H' and A, respectively, on Γ. Then $AKK'H'$ is a rectangle with centre O whose sides are parallel to AH and BC. As a result, AO meets BC at D', the point which is symmetrical to D about the mid-point A' of BC (see the diagram). A similar property holds for the points E' and F' at which BO and CO meet CA and AB, respectively. Since AO, BO, CO are concurrent, Ceva's Theorem yields

$$\frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = 1 .$$

Hence,

$$\frac{-CD}{-DB} \cdot \frac{-BE}{-EC} \cdot \frac{-BF}{-FA} = 1 .$$

By the converse of Ceva's Theorem, it follows that AD, BE, and CF are concurrent (they cannot be parallel, since D, E, F are on the segments BC, CA, AB, respectively).

14. (*Iran*) Ten gangsters are standing on a flat surface. The distances between them are all distinct. Simultaneously each of them shoots at the one among the other nine who is the nearest. At least how many gangsters will be shot at?

Solution supplied by George Evagelopoulos, Athens, Greece.

The problem can be stated mathematically as follows: A set S of ten points in the plane is given. The distances between them are all distinct. For each point $P \in S$, we colour red the point $Q \in S$ nearest to P ($Q \neq P$). Find the least possible number of red points.

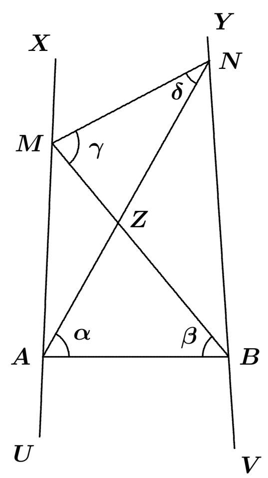
Note that if a red point Q is assigned (as the closest neighbour) to two distinct points P_1 and P_2 in S, then the angle P_1QP_2 must be greater than $60°$, because P_1P_2 must be the longest side in the (non-isosceles) triangle P_1QP_2. It follows that no red point can be assigned to more than five distinct points in S.

Let AB be the shortest segment with end-points A, $B \in S$. Clearly, A and B are both red. We are going to show that there exists at least one more red point. Assume instead that, for each one of the remaining eight points, its closest neighbour is either A or B. In view of the previous observation, A must be assigned to four points M_1, M_2, M_3, M_4 and B must be assigned to the remaining four points, N_1, N_2, N_3, N_4. Choose labelling such that the angles M_iAM_{i+1}, $i = 1, 2, 3$, are successively adjacent, as are the angles N_iBN_{i+1}, with the points M_1 and N_1 on one side of the line AB, and M_4 and N_4 on the opposite side.

The angles M_iAM_{i+1} and N_iBN_{i+1} are each greater than $60°$. Therefore, $\angle M_1AM_4$ and $\angle N_1BN_4$ are each less than $180°$. Hence,

$$(\angle M_1AB + \angle N_1BA) + (\angle M_4AB + \angle N_4BA) \ < \ 360° \, .$$

At least one of the two sums on the left side is less than $180°$, say, $\angle M_1AB + \angle N_1BA < 180°$. From here on, we write M and N in place of M_1 and N_1, for the sake of brevity.



Since $MA < MB$ and $NB < NA$, the points A and M lie on one side of the perpendicular bisector of AB, and the points B and N lie on the other side. Hence, because M and N lie on the same side of AB, the points A, B, N, M are consecutive vertices of a quadrilateral. Since AB is the shortest side of the triangle BNA, the angle BNA is acute. Since MA is not the longest side in the triangle ANM, the angle ANM is acute. Therefore, the internal angle BNM of the quadrilateral $ABNM$ is less than $180°$. Similarly, the internal angle NMA is less than $180°$. Thus, $ABNM$ is a convex quadrilateral.

Choose points U, V, X, Y arbitrarily on the rays MA, NB, AM, BN produced beyond the quadrilateral. Recall that $\angle MAB + \angle NBA < 180^\circ$. This implies that

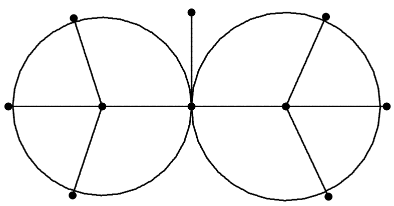
$$\angle UAB + \angle ABV > 180^\circ \quad \text{and} \quad \angle XMN + \angle MNY < 180^\circ .$$

Let $\alpha = \angle NAB$, $\beta = \angle ABM$, $\gamma = \angle BMN$, and $\delta = \angle MNA$. In $\triangle NAB$, we have $AB < NB$, which implies that $\angle ANB < \angle NAB = \alpha$, and, thus, $\angle ABV = \angle NAB + \angle ANB < 2a$. In $\triangle BMN$, we have $MN > BN$, which implies that $\angle MBN > \angle BMN = \gamma$, and, consequently, $\angle MNY = \angle BMN + \angle MBN > 2\gamma$. Analogously, $\angle UAB < 2\beta$ and $\angle XMN > 2\delta$. Hence,

$$2\alpha+2\beta > \angle ABV+\angle UAB > 180^\circ > \angle MNY+\angle XMN > 2\gamma+2\delta .$$

But $\alpha + \beta = \gamma + \delta = \angle AZM$, where Z is the point of intersection of AN and BM. We have a contradiction. Thus, indeed, there exists a third red point.

The following example shows that a fourth red point need not exist, so that *three* is the minimum sought. The two tangent circles in the figure differ slightly in size. The acute central angles are greater than 60°. Six points of S are just a bit outside the circles, two points are the centres of the circles, one point is the point of tangency of the circles, and one point is on the common tangent to the circles at a distance which is slightly greater than the radius of the larger circle. The only three points which will be marked red are the two centres and the point of tangency. (If some of the distances happen to be equal, one can slightly perturb the positions of any points without changing the nearest neighbours.)



15. (*Ireland*) A non-empty set A of real numbers is called a B_3–set if the conditions a_1, a_2, a_3, a_4, a_5, $a_6 \in A$ and $a_1 + a_2 + a_3 = a_4 + a_5 + a_6$ imply that the sequences (a_1, a_2, a_3) and (a_4, a_5, a_6) are identical up to a permutation. For a set X of real numbers, let $D(X)$ denote the difference set $\{|x - y| : x, y \in X\}$. Prove that if $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ and $B = \{0 = b_0 < b_1 < b_2 < \cdots\}$ are infinite sequences of real numbers with $D(A) = D(B)$, and if A is a B_3–set, then $A = B$.

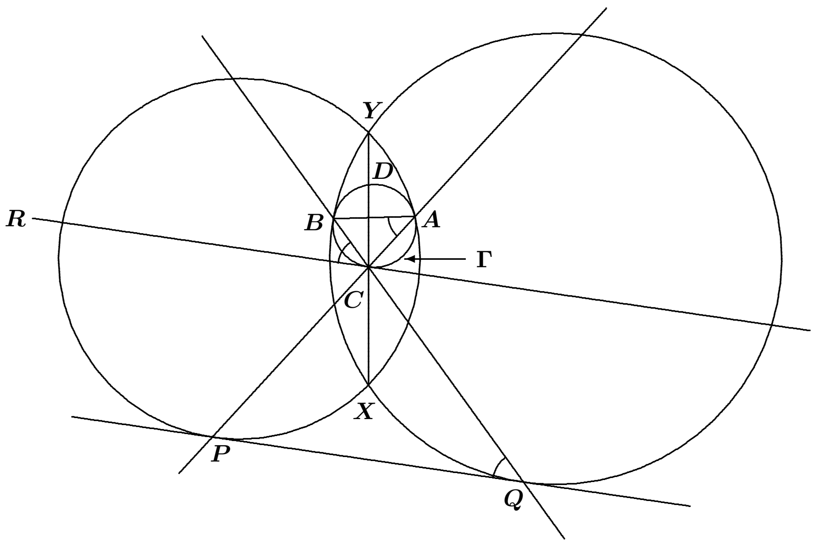
Comment. This result is due to Erdős, Sárőzy, and Sós. A solution due to Fűredi, Jackosch, and Rubel appears in *Combinatorics* 16 (1996), 87–106.

16. (*The Netherlands*) In the plane we are given two circles intersecting at X and Y. Prove that there exist four points such that for every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, and BD passes through one of those four points.

Solution supplied by George Evagelopoulos, Athens, Greece.

The four points of contact of the two given circles with their common tangents have the required property, as we will now prove.

Let Γ be a circle that touches the two given circles at points A and B and meets the line XY at points C and D. Then Γ touches the two given circles either both externally or both internally. In the latter case, the two given circles may be situated inside Γ or outside Γ. The latter subcase is illustrated in the diagram; however, the reasoning that follows is valid for any other situation, without any modifications.



It is enough to consider the lines CA and CB. Let CA meet the circle (AXY) again at P, and let CB meet the circle (BXY) again at Q. Since C lies on XY, we have $CA \cdot CP = CX \cdot CY = CB \cdot CQ$, by the Power of the Point Theorem. Thus, the triangles CAB and CQP are similar, implying that $\angle CAB = \angle CQP$.

Draw the line CR tangent to Γ, with R lying on the same side of line XY as B. Then $\angle BCR = \angle CAB = \angle CQP$, implying that $CR \parallel PQ$. Consider the two homotheties, centred at A and B, respectively, that map Γ onto the two given circles. One of the homotheties transforms the line CR to the line tangent at P to one of these circles, and the other homothety takes CR to the line tangent at Q to the other circle. Both these image lines are parallel to CR; hence, they coincide with the line PQ, which is, therefore, a common tangent to those circles.

18. (*Russia*) Let $A_1A_2\dots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2\dots A_n$ is cyclic if and only if each vertex A_i can be assigned a pair (b_i, c_i) of real numbers so that

$$A_iA_j \;=\; b_jc_i - b_ic_j \qquad (1)$$

for all i and j with $1 \leq i < j \leq n$.

Solution supplied by George Evagelopoulos, Athens, Greece.

Suppose that there are pairs (b_i, c_i) of real numbers satisfying (1). In order for $A_1A_2\dots A_n$ to be cyclic, it suffices that, for each $i = 4, \dots, n$, the points A_1, A_2, A_3, A_i lie on a circle. These points are the consecutive vertices of a convex quadrilateral $A_1A_2A_3A_i$. By the converse of Ptolemy's Theorem, a sufficient condition for $A_1A_2A_3A_i$ to be cyclic is that

$$A_1A_2 \cdot A_3A_i + A_2A_3 \cdot A_1A_i \;=\; A_1A_3 \cdot A_2A_i\,.$$

In view of (1), this translates into

$$\begin{aligned}(b_2c_1 - b_1c_2)(b_ic_3 - b_3c_i) + (b_3c_2 - b_2c_3)(b_ic_1 - b_1c_i)&\\ = \;\; (b_3c_1 - b_1c_3)(b_ic_2 - b_2c_i)&\,.\end{aligned}$$

It is easy to verify that the last equality is an identity.

Conversely, let $A_1A_2\dots A_n$ be cyclic. For $i = 1, 2, \dots, n$, set $b_i = A_2A_i$ and $c_i = \dfrac{A_1A_i}{A_1A_2}$ (a definition suggested by Ptolemy's Theorem). We now check that the numbers b_i, c_i satisfy (1).

If $3 \leq i < j \leq n$, then A_1, A_2, A_i, A_j are the consecutive vertices of the cyclic quadrilateral $A_1A_2A_iA_j$. Hence, by Ptolemy's Theorem,

$$A_1A_2 \cdot A_iA_j \;=\; A_1A_i \cdot A_2A_j - A_2A_i \cdot A_1A_j\,.$$

Dividing both sides by A_1A_2 yields

$$A_iA_j \;=\; A_2A_j \cdot \frac{A_1A_i}{A_1A_2} - A_2A_i \cdot \frac{A_1A_j}{A_1A_2} \;=\; b_jc_i - b_ic_j\,,$$

as desired.

We are left with the cases $i = 1$ and $i = 2$. For $i = 1$ and $2 \leq j \leq n$, the definitions of b_j, c_j give

$$b_jc_1 - b_1c_j \;=\; b_j \cdot 0 - (A_2A_1)c_j \;=\; A_1A_2 \cdot \frac{A_1A_j}{A_1A_2} \;=\; A_1A_j\,,$$

as desired. Similarly, if $i = 2$ and $3 \leq j \leq n$, then

$$b_jc_2 - b_2c_j \;=\; b_j \cdot 1 - 0 \cdot c_j \;=\; A_2A_j\,.$$

The proof is complete.

19. (*United Kingdom*) Let a, b, and c be positive integers such that $c > 2b > 4a$. Prove that there exists a real number λ such that the three numbers λa, λb, and λc all have their fractional parts in the interval $\left(\frac{1}{3}, \frac{2}{3}\right]$.

Solution supplied by George Evagelopoulos, Athens, Greece, modified by the editor.

For each integer n, we define intervals A_n, B_n, C_n as follows:

$$A_n = \left(\frac{n+\frac{1}{3}}{a}, \frac{n+\frac{2}{3}}{a}\right], \quad B_n = \left(\frac{n+\frac{1}{3}}{b}, \frac{n+\frac{2}{3}}{b}\right],$$
$$C_n = \left(\frac{n+\frac{1}{3}}{c}, \frac{n+\frac{2}{3}}{c}\right],$$

For any real number λ, the fractional part of λa lies in $\left(\frac{1}{3}, \frac{2}{3}\right]$ if and only if there is an integer i such that $i + \frac{1}{3} < \lambda a \leq i + \frac{2}{3}$; that is, $\lambda \in A_i$ for some i. The three numbers λa, λb, and λc all have their fractional parts in $\left(\frac{1}{3}, \frac{2}{3}\right]$ if and only if $\lambda \in A_i \cap B_j \cap C_k$ for some i, j, k. Thus it will suffice to prove that there exist i, j, k such that $A_i \cap B_j \cap C_k \neq \emptyset$.

First we observe that, for each integer j, there exists an integer k such that $B_j \cap C_k \neq \emptyset$. To see this, we just have to note that the length of the interval B_j is $\dfrac{n+\frac{2}{3}}{b} - \dfrac{n+\frac{1}{3}}{b} = \dfrac{1}{3b}$, while the distance between any two consecutive intervals C_k and C_{k+1} is $\dfrac{(k+1)+\frac{1}{3}}{c} - \dfrac{k+\frac{2}{3}}{c} = \dfrac{2}{3c} < \dfrac{1}{3b}$ (since $c > 2b$).

To complete the proof, it will be enough to show that there exist integers i and j such that A_i contains B_j. Equivalently, there are integers i and j such that

$$\frac{i+\frac{1}{3}}{a} \leq \frac{j+\frac{1}{3}}{b} \quad \text{and} \quad \frac{j+\frac{2}{3}}{b} \leq \frac{i+\frac{2}{3}}{a}.$$

For $j \geq 0$, these inequalities are equivalent to

$$\frac{3i+1}{3j+1} \leq \frac{a}{b} \leq \frac{3i+2}{3j+2}. \tag{2}$$

Now, a/b is in the interval $\left(0, \frac{1}{2}\right)$, and we want to find integers i, $j \geq 0$ satisfying (2). This can be done, for instance, by representing $\left(0, \frac{1}{2}\right)$ as the union of a two-tailed sequence of consecutively adjacent intervals as follows:

$$\left(0, \tfrac{1}{2}\right) = \ldots \cup \left[\tfrac{1}{25}, \tfrac{2}{26}\right] \cup \left[\tfrac{1}{13}, \tfrac{2}{14}\right] \cup \left[\tfrac{1}{7}, \tfrac{2}{8}\right] \cup \left[\tfrac{1}{4}, \tfrac{2}{5}\right] \cup \left[\tfrac{4}{10}, \tfrac{5}{11}\right] \cup \ldots \tag{3}$$

The end-points of each interval are fractions as in (2), for suitably chosen i and j. (For the interval $[\frac{1}{4}, \frac{2}{5}]$ and the intervals situated to the left of $[\frac{1}{4}, \frac{2}{5}]$, take $i = 0$ and $j = 2^m$ for $m = 0, 1, 2, \ldots$; for the intervals lying to the right of $[\frac{1}{4}, \frac{2}{5}]$, take $i = 2^m - 1$ and $j = 2^{m+1} - 1$ for $m = 1, 2, \ldots$.) The number a/b lies in some interval listed in (3), and hence it satisfies the inequalities (2) for some i and j. This completes the proof.

20. (*United Kingdom*) A function F is defined from the set of non-negative integers to itself such that, for every non-negative integer n, $F(4n) = F(2n) + F(n)$, $F(4n+2) = F(4n) + 1$, and $F(2n+1) = F(2n) + 1$. Prove that, for each positive integer m, the number of integers n with $0 \leq n < 2^m$ and $F(4n) = F(3n)$ is $F(2^{m+1})$.

Solution supplied by George Evagelopoulos, Athens, Greece, modified by the editor.

We have $F(0) = 0$ from the first given equation, and we see that F is uniquely determined by the given equations. The following table shows the values of $F(n)$ for $0 \leq n \leq 16$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$F(n)$	0	1	1	2	2	3	3	4	3	4	4	5	5	6	6	7	5

From the first given equation, by induction, we obtain $F(2^k) = f_{k+1}$, for $k = 0, 1, 2, \ldots$, where $\{f_n\}_{n=1}^{\infty}$ is the Fibonacci sequence, defined by

$$f_1 = f_2 = 1 , \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 3 .$$

In fact, a general formula for $F(n)$ may be given in terms of the Fibonacci numbers: if n has the binary representation

$$n = \varepsilon_k 2^k + \varepsilon_{k-1} 2^{k-1} + \cdots + \varepsilon_1 2 + \varepsilon_0 ,$$

where $\varepsilon_i \in \{0, 1\}$ for each i, then

$$F(n) = \varepsilon_k f_{k+1} + \varepsilon_{k-1} f_k + \cdots + \varepsilon_1 f_2 + \varepsilon_0 f_1 . \tag{1}$$

This formula may be verified simply by checking that it satisfis the given equations defining $F(n)$.

We now prove that $F(3n) \leq F(4n)$ for all non-negative integers n. We proceed by induction on m, the number of digits in the binary representation of n. When $m = 1$, we have $n = 0$ or $n = 1$, in both of which cases $F(3n) = F(4n)$, as can be seen from the table above.

Now let an integer $m \geq 2$ be fixed. As our induction hypothesis, we assume that $F(3n) \leq F(4n)$ whenever n has fewer than m digits in its binary representation. Consider any positive integer n which has exactly m digits in its binary representation. Thus, $n = 2^{m-1} + p$, where $0 \leq p < 2^{m-1}$. By the induction hypothesis, $F(3p) \leq F(4p)$. Using (1), we obtain

$$F(4n) = F(2^{m+1} + 4p) = f_{m+2} + F(4p) \geq f_{m+2} + F(3p) . \tag{2}$$

On the other hand

$$F(3n) = F(3 \cdot 2^{m-1} + 3p) = F(2^m + 2^{m-1} + 3p) . \tag{3}$$

To proceed further with the calculation of $F(3n)$, we consider three cases:

Case 1. $3p < 2^{m-1}$.

Then the number of digits in the binary representation of $3p$ is less than m, and the binary representation of $3p$ does not carry into that of $3 \cdot 2^{m-1}$.

Starting from (3), we can apply (1) and then (2) to get

$$F(3n) \;=\; f_{m+1} + f_m + F(3p) \;=\; f_{m+2} + F(3p) \;\leq\; F(4n)\,. \qquad (4)$$

Case 2. $2^{m-1} < 3p < 2^m$.

Then the number of digits in the binary representation of $3p$ is exactly m, and the binary representation of $3p$ carries 1 into that of $3 \cdot 2^{m-1}$. Starting from (3) and using (1), we calculate

$$\begin{aligned} F(3n) &= F\big(2^m + 2^{m-1} + 2^{m-1} + (3p - 2^{m-1})\big) \\ &= F\big(2^{m+1} + (3p - 2^{m-1})\big) \\ &= f_{m+2} + F(3p) - f_m \;=\; f_{m+1} + F(3p)\,. \end{aligned}$$

Noting that $f_{m+1} < f_{m+2}$ (since $m \geq 2$) and using (2), we find that $F(3n) < F(4n)$.

Case 3. $2^m < 3p < 3 \cdot 2^{m-1}$ $(= 2^m + 2^{m-1})$.

Then the number of digits in the binary representation of $3p$ is exactly $m+1$, and the binary representation of $3p$ carries 10 into that of $3 \cdot 2^{m-1}$. Starting from (3) and using (1), we calculate

$$\begin{aligned} F(3n) &= F\big(2^m + 2^{m-1} + 2^m + (3p - 2^m)\big) \\ &= F\big(2^{m+1} + 2^{m-1} + (3p - 2^m)\big) \\ &= f_{m+2} + f_m + F(3p) - f_{m+1} \;=\; f_{m+2} - f_{m-1} + F(3p)\,. \end{aligned}$$

Noting that $f_{m-1} > 0$ and using (2), we find that $F(3n) < F(4n)$.

In each case, we have shown that $F(3n) \leq F(4n)$. This completes the induction proof.

Let E denote the set consisting of 0 together with all positive integers in whose binary representation the 1s are isolated (meaning that no two 1s appear in adjacent positions). For example, $1 \in E$, $2 \in E$, but $3 \notin E$. We will now prove that $F(3n) = F(4n)$ if and only if $n \in E$.

First suppose that $n \in E$. If $n = 0$, then $F(3n) = F(0) = F(4n)$. Now assume that $n > 0$. On multiplying n by 3 (that is, 11 in binary), each 01 (and the leading 1) is replaced by 11. Consequently, $F(3n)$ is obtained from $F(n)$ by replacing each f_{i+1} in (1) by $f_{i+1} + f_{i+2} = f_{i+3}$. On the other hand, $F(4n)$ is obtained from $F(n)$ by replacing each f_{i+1} in (1) by f_{i+3}. Thus, $F(3n) = F(4n)$.

Conversely, suppose that n is a non-negative integer such that $F(3n) = F(4n)$. If $n = 0$ or $n = 1$, then $n \in E$. Now assume that $n \geq 2$. Let m be the number of digits in the binary representation of n. Upon examining the induction proof above, we observe that the equation $F(3n) = F(4n)$ occurs only in Case 1 and only when $F(3p) = F(4p)$. We note also that, in Case 1, we have $p < 2^{m-1}/3 < 2^{m-1}/2 = 2^{m-2}$, which implies that the second binary digit of n is 0. Therefore, in Case 1, we have $n \in E$ if and only if $p \in E$. By refining the induction proof slightly, we conclude that for all non-negative integers n, the equality $F(3n) = F(4n)$ implies that $n \in E$.

We have proved our claim that $F(3n) = F(4n)$ if and only if $n \in E$. Finally, we will prove that, for each non-negative integer m, the number of integers $n \in E$ such that $0 \leq n < 2^m$ is f_{m+2}. Since $f_{m+2} = F(2^{m+1})$, we will have proved the desired result.

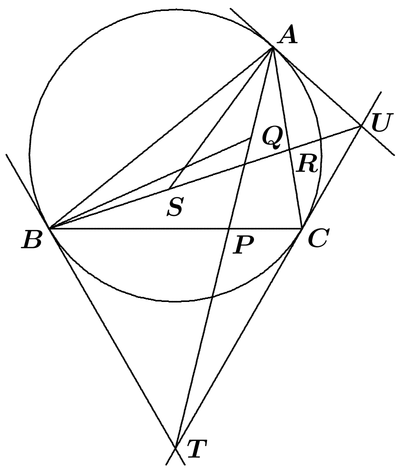
We proceed by induction on m. For $m = 0$, the number of $n \in E$ such that $0 \leq n < 2^m$ is $1 = f_2$; for $m = 1$, the number of $n \in E$ such that $0 \leq n < 2^m$ is $2 = f_3$. Now fix an integer $m \geq 2$, and assume, as the induction hypothesis, that the number of $n \in E$ such that $0 \leq n < 2^k$ is f_{k+2} for all integers k with $0 \leq k < m$. Among the integers $n \in E$ such that $0 \leq n < 2^m$ are those such that $0 \leq n < 2^{m-1}$, the number of which is f_{m+1} (using the induction hypothesis). The remaining $n \in E$ such that $0 \leq n < 2^n$ are those such that $2^{m-1} \leq n < 2^m$. The first two binary digits of any such number n must be 10, and the remaining digits represent a number in E which is less than 2^{m-2}. Thus, the number of such n is f_n (using the induction hypothesis again). The total number of $n \in E$ such that $0 \leq n < 2^n$ is $f_{m+1} + f_m = f_{m+2}$. This completes the proof.

21. (*United Kingdom*) The tangents at B and A to the circumcircle of an acute triangle ABC meet the tangent at C at T and U, respectively. The lines AT and BC meet at P, and Q is the mid-point of AP; the lines BU and CA meet at R, and S is the mid-point of BR.

(a) Prove that $\angle ABQ = \angle BAS$.

(b) Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

Solution supplied by George Evagelopoulos, Athens, Greece.



In $\triangle ABC$, let $\alpha = \angle A$, $\beta = \angle B$, $\gamma = \angle C$. By the tangent-chord theorem, we get $\angle TBC = \angle TCB = \angle BAC = \alpha$. Let R denote the circumradius of $\triangle ABC$. The isosceles triangle BCT gives

$$BT = \frac{BC}{2\cos\alpha} = R\tan\alpha\,.$$

By the Law of Sines in $\triangle ABT$, we have $\frac{AB}{BT} = \frac{\sin\angle ATB}{\sin\angle BAT}$. Let $\varphi = \angle BAT$. Noting that $\angle ABT = \alpha + \beta$, we get

$$\frac{AB}{BT} = \frac{\sin(\alpha+\beta+\varphi)}{\sin\varphi}\,.$$

Using the relations $AB = 2R\sin\gamma$, $BT = R\tan\alpha$, and $\alpha + \beta + \gamma = 180^\circ$, we obtain

$$\cot\varphi = \cot\gamma + 2\cot\alpha\,. \tag{1}$$

Next, we set $\angle ABQ = \psi$ and apply the Law of Sines to $\triangle BAQ$ and $\triangle BPQ$. This gives

$$\frac{BQ}{QA} = \frac{\sin\varphi}{\sin\psi} \quad\text{and}\quad \frac{BQ}{PQ} = \frac{\sin\angle APB}{\sin(\beta-\psi)} = \frac{\sin(\beta+\varphi)}{\sin(\beta-\psi)}\,.$$

Since Q is the mid-point of AP, we have $PQ = QA$. Thus, the ratios in the two equations above are equal. This gives $\dfrac{\sin\varphi}{\sin\psi} = \dfrac{\sin(\beta+\varphi)}{\sin(\beta-\psi)}$, or $\dfrac{\sin(\beta+\varphi)}{\sin\varphi} = \dfrac{\sin(\beta-\psi)}{\sin\psi}$. Hence, $\sin\beta\cot\varphi + \cos\beta = \sin\beta\cot\psi - \cos\beta$, leading to $\cot\psi = 2\cot\beta + \cot\varphi$. In this last equation, we use (1) to get

$$\cot\angle ABQ \;=\; \cot\psi \;=\; 2(\cot\alpha + \cot\beta) + \cot\gamma\,. \qquad (2)$$

It follows, by symmetry, that $\cot\angle BAS = \cot\angle ABQ$. Then $\angle ABQ = \angle BAS$ (since the cotangent function is strictly decreasing in the open interval $(0^\circ, 180^\circ)$). We have proved (a).

Maximizing ψ is equivalent to minimizing $\cot\psi$, which can be done using formula (2). Since

$$\begin{aligned}
\cot\alpha + \cot\beta &= \frac{\sin\beta\cos\alpha + \sin\alpha\cos\beta}{\sin\alpha\sin\beta} = \frac{\sin\gamma}{\sin\alpha\sin\beta}\\
&= \frac{2\sin\gamma}{\cos(\alpha-\beta) - \cos(\alpha+\beta)} = \frac{2\sin\gamma}{\cos(\alpha-\beta) + \cos\gamma}\\
&\ge \frac{2\sin\gamma}{1+\cos\gamma}\,,
\end{aligned}$$

we have

$$\begin{aligned}
\cot\psi &\ge \frac{4\sin\gamma}{1+\cos\gamma} + \cot\gamma\\
&= 4\tan\tfrac{\gamma}{2} + \frac{1-\tan^2\frac{\gamma}{2}}{2\tan\frac{\gamma}{2}} = \tfrac{7}{2}\tan\tfrac{\gamma}{2} + \frac{1}{2\tan\frac{\gamma}{2}}\,.
\end{aligned}$$

Applying the AM–GM Inequality, we obtain

$$\cot\psi \;\ge\; \sqrt{7\,\tan\tfrac{\gamma}{2}\cdot\frac{1}{\tan\frac{\gamma}{2}}} \;=\; \sqrt{7}\,.$$

For equality, it is necessary that $\alpha = \beta = 90^\circ - \gamma/2$ and $7\tan\frac{\gamma}{2} = 1/\tan\frac{\gamma}{2}$. These conditions give $\sin\gamma = \sqrt{7}/4$ and $\sin\alpha = \sin\beta = \sqrt{14}/4$. Hence, by the Law of Sines, the angle ψ is a maximum when the ratios of side lengths of $\triangle ABC$ are

$$a : b : c \;=\; \sqrt{2} : \sqrt{2} : 1\,.$$

That completes the *Corner* for this issue. Send me your nice solutions, generalizations, and contest materials.

BOOK REVIEWS

John Grant McLoughlin

Count Down: Six Kids Vie for Glory at the World's Toughest Math Competition
By Steve Olson, published by the Houghton Mifflin Company, New York, 2004
ISBN 0-618-25141-3, hardcover, 244 pages, US$24.
Reviewed by ***Richard Hoshino****, Dalhousie University, Halifax, NS.*

In *Count Down*, the author profiles the six students who represented the United States at the 2001 International Mathematical Olympiad. In the process, he sheds some new insight on important questions such as why Americans are so afraid of mathematics, why so few girls are involved in higher-level mathematics, and whether or not talent is innate.

Olson portrays the six students who are the members of Team USA as exceptionally well-rounded young men who share an incredible passion for learning. They are curious teenagers with a wide range of hobbies from water polo and ultimate frisbee to playing music and chess. The book does an excellent job of presenting and developing the characters of these six individuals. As a member of the Canadian delegation to the 2001 IMO, I had the privilege of meeting all six of these IMO students, and I vividly remember their well-roundedness and diversity of interests. (I am embarrassed to say that in the annual Canada vs USA game of ultimate frisbee, the Canadians lost convincingly.)

At the IMO, the students are given three problems on the first day of competition, and three problems on the following day. Appropriately, there are six main chapters in the book, with a brief chapter entitled "Interlude: An Afternoon to Rest" in the middle. Of course, this is no coincidence. In each of the six core chapters, there is a profile of one of the IMO team members, a detailed description and analysis of one of the six problems that appeared on the IMO, and a solution to the problem by that student.

The six chapters are titled Insight, Competitiveness, Talent, Creativity, Breadth, and a Sense of Wonder. In these chapters, Olson also investigates each of these characteristics and produces a fresh analysis of age-old questions such as "Is mathematical ability innate?", "Is genius inherited?", "Are Asian students inherently more talented in math?", and "Are boys better than girls in math?" He does a wonderful job of presenting the issues. His research is extensive, and he cites well-known pedagogues such as Ellen Winner, Howard Gardner, and Alfie Kohn, and uses the famous TIMSS study to compare models of mathematics teaching in America with various countries in Asia. He also produces an excellent profile of Melanie Wood, a former two-time member of the American team, and the only female ever to represent the USA at the IMO.

Count Down is more than a problem-solving book, or a simple six-chapter biography on the Olympians. Somehow, the author is able to delicately interweave the mathematics, the student profiles, the history of the Olympiad event, and a discussion of these tough issues in mathematics education. Moreover, he succeeds in every chapter! He is able to present the mathematics in a way that is not intimidating to the non-mathematical reader. In fact, he does quite a good job of presenting the six problems and explaining the meaning of each problem. Olson also presents full solutions to several of the IMO problems in a way that is accessible to a general audience. That was impressive.

This book is not perfect, but is a highly enjoyable read. In particular, I would have preferred to see a lengthier profile on the students and more insight on how they overcame personal adversity to make it to the Olympiad. Nevertheless, I found this book to be delightful, and I would recommend it to anyone interested in math contests and mathematics pedagogy, including high school students, mathematics educators, and parents of young children with an interest in mathematics.

Strange Curves, Counting Rabbits, and Other Mathematical Explorations
By Keith Ball, published by Princeton University Press, 2003
ISBN 0-691-11321-1, hardcover, 251 pages, US$29.95.
Reviewed by **Peter G. Liljedahl**, *Simon Fraser University, Burnaby, BC.*

> Pick up a paperback book, any book which was published fairly recently, and on the back you will find a number—the ISBN or International Standard Book Number. . . . The ISBN identifies the title among all titles published internationally. The ISBN sequence of this book is
>
> 0-691-11321-1
>
> . . . This number has a surprising property . . .

And so begins an enticing and wonderful journey through topics you thought you knew. At first glance, *Strange Curves* seems to be, like so many other books for recreational mathematicians, a collection of "common" topics in mathematics. A more careful perusal of the table of contents, however, will reveal that Keith Ball's treatment of these topics is anything but common. Reading the book will only further confirm this revelation. *Strange Curves* is no ordinary collection of topics.

What is even more surprising is that this book was not written for the recreational mathematician. It was written for the uninitiated, the soon-to-be recreational mathematician. It is a collection of topics that have grown out of Ball's popular lectures to high school students. These lectures, like the book, are meant to broaden students' views of what mathematics is and to present to them the rich diversity and aesthetic elements that are so often missing from the high school mathematics curriculum. This, however, should not be a deterrent for the seasoned veterans, for, as already mentioned, the unique combination and extension of the seemingly "common" mathematical topics is not only refreshing, but also invigorating.

There are ten chapters in all, the titles of which are not always informative. These chapters are built around the following themes: code construction, Pick's Theorem, Fermat's Little Theorem and decimal expansion, fractal curves, the normal curve, estimation of $n!$, binary protocols, Fibonacci and the Golden Ratio, curve approximation, and rational and irrational numbers.

In each chapter, Ball takes a seemingly common topic and develops it in very uncommon ways, the whole time building connections across topics and across chapters. For example, in one chapter Ball discusses the coin-weighing problem:

> Suppose you are given nine coins of which eight are genuine and one counterfeit. You know that all the genuine coins weigh the same as one another and that the counterfeit coin is slightly heavier. ...What is the smallest number of weighings [with a balance scale] needed to identify the counterfeit coin?

He extends this problem to the consideration of the real-world problem of testing for minor abnormalities in infants by testing pooled samples of blood. This then leads to a theoretical discussion of binary protocols, estimates, and entropy. In another chapter he evolves Fibonacci and the Golden Ratio into continued fractions, matrices, Newton's method, prime Lucas numbers, and Pick's Theorem. Throughout each chapter Ball poses problems for the reader to solve. The solutions to these problems are found at the end of each chapter along with a list of suggested references for further reading.

Ball has found the delicate balance between providing detail and challenging the reader, and he uses it masterfully in the writing of this book. I constantly felt comfortable with the material while at the same time stretching to make sense of it. His format of introducing a topic with an interesting (and often well-known) mathematical fact and then relating this fact to more mathematics, and his emphasis on approximations, make this an ideal book for an entry-level modeling course. The material is presented in a clear and linear fashion, with minimal details and rich problems to work on.

Maximum Area of a Triangle Subject to Side Constraints

G.D. Chakerian and M.S. Klamkin

An old *Monthly* problem by C.N. Schmall [3] was to show that the maximum area F of a triangle of sides a, b, c, where $a^3 + b^3 + c^3 = 3k^3$, is $\sqrt{3}k^2/4$ and occurs when $a = b = c$. The solution by E. Swift was via calculus and it was noted that the same method shows that the result also holds if the constraint condition is $a^n + b^n + c^n = 3k^n$, $n > 1$.

Here we give generalizations without using calculus (which should not be needed for elementary triangle inequalities). Firstly (see [2]),

$$F^3 \;\le\; 3\sqrt{3}(abc)^2/64\,,$$

since $abc = 4FR$ and $R^2 \ge 4F/(3\sqrt{3})$, which geometrically corresponds to the fact that the triangle of largest area which can be inscribed in a given circle is the equilateral one. We now use the following variants of the AM–GM Inequality which hold for all positive a, b, c:

$$(abc)^{\frac{n}{3}} \;\le\; \sqrt{((bc)^n + (ca)^n + (ab)^n)/3} \;\le\; (a^n + b^n + c^n)/3$$

for $n > 0$. It follows that if any one of the latter three expressions is given, all the other expressions to the left including the area F are maximized when $a = b = c$.

The above results can be generalized to n-dimensional simplexes by starting with the known inequality [2]

$$V^{n-1} \;\le\; n!\sqrt{\frac{(n+1)^{n-1}}{n^{3n}}}\left(\prod_{k=1}^{n+1} F_k\right)^{\frac{n}{n+1}}, \tag{1}$$

where V is the volume and F_k are the volumes of the $(n-1)$-dimensional facets. From this, we will now show that

$$V^{\frac{n+1}{2}} \;\le\; c(n)\prod_{k=1}^{N} e_k\,, \tag{2}$$

where $N = n(n+1)/2$, e_k are the edges of the simplex, and $c(n)$ is the $(n+1)/2$ power of the volume of a regular simplex of unit edge length. Our proof is inductive.

Write inequality (1) in the form $V^{n-1} \;\le\; \lambda(n)\left(\prod\limits_{k=1}^{n+1} F_k\right)^{\frac{n}{n+1}}$, and assume the validity of (2) for $(n-1)$ dimensions. Then $F_i^{\frac{n}{2}} \le c(n-1)\prod\limits_{j} e_{ij}$, where e_{ij}, for $j = 1, 2, \ldots, n(n-1)/2$, are the edge lengths of the i^{th} facet. This implies that

$$\left(\prod_{i=1}^{n+1} F_i\right)^{\frac{n}{2}} \leq c(n-1)^{n+1}\left(\prod_{k=1}^{N} e_k\right)^{n-1},$$

since each edge of the given simplex belongs to $n-1$ facets. Therefore,

$$V^{n-1} \leq \lambda(n)c(n-1)^2\left(\prod_{k=1}^{N} e_k\right)^{\frac{2(n-1)}{n+1}}.$$

Hence,

$$V^{\frac{n+1}{2}} \leq \lambda(n)^{\frac{n+1}{2(n-1)}} c(n-1)^{\frac{n+1}{n-1}}\left(\prod_{k=1}^{N} e_k\right) = c(n)\prod_{k=1}^{N} e_k .$$

Noting that $V^{\frac{n+1}{2}} / \prod_{k=1}^{N} e_k$ is invariant under similarity gives us the fact that $c(n)$ is the constant described above. Thus, the validity for the $(n-1)$ simplexes implies the validity for dimension n.

We now use the Maclaurin Inequalities [1]. Letting

$$\begin{aligned}(x + e_1^m)\cdots(x + e_{n+1}^m) \\ = \; x^{n+1} + \binom{n+1}{1}p_1x^n + \binom{n+1}{2}p_2x^{n-1} + \cdots + p_{n+1},\end{aligned}$$

where $m > 0$, we have $p_{n+1}^{\frac{1}{n+1}} \leq p_n^{\frac{1}{n}} \leq \cdots \leq p_2^{\frac{1}{2}} \leq p_1$, with equality if and only if all the e_js are equal. This implies that if any one of the p_is is given, then all the p_js where $j > i$ and V achieve their maxima if and only if all the e_is are equal.

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A Vector Introduction to the Orthopole

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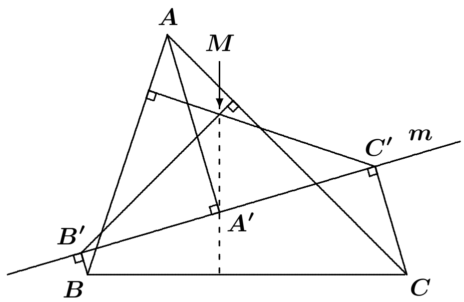
Dedicated to the memory of Murray S. Klamkin

In some geometrical situations, vector proofs are simple and straight to the point. M.S. Klamkin showed this beautifully in [4] by means of several illustrative examples in solid geometry. The purpose of this note is to add a few examples from plane geometry, namely the definition of the orthopole and two results about its location.

In the theorems below, we denote by m a given line in the plane of a triangle ABC, and by A', B', C' the (orthogonal) projections of A, B, C onto m. If U is a point in the plane, we denote by $\overrightarrow{U}$ the vector to U from a fixed origin (chosen arbitrarily).

Theorem 1. The perpendiculars from A' to BC, from B' to CA, and from C' to AB are concurrent lines.



Proof: Let the perpendicular from B' to CA meet the one from C' to AB at M. We just have to show that $A'M$ is perpendicular to BC. Now,

$$(\overrightarrow{A'}-\overrightarrow{M})\cdot(\overrightarrow{C}-\overrightarrow{B}) \;=\; (\overrightarrow{A'}-\overrightarrow{M})\cdot(\overrightarrow{C}-\overrightarrow{A})-(\overrightarrow{A'}-\overrightarrow{M})\cdot(\overrightarrow{B}-\overrightarrow{A})\,.$$

Considering the first term on the right side, we have

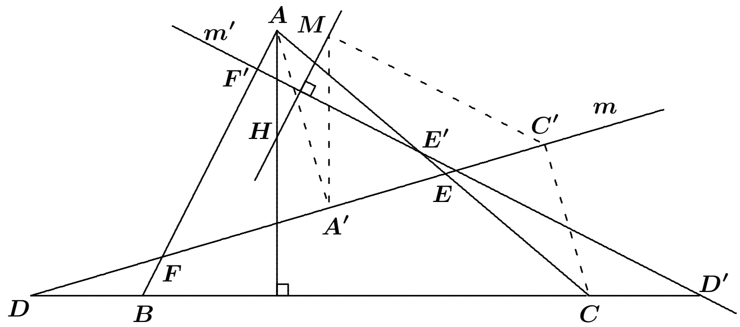
$$(\overrightarrow{A'}-\overrightarrow{M})\cdot(\overrightarrow{C}-\overrightarrow{A}) \;=\; (\overrightarrow{A'}-\overrightarrow{B'})\cdot(\overrightarrow{C}-\overrightarrow{A}) \;=\; (\overrightarrow{A'}-\overrightarrow{B'})\cdot(\overrightarrow{C'}-\overrightarrow{A'})\,,$$

where the first step follows from the fact that $MB' \perp CA$ and the second from the fact that both AA' and CC' are orthogonal to $A'B'$. Similarly, we find that $(\overrightarrow{A'}-\overrightarrow{M})\cdot(\overrightarrow{B}-\overrightarrow{A}) = (\overrightarrow{A'}-\overrightarrow{C'})\cdot(\overrightarrow{B'}-\overrightarrow{A'})$. Therefore, $(\overrightarrow{A'}-\overrightarrow{M})\cdot(\overrightarrow{C}-\overrightarrow{B}) = 0$. ∎

The point M is called the *orthopole* of m (with respect to $\triangle ABC$). It is interesting to compare the short proof above with synthetic proofs ([3], [1]) or a proof using complex numbers ([2]). The simplicity of vector proofs also stands out in the following two lesser-known theorems that link the orthopole to the notion of isotomic transversal.

Let the line m meet the extended sides BC, CA, AB at D, E, F, respectively. Then the *isotomic transversal* m' of m is the line passing through the points D', E', F' symmetric with D, E, F about the mid-points of the sides BC, CA, AB, respectively. (The collinearity of D', E', F' results at once from Menelaus' Theorem.)

Theorem 2. The orthopole M of m lies on the perpendicular from the orthocentre H of $\triangle ABC$ to the isotomic transversal m' of m.



Proof: The method is analogous to Theorem 1. First we write

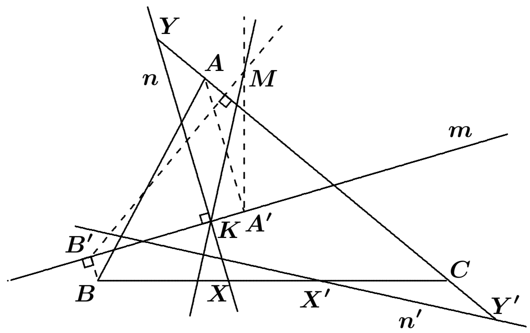
$$(\overrightarrow{M} - \overrightarrow{H}) \cdot (\overrightarrow{F'} - \overrightarrow{D'}) \;=\; (\overrightarrow{M} - \overrightarrow{H}) \cdot (\overrightarrow{F'} - \overrightarrow{B}) - (\overrightarrow{M} - \overrightarrow{H}) \cdot (\overrightarrow{D'} - \overrightarrow{B})\,.$$

Considering the first term on the right side, we have

$$\begin{aligned}(\overrightarrow{M} - \overrightarrow{H}) \cdot (\overrightarrow{F'} - \overrightarrow{B}) &= (\overrightarrow{M} - \overrightarrow{H}) \cdot (\overrightarrow{A} - \overrightarrow{F}) \;=\; (\overrightarrow{C'} - \overrightarrow{C}) \cdot (\overrightarrow{A} - \overrightarrow{F}) \\ &= (\overrightarrow{C'} - \overrightarrow{C}) \cdot (\overrightarrow{A} - \overrightarrow{A'})\,,\end{aligned}$$

where the first step follows from the definition of F', the second from the fact that $C'M$ and CH are orthogonal to AF, and the third because $FA' \perp CC'$. Similarly, we see that $(\overrightarrow{M} - \overrightarrow{H}) \cdot (\overrightarrow{D'} - \overrightarrow{B}) = (\overrightarrow{A'} - \overrightarrow{A}) \cdot (\overrightarrow{C} - \overrightarrow{C'})$. Thus, $(\overrightarrow{M} - \overrightarrow{H}) \cdot (\overrightarrow{F'} - \overrightarrow{D'}) = 0$. ∎

Theorem 3. Choose any point K on m and let n be the perpendicular to m through K. The orthopole M of m lies on the perpendicular from K to the isotomic transversal n' of n.



Proof: Let n, n' meet BC at X, X' and CA at Y, Y', respectively. Again,

$$(\overrightarrow{M}-\overrightarrow{K})\cdot(\overrightarrow{Y'}-\overrightarrow{X'}) = (\overrightarrow{M}-\overrightarrow{K})\cdot(\overrightarrow{Y'}-\overrightarrow{C}) - (\overrightarrow{M}-\overrightarrow{K})\cdot(\overrightarrow{X'}-\overrightarrow{C})$$

with

$$\begin{aligned}(\overrightarrow{M}-\overrightarrow{K})\cdot(\overrightarrow{Y'}-\overrightarrow{C}) &= (\overrightarrow{M}-\overrightarrow{K})\cdot(\overrightarrow{A}-\overrightarrow{Y}) = (\overrightarrow{B'}-\overrightarrow{K})\cdot(\overrightarrow{A}-\overrightarrow{Y}) \\ &= (\overrightarrow{B'}-\overrightarrow{K})\cdot(\overrightarrow{A'}-\overrightarrow{K})\,.\end{aligned}$$

Similarly, we have $(\overrightarrow{M}-\overrightarrow{K})\cdot(\overrightarrow{X'}-\overrightarrow{C}) = (\overrightarrow{A'}-\overrightarrow{K})\cdot(\overrightarrow{B'}-\overrightarrow{K})$. ∎

In conclusion, we propose the following problem, easily solved with the help of the previous theorems: Let s_P denote the Simson line of a point P on the circumcircle of $\triangle ABC$.

(a) show that the orthopole of a line perpendicular to s_P lies on s_P;

(b) show that the orthopole of s_P lies on the Steiner line of P (the line through the reflections of P in the sides BC, CA, AB). What is its exact position on this line? [*Hint:* What is the isotomic transversal of s_P? ...]

For readers wishing to pursue this further, there are many interesting results connecting the orthopole to Simson lines in [3] and [2], and to the complete quadrilateral in [1].

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Two Generalizations of Popoviciu's Inequality

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Dedicated to the memory of Murray S. Klamkin

In 1965 the Romanian mathematician T. Popoviciu proved the following inequality [8]

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right) \geq 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right),$$

where f is a convex function on an interval I and x, y, $z \in I$. A. Lupas [7] generalized this in 1982 as follows (where p, q, and r are positive numbers)

$$\begin{aligned} & pf(x)+qf(y)+rf(z)+(p+q+r)f\left(\frac{px+qy+rz}{p+q+r}\right) \\ \geq\ & (p+q)f\left(\frac{px+qy}{p+q}\right)+(q+r)f\left(\frac{qy+rz}{q+r}\right)+(r+p)f\left(\frac{rz+px}{r+p}\right). \end{aligned}$$

In this paper we present two generalizations of Popoviciu's Inequality to n variables with some applications. The first generalization was published in [1] with a more difficult proof than the one below. The second generalization (without solution) was posted on the Mathlinks Site – Inequalities Forum in 2004. It is possible that one or both generalizations might have been previously published elsewhere, but not as far as we know.

Our proof relies on Karamata's Inequality for convex functions, see [4] and [3], which we now recall. We say that a vector $\overrightarrow{A} = [a_1, a_2, \ldots, a_n]$ with $a_1 \geq a_2 \geq \ldots \geq a_n$ majorizes a vector $\overrightarrow{B} = [b_1, b_2, \ldots, b_n]$ with $b_1 \geq b_2 \geq \ldots \geq b_n$ and write this as $\overrightarrow{A} \succeq \overrightarrow{B}$, if

$$\begin{aligned} a_1 &\geq b_1 \\ a_1+a_2 &\geq b_1+b_2 \\ &\vdots \\ a_1+a_2+\cdots+a_{n-1} &\geq b_1+b_2+\cdots+b_{n-1} \\ a_1+a_2+\cdots+a_n &= b_1+b_2+\cdots+b_n\,. \end{aligned}$$

The Karamata Inequality states that if f is any convex function and $\overrightarrow{A} \succeq \overrightarrow{B}$, then one has the following

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq f(b_1)+f(b_2)+\cdots+f(b_n)\,.$$

The First Generalization.

Theorem 1. If f is a convex function on an interval I and $a_1, a_2, \ldots, a_n \in I$, then

$$\begin{aligned} f(a_1) + f(a_2) + \cdots + f(a_n) + n(n-2) f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \\ \geq \ (n-1)\bigl(f(b_1) + f(b_2) + \cdots + f(b_n)\bigr), \end{aligned}$$

where $b_i = \dfrac{1}{n-1} \sum\limits_{j \neq i} a_j$ for all i.

Proof: Without loss of generality, we may assume that $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. Then there is an integer m with $1 \leq m \leq n-1$ and $a_1 \leq \cdots \leq a_m \leq a \leq a_{m+1} \leq \cdots \leq a_n$, where $a = (a_1 + \cdots + a_n)/n$. We also have $b_1 \geq \cdots \geq b_m \geq a \geq b_{m+1} \geq \cdots \geq b_n$. It is clear that the inequality that we are trying to prove is the sum of the following two inequalities:

$$\begin{aligned} f(a_1) + f(a_2) + \cdots + f(a_m) + n(n-m-1) f(a) \\ \geq \ (n-1)\bigl(f(b_{m+1}) + f(b_{m+2}) + \cdots + f(b_n)\bigr), \quad (1) \\ f(a_{m+1}) + f(a_{m+2}) + \cdots + f(a_n) + n(m-1) f(a) \\ \geq \ (n-1)\bigl(f(b_1) + f(b_2) + \cdots + f(b_m)\bigr). \quad (2) \end{aligned}$$

In order to prove (1), we apply Jensen's Inequality to get

$$f(a_1) + f(a_2) + \cdots + f(a_m) + (n-m-1) f(a) \ \geq \ (n-1) f(b),$$

where $b = \dfrac{a_1 + a_2 + \cdots + a_m + (n-m-1)a}{n-1}$. Thus, we still have to show

$$(n-m-1) f(a) + f(b) \ \geq \ f(b_{m+1}) + f(b_{m+2}) + \cdots + f(b_n).$$

Since $a \geq b_{m+1} \geq b_{m+2} \geq \cdots \geq b_n$ and

$$(n-m-1)a + b \ = \ b_{m+1} + b_{m+2} + \cdots + b_n,$$

we see that $\overrightarrow{A}_{n-m} = [a, \ldots, a, b]$ majorizes $\overrightarrow{B}_{n-m} = [b_{m+1}, b_{m+2}, \ldots, b_n]$. The inequality follows by Karamata's Inequality for convex functions.

The inequality (2) can be proved similarly by adding Jensen's Inequality

$$\frac{f(a_{m+1}) + f(a_{m+2}) + \cdots + f(a_n) + (m-1) f(a)}{n-1} \ \geq \ f(c)$$

to the inequality

$$f(c) + (m-1) f(a) \ \geq \ f(b_1) + f(b_2) + \cdots + f(b_m),$$

where $c = (a_{m+1} + a_{m+2} + \cdots + a_n + (m-1)a)/(n-1)$. The last inequality follows from Karamata's Inequality, because $b_1 \geq \cdots \geq b_m \geq a$ and $c + (m-1)a = b_1 + b_2 + \cdots + b_m$, and therefore $\overrightarrow{C}_m = [c, a, \ldots, a]$ majorizes $\overrightarrow{D}_m = [b_1, b_2, \ldots, b_m]$.

The Second Generalization.

Theorem 2. If f is a convex function on an interval I and $a_1, a_2, \ldots, a_n \in I$, then

$$\begin{aligned}(n-2)\big(f(a_1)+f(a_2)+\cdots+f(a_n)\big)+nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)&\\ \geq\ 2\sum_{1\leq i<j\leq n} f\left(\frac{a_i+a_j}{2}\right)&.\end{aligned}$$

Proof: We will prove this using induction. For $n=2$, one has equality. Suppose now that $n \geq 3$ and that the inequality is valid for $n-1$. We will show that it holds for n. Let $a=(a_1+a_2+\cdots+a_n)/n$ and let $x=(a_1+a_2+\cdots+a_{n-1})/(n-1)$. By the induction hypothesis, we have

$$\begin{aligned}(n-3)\big(f(a_1)+f(a_2)+\cdots+f(a_{n-1})\big)+(n-1)f(x)&\\ \geq\ 2\sum_{1\leq i<j\leq n-1} f\left(\frac{a_i+a_j}{2}\right)&.\end{aligned}$$

Thus, it suffices to show that

$$\begin{aligned}f(a_1)+f(a_2)+\cdots+f(a_{n-1})+(n-2)f(a_n)+nf(a)&\\ \geq\ (n-1)f(x)+2\sum_{i=1}^{n-1} f\left(\frac{a_i+a_n}{2}\right)&.\end{aligned}$$

From Jensen's Inequality, we have

$$f(a_1)+f(a_2)+\cdots+f(a_{n-1})\ \geq\ (n-1)f(x)\,.$$

Hence, we just have to show that

$$(n-2)f(a_n)+nf(a)\ \geq\ 2\sum_{i=1}^{n-1} f\left(\frac{a_i+a_n}{2}\right).$$

Since $(n-2)a_n+na=2\sum_{i=1}^{n-1}\frac{a_i+a_n}{2}$, we will again use Karamata's Inequality for two cases.

Case 1. $2a \geq \min\{a_1, a_2, \ldots, a_n\}+\max\{a_1, a_2, \ldots, a_n\}$.

Without loss of generality, we may assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. Then $a \geq (a_1+a_n)/2$. According to Karamata's Inequality, it is enough to show that $a_n \leq \min\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$ and $a \geq \max\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$. The first condition is clearly true and the second condition reduces to $a \geq (a_1+a_n)/2$.

Case 2. $2a < \min\{a_1, a_2, \ldots, a_n\}+\max\{a_1, a_2, \ldots, a_n\}$.

Without loss of generality, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then $a \leq (a_1+a_n)/2$. According to Karamata's Inequality, it is enough to show that $a \leq \min\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$ and $a_n \geq \max\{(a_1+a_n)/2, (a_2+a_n)/2, \ldots, (a_{n-1}+a_n)/2\}$. The second condition is clearly true and the first condition reduces to $a \leq (a_1+a_n)/2$.

Some Applications.

Proposition 1. Let $a_1, a_2, \ldots, a_n$ be positive numbers with $a_1a_2\cdots a_n = 1$. Then

$$a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}+n(n-2) \;\geq\; (n-1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

Proof: This inequality follows from Theorem 1 if we consider the convex function $f(x) = e^x$ and replace $a_1, a_2, \ldots, a_n$ with $(n-1)\ln a_1$, $(n-1)\ln a_2, \ldots, (n-1)\ln a_n$, respectively.

Remark. For $n=3$ and $a_1 = \frac{x^2}{yz}$, $a_2 = \frac{y^2}{zx}$, $a_3 = \frac{z^2}{xy}$, this inequality reduces to

$$x^6+y^6+z^6+3(xyz)^2 \;\geq\; 2(y^3z^3+z^3x^3+x^3y^3)\,.$$

This inequality was proposed by Murray Klamkin in [5].

Proposition 2. If $a_1, a_2, \ldots, a_n$ are positive numbers satisfying $a_1+a_2+\cdots+a_n = n$, then

$$(n-a_1)(n-a_2)\cdots(n-a_n) \;\geq\; (n-1)^n(a_1a_2\cdots a_n)^{\frac{1}{n-1}}\,.$$

Proof: We apply Theorem 1 to the convex function $f(x) = -\ln x$ for $x > 0$.

Remark. Since $a_1+a_2+\cdots+a_n = n$ implies that $a_1a_2\cdots a_n \leq 1$ (the AM–GM Inequality), the above inequality is sharper than the inequality

$$(n-a_1)(n-a_2)\cdots(n-a_n) \;\geq\; (n-1)^n a_1a_2\cdots a_n\,,$$

which easily follows by multiplying the inequalities

$$\begin{aligned}
n-a_1 &= a_2+a_3+\cdots+a_n \;\geq\; (n-1)(a_2a_3\cdots a_n)^{\frac{1}{n-1}}\,,\\
n-a_2 &= a_1+a_3+\cdots+a_n \;\geq\; (n-1)(a_1a_3\cdots a_n)^{\frac{1}{n-1}}\,,\\
&\;\;\vdots\\
n-a_n &= a_1+a_2+\cdots+a_{n-1} \;\geq\; (n-1)(a_1a_2\cdots a_{n-1})^{\frac{1}{n-1}}\,.
\end{aligned}$$

Proposition 3. If $a_1, a_2, \ldots, a_n$ are positive numbers and $b_i = \frac{1}{n-1}\sum_{j\neq i} a_j$ for all i, then

$$\frac{b_1}{a_1}+\frac{b_2}{a_2}+\cdots+\frac{b_n}{a_n} \;\geq\; \frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\,. \tag{3}$$

Proof: We can prove this well-known inequality by applying Theorem 1. Let $a = (a_1+a_2+\cdots+a_n)/n$. Using the relations

$$\frac{(n-1)b_i}{a_i} = \frac{na}{a_i}-1 \qquad\text{and}\qquad \frac{a_i}{b_i} = \frac{na}{b_i}-n+1$$

for $i = 1, 2, \ldots, n$, we see that (3) is equivalent to the inequality

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\frac{n(n-2)}{a} \;\geq\; (n-1)\left(\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_n}\right).$$

But this easily follows from Theorem 1 by using the convex function $f(x) = 1/x$ for $x > 0$.

Proposition 4. Suppose that $x_1, x_2, \ldots, x_n$ are positive numbers such that

$$x_1 + x_2 + \cdots + x_n \;=\; \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\,. \tag{4}$$

Then

$$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \;\geq\; 1\,. \tag{5}$$

Proof: This inequality may be derived from (3) in the following way. Suppose that

$$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \;<\; 1\,; \tag{6}$$

in other words, we suppose that (5) is false. Then we will show that (4) also does not hold. To this end, let $a_i = \dfrac{1}{1+(n-1)x_i}$ for $i = 1, 2, \ldots, n$. Note that $a_i > 0$ and that $x_i = \dfrac{1-a_i}{(n-1)a_i}$ for all $i = 1, 2, \ldots, n$. We also have $\sum a_i < 1$ by (6). Hence,

$$1 - a_i \;>\; \sum_{j \neq i} a_j \;=\; (n-1)b_i \tag{7}$$

for all $i = 1, 2, \ldots, n$. Thus,

$$\begin{aligned}
x_1 + \cdots + x_n \;=\; \sum_{i=1}^{n} \frac{1-a_i}{(n-1)a_i} \quad &> \quad \sum_{i=1}^{n} \frac{b_i}{a_i} && \text{by (7)}\\
&\geq \quad \sum_{i=1}^{n} \frac{a_i}{b_i} && \text{by (3)}\\
&> \quad \sum_{i=1}^{n} \frac{(n-1)a_i}{1-a_i} && \text{by (7)}\\
&= \quad \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\,.
\end{aligned}$$

Hence, we have proved that if (5) does not hold, then (4) does not hold. Therefore, (4) implies (5), by the contrapositive.

Remark. Substituting $1/x_i$ for x_i in (5) and noting that (4) is still satisfied gives us

$$\frac{x_1}{n-1+x_1} + \frac{x_2}{n-1+x_2} + \cdots + \frac{x_n}{n-1+x_n} \;\geq\; 1\,.$$

Since

$$\frac{x_i}{n-1+x_i} = 1 - \frac{n-1}{n-1+x_i}$$

for each $i = 1, 2, \ldots, n$, the inequality can be rewritten in the form

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1 .$$

This is an inequality in [2] that was derived using Lagrange Multipliers.

Proposition 5. If $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$(n-1)(x_1^2 + x_2^2 + \cdots + x_n^2) + n(x_1^2 x_2^2 \cdots x_n^2)^{\frac{1}{n}} \geq (x_1 + x_2 + \cdots + x_n)^2 .$$

Proof: This is a known inequality [6] which follows from Theorem 2 using the convex function $f(x) = e^x$ and replacing $a_1, a_2, \ldots, a_n$ with $2\ln x_1$, $2\ln x_2, \ldots, 2\ln x_n$, respectively.

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A New Proof and Extension of Problem 2620

Eric Lenza and Bill Sands

Dedicated to the memory of Murray S. Klamkin

Here is the problem in the title:

Three cards are handed to you. On each card are three non-negative real numbers, written one below the other, so that the sum of the numbers on each card is 1. *You are allowed to put the three cards in any order you like, then you write down the first number from the first card, the second number from the second card, and the third number from the third card, and you add these three numbers together. Prove that you can always arrange the cards so that your sum is in the interval* $[1/2, 3/2]$.

Actually, this problem first appeared in a weaker form on the 2000 Calgary Junior Mathematics Contest (see [1]). It was later submitted to ***CRUX with MAYHEM*** by the second author in 2001 (and was dedicated to Murray on the occasion of his 80^{th} birthday, in fact), and a solution (by Michel Bataille) was published in March 2002. At the end of this solution, the second author ventured a conjecture about the analogous problem with four or more cards.

In 2003 the first author (who is an undergraduate mathematics student at the University of Calgary) was awarded an NSERC Summer Undergraduate Research Fellowship under the second author's supervision, and was given the more general problem to work on. This paper is the result. Specifically, we will:

1. give a new proof, due to the first author, of the original problem;
2. demolish the second author's conjecture;
3. prove a correct result for the case of four cards;
4. suggest some new problems.

First, we present the first author's new proof of the original problem.

Proof of Crux problem 2620 (Eric Lenza): Consider the following three cards:

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \quad \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix},$$

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where $0 \leq a_i,\, b_i,\, c_i \leq 1$ and $a_i + b_i + c_i = 1$ for each i. Each permutation of these cards results in a sum of three numbers, and we will denote the permutation by this sum. For instance, if the cards are ordered as given, the permutation is denoted by the sum $a_1 + b_2 + c_3$, while if we switch the first two cards the permutation becomes $b_1 + a_2 + c_3$. We wish to prove that at least one of the six possible permutations yields a sum between $1/2$ and $3/2$, with either extreme value being allowed. This follows immediately from the following two claims:

1. At most two of the six permutations have a sum strictly greater than $3/2$;
2. At most three of the six permutations have a sum strictly less than $1/2$.

Proof of 1: Choose any three permutations. Note that all six permutations can be divided into two sets of three as follows:

$$a_1 + b_2 + c_3,\;\; b_1 + c_2 + a_3,\;\; c_1 + a_2 + b_3;$$
$$a_1 + c_2 + b_3,\;\; b_1 + a_2 + c_3,\;\; c_1 + b_2 + a_3.$$

The permutations in each set of three are pairwise disjoint (since they are cyclic shifts of each other). Two of our three chosen permutations must belong to the same set of three, and are therefore disjoint. Thus, the sum of their six entries is together at most the sum of the three cards, which is just 3. Therefore, one of these two permutations must have sum at most $3/2$. ∎

Proof of 2: Choose any four permutations. We claim that these permutations must use all the entries of at least two of the cards. Since there are only three entries on each card, some entry on the first card must appear twice. By symmetry and relabelling, we may suppose that it is a_1, and thus two of the chosen permutations are $a_1 + b_2 + c_3$ and $a_1 + c_2 + b_3$. If our four permutations do not include all entries of the first card, then by symmetry we may suppose that the other two chosen permutations are $b_1 + a_2 + c_3$ and $b_1 + c_2 + a_3$, and all entries of cards 2 and 3 are included. On the other hand, if we do include all entries of the first card, then we may assume that one of the other two chosen permutations is $b_1 + a_2 + c_3$, and now we have all entries of cards 1 and 2.

Since the sum S of the four chosen permutations must contain all the entries from at least two of the cards, and the sum of each card is 1, it follows that $S \geq 2$. Thus not all of the four chosen permutations can have sum strictly less than $1/2$. ∎

The interval $[1/2, 3/2]$ is the best possible, as stated in [2002 : 128]. The above two claims are the best possible too, in the sense that it is possible for two permutation sums to be greater than $3/2$, or for three sums to be less than $1/2$. For example, the first set of cards below has two permutation sums equal to 2, and the second set has three sums equal to $1/3$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\; \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix},\; \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}; \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\; \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}.$$

We now generalize the problem.

The n-card problem. Let $n \geq 4$ be an integer, and suppose you are given n cards, each containing n non-negative real numbers, written one below the other, so that the sum of the numbers on each card is 1. You are allowed to order the cards arbitrarily. Then you write down the first number from the first card, the second number from the second card, ..., and the n^{th} number from the n^{th} card, and you add these n numbers together. What is the smallest interval $[a, b]$ containing 1 such that the sum will always lie in $[a, b]$?

The second author had suggested ([2002 : 128]) that the answer to the n-card problem might be the intervals

$$\left[1 - \frac{2}{n}, 1 + \frac{2}{n}\right] \quad \text{for } n \text{ even}\,, \qquad \left[1 - \frac{2}{n+1}, 1 + \frac{2}{n+1}\right] \quad \text{for } n \text{ odd}\,,$$

which would imply in particular that the length of the interval approaches 0 as $n \to \infty$. It turns out this conjecture is badly out of whack! The following examples, found by the first author, show this convincingly. And we only need two different cards to do it.

Example 1. For n odd, consider the cards

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{(n-1)/2}, \underbrace{\begin{pmatrix} 0 \\ 1/(n-1) \\ 1/(n-1) \\ \vdots \\ 1/(n-1) \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 1/(n-1) \\ 1/(n-1) \\ \vdots \\ 1/(n-1) \end{pmatrix}}_{(n+1)/2},$$

where every card has n entries. It is easy to check that any permutation of these cards will result in a sum of either

$$\left(\frac{n+1}{2} - 1\right)\left(\frac{1}{n-1}\right) = \frac{1}{2} \qquad \text{or} \qquad 1 + \left(\frac{n+1}{2}\right)\left(\frac{1}{n-1}\right) = \frac{3}{2} + \frac{1}{n-1}\,.$$

If we replace one of the right-hand cards by another left-hand card, these sums become

$$\left(\frac{n-1}{2} - 1\right)\left(\frac{1}{n-1}\right) = \frac{1}{2} - \frac{1}{n-1} \qquad \text{or} \qquad 1 + \left(\frac{n-1}{2}\right)\left(\frac{1}{n-1}\right) = \frac{3}{2}\,.$$

Example 2. For n even, we just use $n/2$ of each kind of card in Example 1. Then the sum for any permutation of these n cards will be either

$$\left(\frac{n}{2} - 1\right)\left(\frac{1}{n-1}\right) = \frac{1}{2} - \frac{1}{2n-2} \quad \text{or} \quad 1 + \left(\frac{n}{2}\right)\left(\frac{1}{n-1}\right) = \frac{3}{2} + \frac{1}{2n-2}\,.$$

For future reference, we also mention two variants of Example 2. For the first, use $(n/2) - 1$ of the left-hand cards of Example 1 and $(n/2) + 1$ of the right-hand cards. Then the sums are

$$\left(\frac{n}{2}\right)\left(\frac{1}{n-1}\right) = \frac{1}{2} + \frac{1}{2n-2} \qquad \text{or} \qquad 1 + \left(\frac{n}{2} + 1\right)\left(\frac{1}{n-1}\right) = \frac{3}{2} + \frac{3}{2n-2}\,.$$

For the second, use $(n/2)+1$ of the left-hand cards and $(n/2)-1$ of the right-hand cards; then the sums are

$$\left(\frac{n}{2}-2\right)\left(\frac{1}{n-1}\right) = \frac{1}{2}-\frac{3}{2n-2} \quad \text{or} \quad 1+\left(\frac{n}{2}-1\right)\left(\frac{1}{n-1}\right) = \frac{3}{2}-\frac{1}{2n-2}.$$

Examples 1 and 2 show that for any $n \geq 3$, no interval satisfying the n-card problem can be strictly contained in the interval $[1/2, 3/2]$; thus, the conjecture on [2002 : 128] certainly fails for large n (for all $n \geq 4$, in fact).

Here is the reality for $n = 4$.

4-**Card Theorem**. The interval $[1/3, 4/3]$ satisfies the 4-card problem.

Proof: Denote the cards by $C_i = (a_i, b_i, c_i, d_i)$, for $i = 1, 2, 3, 4$. The idea of the proof is the same as for three cards. Namely, the theorem follows immediately from the following two claims:

1. At most 12 of the 24 permutation sums are strictly greater than $4/3$.

2. At most 11 of the 24 permutation sums are strictly less than $1/3$.

Proof of Claim 1: If we group together permutations which are cyclic shifts of each other, as we did in the 3-card case, we get six sets of four permutations each. Hence, given any 13 permutations, three of them must belong to the same cyclic set, and thus will be pairwise disjoint. The sum of the entries in these three permutations will therefore be at most the sum of the entries in the four cards, which is 4. Thus, not all three of the disjoint permutations can have sum greater than $4/3$. ■

Claim 2 is harder to prove, and we first need some new ideas.

We let $[n] = \{1, 2, \ldots, n\}$, and denote the set of all functions $f : [n] \to [n]$ by $\mathcal{F}[n]$. The set of all one-to-one onto functions in $\mathcal{F}[n]$ is, of course, the set of all the permutations of $[n]$, and is denoted S_n. Given two functions $f, g \in \mathcal{F}[n]$, we say that f *avoids* g if $f(x) \neq g(x)$ for all $x \in [n]$. For an arbitrary $f \in \mathcal{F}[n]$, let $A(f)$ be the set of *permutations* of $[n]$ which avoid f; that is,

$$A(f) = \{\varphi \in S_n \mid \varphi(x) \neq f(x) \text{ for all } x \in [n].\}$$

If $f \in S_n$, then $|A(f)|$ is just $d(n)$, the number of *derangements* of $[n]$. (See [2].)

The least possible value of $|A(f)|$ over all functions f is, of course, 0, which is achieved when f is a constant function. (In fact, only when f is a constant function, as readers may like to prove.) But what functions f maximize $|A(f)|$? Here is the not-too-surprising answer.

Lemma. For $n \geq 4$, $|A(f)|$ is maximized if and only if $f \in S_n$.

Proof: Let $f : [n] \to [n]$ be fixed and $f \notin S_n$. Then $f(a) = f(b) = c$ for some $a, b, c \in [n]$, where $a \neq b$. Since f is not one-to-one, it is not onto either; whence, there is some $d \in [n]$ which is not in $\mathbf{Im}(f)$, the image of f. Construct $g : [n] \longrightarrow [n]$ as follows:

$$g(x) \;=\; \begin{cases} f(x) & \text{if } x \neq b, \\ d & \text{if } x = b. \end{cases} \tag{1}$$

Then $\mathbf{Im}(g) = \mathbf{Im}(f) \cup \{d\}$; thus, g is "nearer" to being a permutation than f is.

We claim that $|A(g)| \geq |A(f)|$. To prove this, to each $\varphi \in A(f)$ we will associate a function $\varphi' \in A(g)$, as follows. Let $\varphi \in A(f)$. If $\varphi \in A(g)$ also, then put $\varphi' = \varphi$. Otherwise, by (1), it must be true that $\varphi(b) = d$. Then find $e \in [n]$ such that $\varphi(e) = c$, and construct φ' by

$$\varphi'(x) \;=\; \begin{cases} \varphi(x) & \text{if } x \neq b \text{ and } x \neq e, \\ c & \text{if } x = b, \\ d & \text{if } x = e. \end{cases}$$

It is easy to see that $\varphi' \in S_n$ and that $\varphi' \in A(g)$. Thus, $|A(g)| \geq |A(f)|$ will be proved provided that the pairing $\varphi \longrightarrow \varphi'$ is one-to-one. To that end, suppose that $\varphi_1 \longrightarrow \varphi_1'$ and $\varphi_2 \longrightarrow \varphi_2'$, and that

$$\varphi_1' \;=\; \varphi_2'\,. \tag{2}$$

We want to show that $\varphi_1 = \varphi_2$. We have three cases.

Case 1. $\varphi_1, \varphi_2 \in A(g)$.

Then $\varphi_1 = \varphi_1'$ and $\varphi_2 = \varphi_2'$, and it follows from (2) that $\varphi_1 = \varphi_2$.

Case 2. $\varphi_1, \varphi_2 \notin A(g)$.

Then $\varphi_1(b) = d = \varphi_2(b)$ and

$$\varphi_1'(x) \;=\; \begin{cases} \varphi_1(x) & \text{if } x \neq b \text{ and } x \neq e_1, \\ c & \text{if } x = b, \\ d & \text{if } x = e_1, \end{cases}$$

and

$$\varphi_2'(x) \;=\; \begin{cases} \varphi_2(x) & \text{if } x \neq b \text{ and } x \neq e_2, \\ c & \text{if } x = b, \\ d & \text{if } x = e_2, \end{cases} \tag{3}$$

where $\varphi_1(e_1) = c$ and $\varphi_2(e_2) = c$. Since φ_1 and φ_2 are permutations, (2) implies that $e_1 = e_2$. Also, by (2) and the above equations, we have $\varphi_1(x) = \varphi_2(x)$ for all $x \neq e_1, e_2$. It follows that $\varphi_1 = \varphi_2$.

Case 3. Without loss of generality, $\varphi_1 \in A(g)$ and $\varphi_2 \notin A(g)$.

Then $\varphi_1 = \varphi_1'$ and (3) holds as well. By (2), we have $\varphi_1 = \varphi_1' = \varphi_2'$, and in particular, by (3), we have $\varphi_1(b) = c$. This implies that $\varphi_1 \notin A(f)$, and from this contradiction we conclude that Case 3 is not realizable.

By repeated applying the above construction, any $f \notin S_n$ with which we start will be transformed into a permutation, and thus we have shown that for any $f \notin S_n$ there is a $g \in S_n$ such that $|A(f)| \leq |A(g)| = d(n)$, the number of derangements of $[n]$. The proof will be completed by showing that if $|\mathrm{Im}(f)| = n - 1$, then $|A(g)| > |A(f)|$ for the permutation g obtained from f.

Suppose that $\mathrm{Im}(f) = [n] - \{d\}$ for some $d \in [n]$. This means that $f(a) = f(b) = c$ for some $a \neq b$ and c, and f restricted to $[n] - \{b\}$ is a bijection onto $[n] - \{d\}$. Then, constructing $g : [n] \to [n]$ as in (1), we get $g(b) = d$ and $g(x) = f(x)$ for all $x \neq b$, and that $g \in S_n$ in this specific case. We need to find some function $\varphi'' \in A(g)$ which is not matched with any $\varphi \in A(f)$.

Let

$$\begin{aligned} [n] - \{a, b\} &= \{v_1, v_2, \ldots, v_{n-2}\} \\ \text{and} \quad [n] - \{c, d\} &= \{w_1, w_2, \ldots, w_{n-2}\} \end{aligned}$$

so that $g(v_i) = f(v_i) = w_i$ for all i. Define $\varphi''(v_i) = w_{i+1}$ for $1 \leq i \leq n-3$ and $\varphi''(v_{n-2}) = w_1$, and also $\varphi''(a) = d$ and $\varphi''(b) = c$. Then φ'' is a permutation and $\varphi'' \in A(g)$. We see that $\varphi'(e) = d$ for all φ' such that $\varphi \to \varphi'$ for some $\varphi \in A(f)$. Since $a \neq e$ and $\varphi''(a) = d$, it must be the case that $\varphi \not\to \varphi''$ for all $\varphi \in A(f)$. That is, $|A(g)| > |A(f)|$. ∎

Now we can handle Claim 2 and complete our proof of the 4-Card Theorem.

Proof of 2: We begin by assuming that 12 permutation sums are strictly less than $1/3$. For each card C_i ($i = 1, 2, 3, 4$), choose an entry e_i occurring a least number of times in these 12 sums. We call e_i the *minimal entry* for card C_i. Thus, $e_i \in \{a_i, b_i, c_i, d_i\}$ for each i, and e_i occurs in the 12 permutation sums at most the same number of times as each of the other three entries on card C_i. We wish to compare the sum $e_1 + e_2 + e_3 + e_4$ with each of the 12 permutation sums. Note that a permutation sum will have no elements in common with $e_1 + e_2 + e_3 + e_4$ precisely if the permutation (considered as a map from [4] to itself) avoids the function $\alpha : [4] \to [4]$ defined by

$$\alpha(i) = \begin{cases} 1 & \text{if } e_i = a_i, \\ 2 & \text{if } e_i = b_i, \\ 3 & \text{if } e_i = c_i, \\ 4 & \text{if } e_i = d_i. \end{cases}$$

If α happens to be a permutation of [4], then the number of permutations which avoid α is simply $d(4) = 9$. Otherwise, by the lemma, the number of permutations avoiding α is at most 8.

Case 1. $\alpha \notin S_4$.

The maximum number of permutation sums which avoid α in this case is 8; it follows that in any choice of 12 permutation sums, at least four must use at least one minimal entry e_i. Since e_i is the entry on card C_i which appears the least number of times, it follows that every entry of that respective card is accounted for at least as many times as the minimal entry (within the 12 chosen permutation sums). That is, for each minimal entry, we are guaranteed one sum $a_i + b_i + c_i + d_i$ contained in the sum of the 12 chosen permutations, and this sum corresponds to the card to which the minimal entry belongs. Since there are at least 4 minimal entries, it follows that there are four sums $a_i + b_i + c_i + d_i$ in the sum of all 12 permutations. Therefore, the sum of all 12 permutations must be at least 4, which means that the 12 permutations cannot all have sum strictly less than 1/3.

Case 2. $\alpha \in S_4$.

Suppose, without loss of generality, that $\alpha(i) = i$ for $i = 1, 2, 3, 4$. There are 9 permutation sums which are composed entirely of non-minimal entries, namely those corresponding to the derangements of $\{1, 2, 3, 4\}$. Thus, in any set of 12 permutation sums, at least 3 of them must use minimal entries. We note that if four permutation sums use minimal entries, then the above contradiction holds. Thus, we assume we have exactly 3 permutation sums which use minimal entries, and that all 9 derangements are included among the 12 permutation sums. Since derangements cannot include a_1, b_2, c_3, or d_4, by symmetry the sum of the 9 derangements is easily seen to be

$$3(b_1 + c_1 + d_1 + a_2 + c_2 + d_2 + a_3 + b_3 + d_3 + a_4 + b_4 + c_4)\,.$$

In the remaining three permutation sums, we may assume that a_1 appears. But since it appears in at most two sums not containing b_2, c_3, or d_4 (namely $a_1+c_2+d_3+b_4$ and $a_1+d_2+b_3+c_4$), we may assume that b_2 appears as well. Now since a_1 appears in some permutation sum less than 1/3, we know that $a_1 < 1/3$ which means $b_1 + c_1 + d_1 > 2/3$. Similarly, $a_2 + c_2 + d_2 > 2/3$. But now the sum of the nine derangements must be at least

$$3(b_1 + c_1 + d_1 + a_2 + c_2 + d_2) \;>\; 3\left(\frac{2}{3} + \frac{2}{3}\right) \;=\; 4\,,$$

which contradicts the assumption that they each have sum less than 1/3. ∎

Note that the case where 11 permutation sums are strictly less than 1/3 is realizable with the following card-set, which has 11 permutation sums equal to 1/4.

$$\begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Also, Example 2 for $n = 4$ has 12 sums greater than 4/3. In fact it has 12 sums equal to 5/3 and 12 sums equal to 1/3. The second variant of Example 2 for $n = 4$ has sums equal to 0 or 4/3. These examples together show that the interval $[1/3, 4/3]$ in our theorem is best possible.

However, many questions remain.

Problem 1. Is the interval $[1/2, 3/2]$ a solution to the n-card problem for all odd $n \geq 3$?

Example 1 shows that, if true, this result would be best possible. For even n, Example 2 and its two variants show that the intervals

$$\left[\frac{1}{2} - \frac{1}{2n-2}, \frac{3}{2} - \frac{1}{2n-2}\right] \qquad \text{and} \qquad \left[\frac{1}{2} + \frac{1}{2n-2}, \frac{3}{2} + \frac{1}{2n-2}\right] \qquad (4)$$

would both be best possible if they were solutions of the n-card problem.

Problem 2. Are the two intervals in (4) solutions of the n-card problem for all even $n \geq 4$? In particular, is $[2/3, 5/3]$ a solution of the 4-card problem?

Problem 2 suggests that there may be more than one solution to the n-card problem for some n. In fact, it is easy to see that the interval $[0, 1]$ is a solution for any n. The reason is simply that the average value of all permutation sums is 1; hence, there is always some permutation whose sum is at most 1. Once again, one can show that $[0, 1]$ is best possible.

For the case of $n = 3$, it is not hard to show that the only minimal solutions are the intervals $[0, 1]$, $[1/2, 3/2]$, and $[1, 2]$. The proof that $[1, 2]$ works is similar to the proof for $[1/2, 3/2]$ and is left to the reader, with the hint: show that there exists at most one permutation with sum greater than 2, and at most 4 permutations with sum less than 1.

Problem 3. Is $[1, 2]$ also a solution for $n = 4$? What is the situation for larger n?

Problem 4. Finally, it seems that, for $n > 2$, minimal intervals satisfying the n-card problem always have length exactly 1. Is this true? If so, why? Or is it just wishful thinking?

References

[1] http://www.math.ucalgary.ca/education/JrMath/exam2000/jrmath2000.html

[2] http://mathworld.wolfram.com/Derangement.html

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KLAMKIN PROBLEMS

The following problems have all been identified by the proposers to be dedicated to the lasting memory of Murray S. Klamkin. Solutions should arrive no later than **1 March 2006**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

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KLAMKIN–01. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x and y be positive real numbers from the interval $\left[0,\frac{1}{2}\right]$. Prove that

$$2 \;\le\; \left(\frac{1-x}{1-y}\right)^{\frac{1}{4}} + \left(\frac{1-y}{1-x}\right)^{\frac{1}{4}} \;\le\; \frac{2}{\left(\sqrt{x}\sqrt{y}+\sqrt{1-x}\sqrt{1-y}\right)^{\frac{1}{2}}}\,.$$

(b)★ Is there a generalization of the above inequality to three or more numbers?

KLAMKIN–02. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \;\ge\; \frac{28}{27}\,.$$

(b)★ Prove or disprove the following generalization involving n positive real numbers x_1, x_2, $\ldots$, x_n which sum to 1:

$$\left(\prod_{i=1}^{n} x_i\right)\left(1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right) \;\ge\; \frac{n^3+1}{n^n}\,.$$

KLAMKIN–03. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

If a, b, c are positive real numbers, prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2}+\frac{1}{2}\left(\frac{a^3+b^3+c^3}{abc}-\frac{a^2+b^2+c^2}{ab+bc+ca}\right) \;\ge\; 4\,.$$

KLAMKIN–04. *Proposed by Mihály Bencze, Brasov, Romania.*

Let f_n denote the Fibonacci sequence (that is, $f_0=f_1=1$ and $f_n=f_{n-1}+f_{n-2}$ for $n\ge 2$). For all integers $k\ge 1$, determine the remainder when f_{kn-r} is divided by f_n^2 for the following cases:

(a) $r=1$; (b) $r=2$; (c)★ $r\in\{3,4,\ldots,k-1\}$.

KLAMKIN–05. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let k and n be positive integers with $k < n$, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \le a_2 \le \cdots \le a_n$. Prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \;\ge\; n(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_n a_{n+k})$$

(where the subscipts are taken modulo n) in the following cases:

(a) $n = 2k$; (b) $n = 4k$; (c)★ $2 < \frac{n}{k} < 4$.

KLAMKIN–06. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let Γ be the circumcircle of $\triangle ABC$.

(a) Suppose that the median and the interior angle bisector from A intersect BC at M and N, respectively. Extend AM and AN to intersect Γ at M' and N', respectively. Prove that $MM' \ge NN'$.

(b)★ Suppose that P is a point in the interior of side BC and AP intersects Γ at P'. Find the location of P where PP' is maximal. Is this maximal P constructible by straightedge and compass?

KLAMKIN–07. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c, d be real numbers such that $a > b \ge c > d > 0$. If $ad - bc > 0$, prove that

$$\prod_{k=1}^{n} \left(\frac{a^{\binom{n}{k}} - b^{\binom{n}{k}}}{c^{\binom{n}{k}} - d^{\binom{n}{k}}} \right)^{k} \;\ge\; \left(\frac{a^{\frac{2^n}{n+1}} - b^{\frac{2^n}{n+1}}}{c^{\frac{2^n}{n+1}} - d^{\frac{2^n}{n+1}}} \right)^{\binom{n+1}{2}} .$$

KLAMKIN–08. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let m and n be positive integers, and let $x_1, x_2, \ldots, x_m$ be positive real numbers. If λ is a real number, $\lambda \ge 1$, prove that

$$\left(\prod_{i=1}^{m} x_i \right)^{\frac{1}{m}} \;\le\; \left(\frac{\lambda \left(\sum\limits_{i=1}^{m} x_i \right)^{n} + (1-\lambda) \sum\limits_{i=1}^{m} x_i^n}{\lambda m^n + (1-\lambda) m} \right)^{\frac{1}{n}} \;\le\; \frac{1}{m} \sum_{i=1}^{m} x_i \,.$$

KLAMKIN–09. *Proposed by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.*

For $0 < x < \pi/2$, prove or disprove that

$$\frac{\ln(1 - \sin x)}{\ln(\cos x)} \;<\; \frac{2 + x}{x} \,.$$

KLAMKIN–10. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $P(x) = \sum\limits_{i=0}^{n} a_i x^i$ be a polynomial with real coefficients and simple roots. Prove that

(a) $\displaystyle\sum_{i=1}^{n} \frac{1}{x_i P'(x_i)} \;=\; -\frac{1}{a_0}$, and (b) $\displaystyle\sum_{i=1}^{n} \frac{x_i^{n-1}}{P'(x_i)} \;=\; \frac{1}{a_n}$.

KLAMKIN–11. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let P be an interior point of a triangle ABC, and let r_1, r_2, and r_3 be the inradii of the triangles APB, BPC, and CPA, respectively. Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \;\geq\; \frac{6 + 4\sqrt{3}}{R}\,,$$

where R is the circumradius of triangle ABC. When does equality hold?

KLAMKIN–12. *Proposed by Michel Bataille, Rouen, France.*

Let a, b, c be the sides of a spherical triangle. Show that

$$3\cos a\cos b\cos c \;\leq\; \cos^2 a + \cos^2 b + \cos^2 c \;\leq\; 1 + 2\cos a\cos b\cos c\,.$$

KLAMKIN–13. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $\mathcal{C}$ be a smooth closed convex curve in the plane. Theorems in analysis assure us that there is at least one circumscribing triangle $A_0B_0C_0$ to $\mathcal{C}$ having minimum perimeter. Prove that the excircles of $A_0B_0C_0$ are tangent to $\mathcal{C}$.

KLAMKIN–14. *Proposed by Andy Liu, University of Alberta, Edmonton, AB.*

A vertical wall OY meets the horizontal floor OX at the corner O. Initially, a ladder AB is placed so that its bottom B is at O while its apex A is on the wall OY. A cat jumps onto the ladder and clings to the point C where $BC = \lambda AC$ for some real number $0 < \lambda < 1$. This jiggles the ladder so that it begins to slide, with A moving down towards O along YO and B moving away from O along OX, until it comes to rest with A at O and B on OX. What is the curve traced out by the cat?

KLAMKIN–15. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

A square $ABCD$ sits in the plane with corners A, B, C, D initially located at positions $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, respectively. The square is rotated counterclockwise through an angle θ ($0^\circ \leq \theta < 360^\circ$) four times, with the centre of rotation at the points A, B, C, D in successive rotations. Suppose point A ends up on the x-axis or y-axis. Find all possible values of θ.

KLAMKIN–01. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

(a) Soit x et y deux nombres réels positifs dans l'intervalle $\left[0, \frac{1}{2}\right]$. Montrer que

$$2 \leq \left(\frac{1-x}{1-y}\right)^{\frac{1}{4}} + \left(\frac{1-y}{1-x}\right)^{\frac{1}{4}} \leq \frac{2}{\left(\sqrt{x}\sqrt{y}+\sqrt{1-x}\sqrt{1-y}\right)^{\frac{1}{2}}}\,.$$

(b)★ Y a-t-il une généralisation de l'inégalité ci-dessus à plus de deux nombres ?

KLAMKIN–02. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Autriche.*

(a) Soit x, y et z trois nombres réels positifs tels que $x+y+z=1$. Montrer que

$$xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \geq \frac{28}{27}\,.$$

(b)★ Prouver ou réfuter la généralisation suivante portant sur n nombres réels positifs x_1, x_2, $\ldots$, x_n dont la somme vaut 1 :

$$\left(\prod_{i=1}^{n} x_i\right)\left(1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right) \geq \frac{n^3+1}{n^n}\,.$$

KLAMKIN–03. *Proposé par Pham Van Thuan, Hanoi, Viêt Nam.*

Si a, b et c sont des nombres réels positifs, montrer que

$$\frac{(a+b+c)^2}{a^2+b^2+c^2}+\frac{1}{2}\left(\frac{a^3+b^3+c^3}{abc}-\frac{a^2+b^2+c^2}{ab+bc+ca}\right) \geq 4\,.$$

KLAMKIN–04. *Proposé par Mihály Bencze, Brasov, Roumanie.*

On désigne par f_n la suite de Fibonacci (c-à-d. $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ pour $n \geq 2$). Pour tout entier $k \geq 1$, déterminer le reste de la division de f_{kn-r} par f_n^2 dans les cas suivants :

(a) $r = 1$; (b) $r = 2$; (c)★ $r \in \{3, 4, \ldots, k-1\}$.

KLAMKIN–05. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit k et n deux nombres entiers positifs avec $k < n$, et soit a_1, a_2, $\ldots$, a_n des nombres réels tels que $a_1 \leq a_2 \leq \cdots \leq a_n$. Montrer que

$$(a_1+a_2+\cdots+a_n)^2 \geq n(a_1a_{k+1}+a_2a_{k+2}+\cdots+a_na_{n+k})$$

(où les indices sont pris modulo n) dans les cas suivants :

(a) $n = 2k$; (b) $n = 4k$; (c)★ $2 < \frac{n}{k} < 4$.

KLAMKIN–06. *Proposé par Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Soit Γ le cercle circonscrit d'un triangle ABC.

(a) On suppose que la médiane et la bissectrice de l'angle intérieur au sommet A coupent respectivement BC en M et en N. Prolonger AM et AN pour qu'elles coupent respectivement Γ en M' and N'. Montrer que $MM' \geq NN'$.

(b)★ On suppose que P est un point intérieur du côté BC et que AP coupe Γ en P'. Trouver la position de P de sorte que PP' soit maximal. Cette position de P est-elle constructible avec la règle et le compas ?

KLAMKIN–07. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b, c et d trois nombres réels tels que $a > b \geq c > d > 0$. Si $ad - bc > 0$, montrer que

$$\prod_{k=1}^{n} \left(\frac{a^{\binom{n}{k}} - b^{\binom{n}{k}}}{c^{\binom{n}{k}} - d^{\binom{n}{k}}} \right)^{k} \geq \left(\frac{a^{\frac{2^n}{n+1}} - b^{\frac{2^n}{n+1}}}{c^{\frac{2^n}{n+1}} - d^{\frac{2^n}{n+1}}} \right)^{\binom{n+1}{2}} .$$

KLAMKIN–08. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit m et n deux entiers positifs, et soit $x_1, x_2, \dots, x_m$ des nombres réels positifs. Si λ est un nombre réel, $\lambda \geq 1$, montrer que

$$\left(\prod_{i=1}^{m} x_i \right)^{\frac{1}{m}} \leq \left(\frac{\lambda \left(\sum\limits_{i=1}^{m} x_i \right)^{n} + (1-\lambda) \sum\limits_{i=1}^{m} x_i^n}{\lambda m^n + (1-\lambda) m} \right)^{\frac{1}{n}} \leq \frac{1}{m} \sum_{i=1}^{m} x_i .$$

KLAMKIN–09. *Proposé par Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.*

Pour $0 < x < \pi/2$, démontrer ou réfuter l'inégalité

$$\frac{\ln(1 - \sin x)}{\ln(\cos x)} < \frac{2 + x}{x} .$$

KLAMKIN–10. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $P(x) = \sum\limits_{i=0}^{n} a_i x^i$ a polynôme à coefficients réels et racines simples. Montrer que

(a) $\displaystyle\sum_{i=1}^{n} \frac{1}{x_i P'(x_i)} = -\frac{1}{a_0}$, et (b) $\displaystyle\sum_{i=1}^{n} \frac{x_i^{n-1}}{P'(x_i)} = \frac{1}{a_n}$.

KLAMKIN–11. *Proposé par Mohammed Aassila, Strasbourg, France.*

Soit P un point intérieur d'un triangle ABC, et soit r_1, r_2, et r_3 les rayons respectifs des cercles incrits des triangles APB, BPC et CPA. Montrer que

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \;\geq\; \frac{6+4\sqrt{3}}{R}\,,$$

où R est le rayon du cercle circonscrit du triangle ABC. Quant y a-t-il égalité ?

KLAMKIN–12. *Proposé par Michel Bataille, Rouen, France.*

Soit a, b et c les côtés d'un triangle sphérique. Montrer que

$$3\cos a\cos b\cos c \;\leq\; \cos^2 a+\cos^2 b+\cos^2 c \;\leq\; 1+2\cos a\cos b\cos c\,.$$

KLAMKIN–13. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit $\mathcal{C}$ une courbe plane fermée, lisse et convexe. Des théorèmes d'analyse assurent l'existence d'au moins un triangle de périmètre minimal $A_0B_0C_0$ circonscrit à $\mathcal{C}$. Montrer que les cercles exinscrits de $A_0B_0C_0$ sont tangents à $\mathcal{C}$.

KLAMKIN–14. *Proposé par Andy Liu, Université d'Alberta, Edmonton, AB.*

Une paroi verticale OY repose sur un sol horizontal OX au point O. Une échelle AB est placée en équilibre contre la paroi avec sa base en O et son sommet A sur la paroi OY. Un chat saute sur l'échelle et se cramponne au point C où $BC = \lambda AC$ pour un certain nombre réel $0 < \lambda < 1$. Cette maneuvre fait glisser l'échelle par terre de sorte que le point A descend jusqu'en O le long de YO, tandis que B s'éloigne de la paroi le long de OX. Quelle est la courbe tracée par le chat ?

KLAMKIN–15. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Un carré $ABCD$ est donné dans le plan par ses sommets A, B, C et D de position initiale respectives $(0,0)$, $(1,0)$, $(1,1)$ et $(0,1)$. On tourne le carré dans le sens anti-horaire d'un angle θ ($0^\circ \leq \theta < 360^\circ$) quatre fois de suite, avec les points A, B, C et D comme centres successifs de rotation. On suppose que le point A finit sa course sur l'axe des x ou sur l'axe des y. Trouver toutes les valeurs possibles de θ.

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 March 2006**. *An asterisk* (⋆) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3051. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let x, y, $z \in [0,1)$ such that $x+y+z=1$. Prove that

(a) $\sqrt{\dfrac{x}{x+yz}}+\sqrt{\dfrac{y}{y+zx}}+\sqrt{\dfrac{z}{z+xy}} \;\le\; 3\sqrt{\dfrac{3}{2}}$;

(b) $\dfrac{\sqrt{xyz}}{(1-x)(1-y)(1-z)} \;\le\; \dfrac{3\sqrt{3}}{8}$.

3052. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let G be the centroid of $\triangle ABC$, and let A_1, B_1, C_1 be the mid-points of BC, CA, AB, respectively. If P is an arbitrary point in the plane of $\triangle ABC$, show that

$$PA+PB+PC+3PG \;\ge\; 2(PA_1+PB_1+PC_1)\,.$$

3053. *Proposed by Avet A. Grigoryan and Hayk N. Sedrakyan, students, A. Shahinyan Physics and Mathematics School, Yerevan, Armenia.*

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers whose sum is 1. Prove that

$$n-1 \;\le\; \sqrt{\frac{1-a_1}{1+a_1}}+\sqrt{\frac{1-a_2}{1+a_2}}+\cdots+\sqrt{\frac{1-a_n}{1+a_n}} \;\le\; n-2+\frac{2}{\sqrt{3}}\,.$$

3054. *Proposed by Michel Bataille, Rouen, France.*

For $n=0,1,2,\ldots$, let $U_n=\sum\limits_{k=0}^{n}\binom{2k}{k}$ and $V_n=\sum\limits_{k=0}^{n}(-1)^k\binom{2k}{k}$. Evaluate the following in closed form:

(a) $U_n^2+2\sum\limits_{k=1}^{n}\binom{2n+2k}{n+k}U_{n-k}$; (b) $V_n^2+2\sum\limits_{k=1}^{n}(-1)^{n+k}\binom{2n+2k}{n+k}V_{n-k}$.

3055. *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of an acute-angled triangle ABC be tangent to BC, CA, AB at D, E, F, respectively. Let D_0 be the reflection of D through the incentre of $\triangle ABC$, and let D_1 and D_2 be the reflections of D across the diameters of the incircle through E and F. Define E_0, E_1, E_2 and F_0, F_1, F_2 analogously. Show that

$$\begin{aligned}&[D_0D_1D_2]+[E_0E_1E_2]+[F_0F_1F_2]\\&\qquad = [DD_1D_2] = [EE_1E_2] = [FF_1F_2] \le \tfrac{1}{4}[ABC],\end{aligned}$$

where $[XYZ]$ denotes the area of $\triangle XYZ$.

3056. *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*

If $f(x)$ is a non-negative, continuous, convex function on the closed interval $[0,1]$ such that $f(0)=1$, show that

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \le \left[\int_0^1 f(x)\,dx\right]^2 .$$

3057. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be non-negative real numbers, and let $p \ge \dfrac{\ln 3}{\ln 2} - 1$. Prove that

$$\left(\frac{2a}{b+c}\right)^p + \left(\frac{2b}{c+a}\right)^p + \left(\frac{2c}{a+b}\right)^p \ge 3 .$$

3058. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let A, B, C be the angles of a triangle. Prove that

(a) $\dfrac{1}{2-\cos A} + \dfrac{1}{2-\cos B} + \dfrac{1}{2-\cos C} \ge 2$;

(b) $\dfrac{1}{5-\cos A} + \dfrac{1}{5-\cos B} + \dfrac{1}{5-\cos C} \le \dfrac{2}{3}$.

3059. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2 \le 1$. Prove that

$$ab+bc+cd+da+ac+bd \le 4abcd + \frac{5}{4} .$$

3060. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Let a and b be positive real numbers such that $a<2$. For each integer $n \ge 1$, let $x_n = \lfloor an+b \rfloor$. Prove that the sequence $\{x_n\}_{n\ge 1}$ has an infinite number of terms whose sum of digits is even. (Note: $\lfloor z \rfloor$ is the greatest integer not exceeding z.)

3061. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Find the smallest non-negative integer n for which there exists a non-constant function $f : \mathbb{Z} \to [0, \infty)$ such that for all integers x and y,

(a) $f(xy) = f(x)f(y)$, and

(b) $2f(x^2 + y^2) - f(x) - f(y) \in \{0, 1, \ldots, n\}$.

For this value of n, find all the functions f which satisfy (a) and (b).

3062. *Proposed by Gabriel Dospinescu, Onesti, Romania.*

Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$(ab + bc + ca)\left(\frac{a}{b^2 + b} + \frac{b}{c^2 + c} + \frac{c}{a^2 + a}\right) \;\geq\; \frac{3}{4}\,.$$

3063. *Proposed by Mohammed Aassila, Strasbourg, France.*

Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f\big(f(x)\big) + f(x) \;=\; 2x + a\,,$$

where a is a real constant.

. . . .

3051. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit x, y, $z \in [0, 1)$ tels que $x + y + z = 1$. Montrer que

(a) $\displaystyle \sqrt{\frac{x}{x + yz}} + \sqrt{\frac{y}{y + zx}} + \sqrt{\frac{z}{z + xy}} \;\leq\; 3\sqrt{\frac{3}{2}}\,;$

(b) $\displaystyle \frac{\sqrt{xyz}}{(1 - x)(1 - y)(1 - z)} \;\leq\; \frac{3\sqrt{3}}{8}.$

3052. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, USA.*

Soit G le centre de gravité du triangle ABC, et soit respectivement A_1, B_1 et C_1 les point milieu de BC, CA et AB. Si P est un point arbitraire dans le plan du triangle ABC, montrer que

$$PA + PB + PC + 3PG \;\geq\; 2(PA_1 + PB_1 + PC_1)\,.$$

3053. *Proposé par Avet A. Grigoryan et Hayk N. Sedrakyan, étudiants, A. Shahinyan Physics and Mathematics School, Yerevan, Arménie.*

Soit a_1, a_2, $\ldots$, a_n des nombres réels non négatifs dont la somme vaut 1. Montrer que

$$n - 1 \;\leq\; \sqrt{\frac{1 - a_1}{1 + a_1}} + \sqrt{\frac{1 - a_2}{1 + a_2}} + \cdots + \sqrt{\frac{1 - a_n}{1 + a_n}} \;\leq\; n - 2 + \frac{2}{\sqrt{3}}\,.$$

3054. *Proposé par Michel Bataille, Rouen, France.*

Pour $n = 0, 1, 2, \ldots$, soit $U_n = \sum\limits_{k=0}^{n} \binom{2k}{k}$ et $V_n = \sum\limits_{k=0}^{n} (-1)^k \binom{2k}{k}$. Trouver une formule pour

(a) $U_n^2 + 2\sum\limits_{k=1}^{n} \binom{2n+2k}{n+k} U_{n-k}$; (b) $V_n^2 + 2\sum\limits_{k=1}^{n} (-1)^{n+k} \binom{2n+2k}{n+k} V_{n-k}$.

3055. *Proposé par Michel Bataille, Rouen, France.*

On suppose que le cercle inscrit d'un triangle acutangle ABC est respectivement tangent aux côtés BC, CA, AB aux points D, E, F. Soit D_0 le symétrique de D par rapport au centre de ce cercle et soit D_1 et D_2 les symétriques de D par rapport aux diamètres passant par E and F. On définit E_0, E_1 et E_2 de même que F_0, F_1 et F_2 de manière analogue. Montrer que

$$\begin{aligned}&[D_0D_1D_2] + [E_0E_1E_2] + [F_0F_1F_2] \\ &\qquad = [DD_1D_2] = [EE_1E_2] = [FF_1F_2] \leq \tfrac{1}{4}[ABC]\,,\end{aligned}$$

où $[XYZ]$ désigne l'aire du triangle XYZ.

3056. *Proposé par Paul Bracken, Université du Texas, Edinburg, TX, USA.*

Si $f(x)$ est une fonction continue, convexe et non négative sur l'intervalle fermé $[0,1]$ telle que $f(0) = 1$, montrer que

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \leq \left[\int_0^1 f(x)\,dx\right]^2 .$$

3057. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a, b et c des nombres réel non négatifs, et soit $p \geq \dfrac{\ln 3}{\ln 2} - 1$. Montrer que

$$\left(\frac{2a}{b+c}\right)^p + \left(\frac{2b}{c+a}\right)^p + \left(\frac{2c}{a+b}\right)^p \geq 3\,.$$

3058. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit A, B et C les angles d'un triangle. Montrer que

(a) $\dfrac{1}{2-\cos A} + \dfrac{1}{2-\cos B} + \dfrac{1}{2-\cos C} \geq 2$;

(b) $\dfrac{1}{5-\cos A} + \dfrac{1}{5-\cos B} + \dfrac{1}{5-\cos C} \leq \dfrac{2}{3}$.

3059. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Soit a, b, c et d des nombres réels tels que $a^2 + b^2 + c^2 + d^2 \leq 1$. Montrer que

$$ab + bc + cd + da + ac + bd \leq 4abcd + \frac{5}{4}.$$

3060. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Soit a et b deux nombres réels positifs tels que $a < 2$. Pour tout entier $n \geq 1$, soit $x_n = \lfloor an + b \rfloor$. Montrer que la suite $\{x_n\}_{n \geq 1}$ a une infinité de termes dont la somme des chiffres est paire. (Note : $\lfloor z \rfloor$ est le plus grand entier n'excédant pas z.)

3061. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Trouver le plus petit entier non négatif n pour lequel il existe une fonction non constante $f : \mathbb{Z} \to [0, \infty)$ telle que, pour tous les entiers x et y,

(a) $f(xy) = f(x)f(y)$, et

(b) $2f(x^2 + y^2) - f(x) - f(y) \in \{0, 1, \ldots, n\}$.

Pour cette valeur de n, trouver toutes les fonctions f satisfaisant à (a) et (b).

3062. *Proposé par Gabriel Dospinescu, Onesti, Roumanie.*

Soit a, b et c trois nombres réels positifs tels que $a+b+c = 1$. Montrer que

$$(ab + bc + ca)\left(\frac{a}{b^2 + b} + \frac{b}{c^2 + c} + \frac{c}{a^2 + a}\right) \geq \frac{3}{4}.$$

3063. *Proposé par Mohammed Aassila, Strasbourg, France.*

Déterminer toutes les fonctions continues $f : \mathbb{R} \to \mathbb{R}$ telles que, pour tout $x \in \mathbb{R}$,

$$f(f(x)) + f(x) = 2x + a,$$

où a est une constante réelle.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2951. [2004 : 296, 298] *Proposed by Nevena Sybeva, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

Let M and N be interior points of $\triangle ABC$. Define the point T_A to be the point on BC such that light travelling from M to T_A, and undergoing perfect reflection at T_A, will pass through N. Define T_B and T_C similarly.

Prove that if the three possible light paths MT_AN, MT_BN, MT_CN have equal length, then the lines AT_A, BT_B, and CT_C are concurrent.

Combination of similar solutions by Michel Bataille, Rouen, France; John G. Heuver, Grande Prairie, AB; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $2a$ be the common length of the three light paths. Since $T_AM + T_AN = 2a$, the point T_A lies on the ellipse $\mathcal{E}$ with foci M and N and major axis $2a$. Moreover, from the hypothesis of perfect reflection at T_A, we have $\angle MT_AB = \angle NT_AC$, which implies that BC is the tangent to $\mathcal{E}$ at T_A. Similar results hold for T_B and T_C. Thus, $\mathcal{E}$ is inscribed in $\triangle ABC$. The desired concurrency of AT_A, BT_B, CT_C follows immediately from Brianchon's Theorem applied to the (degenerate) hexagon with sides AT_C, T_CB, BT_A, T_AC, CT_B, T_BA. For more about Brianchon's Theorem applied to triangles see N. Sato, "Ellipses in Polygons" [2000 : 361–371].

Also solved by Toshio Seimiya, Kawasaki, Japan; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

The argument used in the three other solutions came down to an application of Ceva's Theorem. Zhao added the observation, which he easily verified, that M is the isogonal conjugate of N with respect to $\triangle ABC$.

2952. [2004 : 296, 299] *Proposed by C.R. Pranesachar and Prithu Bharti, Indian Institute of Science, Bangalore, India.*

Find a closed form for the real series

$$\sum_{\substack{r\geq 0\\ r\geq -n}} \binom{n+2r}{r} x^r\,, \qquad x\in\left(-\tfrac{1}{4},\tfrac{1}{4}\right)\,,$$

where n is an integer (positive, negative, or zero).

I. *Solution by Michel Bataille, Rouen, France, modified by the editor.*

First, suppose that $n \geq 0$. The given sum is then $\sum\limits_{r\geq 0} \binom{n+2r}{r} x^r$. We will use the following Taylor series expansion:

$$\frac{1}{(1-z)^{m+1}} \;=\; \sum_{k=0}^{\infty} \binom{m+k}{k} z^k\,.$$

Here m may be any non-negative integer, and the expansion is valid for all complex numbers z such that $|z| < 1$.

Let an integer r be fixed, and let $f(z) = \dfrac{1}{z^{r+1}(1-z)^{n+r+1}}$. Then, for $0 < |z| < 1$, we have

$$f(z) \;=\; \sum_{k=0}^{\infty} \binom{n+r+k}{k} z^{k-r-1}\,.$$

In this Laurent series for $f(z)$, the coefficient of z^{-1} is $\dbinom{n+2r}{r}$. This is the residue of $f(z)$ at 0. By the Residue Theorem,

$$\binom{n+2r}{r} \;=\; \frac{1}{2\pi i} \oint_C f(z)\,dz\,,$$

where C denotes the circle $|z| = \frac{1}{2}$ traversed once counterclockwise.

Let $x \in \left(-\frac{1}{4}, \frac{1}{4}\right)$ be fixed. For all z on C, we have $|1-z| \geq \frac{1}{2}$, and hence,

$$\left|\frac{x}{z(1-z)}\right| \;\leq\; \frac{|x|}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} \;=\; 4|x| \;<\; 1\,.$$

Then, for all z on C,

$$\sum_{r=0}^{\infty} \left(\frac{x}{z(1-z)}\right)^r \;=\; \frac{1}{1 - \dfrac{x}{z(1-z)}} \;=\; \frac{-z(1-z)}{z^2 - z + x}\,,$$

and the convergence is uniform with respect to z on C. The uniform convergence allows us to interchange the integration with the summation in the following calculation:

$$\begin{aligned}
\sum_{r=0}^{\infty} \binom{n+2r}{r} x^r &= \sum_{r=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C f(z)\,dz\right) x^r \\
&= \frac{1}{2\pi i} \oint_C \frac{1}{z(1-z)^{n+1}} \sum_{r=0}^{\infty} \left(\frac{x}{z(1-z)}\right)^r dz \\
&= \frac{1}{2\pi i} \oint_C \frac{-1}{(1-z)^n (z^2 - z + x)}\,dz\,.
\end{aligned}$$

The polynomial $p(z) = z^2 - z + x$ has roots $\zeta_+ = \dfrac{1+\sqrt{1-4x}}{2}$ and $\zeta_- = \dfrac{1-\sqrt{1-4x}}{2}$. Note that $\zeta_+ + \zeta_- = 1$ and $\zeta_+ - \zeta_- = \sqrt{1-4x}$. The integrand above has simple poles at ζ_+ and ζ_-, but only ζ_- is inside C. By the Residue Theorem,

$$\begin{aligned}
\sum_{r=0}^{\infty} \binom{n+2r}{r} x^r &= \operatorname{Res}\left(\frac{-1}{(1-z)^n p(z)}, \zeta_-\right) = \lim_{z\to\zeta_-} \frac{-(z-\zeta_-)}{(1-z)^n p(z)} \\
&= \left.\frac{-1}{(1-z)^n (z-\zeta_+)}\right|_{z=\zeta_-} = \frac{1}{\zeta_+^n(\zeta_+ - \zeta_-)} \\
&= \frac{1}{\sqrt{1-4x}} \left(\frac{2}{1+\sqrt{1-4x}}\right)^n . \qquad (1)
\end{aligned}$$

This is the required closed form for the case $n \geq 0$.

Now suppose that $n < 0$. Letting $m = -n$ and $s = r - m$, we have, from (1),

$$\begin{aligned}
\sum_{r\geq -n} \binom{n+2r}{r} x^r &= x^m \sum_{r=m}^{\infty} \binom{-m+2r}{r} x^{r-m} = x^m \sum_{s=0}^{\infty} \binom{m+2s}{m+s} x^s \\
&= x^m \sum_{s=0}^{\infty} \binom{m+2s}{s} x^s = \frac{(2x)^m}{\left(1+\sqrt{1-4x}\right)^m \sqrt{1-4x}} \\
&= \frac{(2x)^{-n}}{\left(1+\sqrt{1-4x}\right)^{-n} \sqrt{1-4x}} .
\end{aligned}$$

II. *Solution by the proposers, modified slightly by the editor.*

Observe that

$$\lim_{r\to\infty} \frac{\binom{n+2r+2}{r+1}}{\binom{n+2r}{r}} = \lim_{r\to\infty} \frac{(n+2r+2)(n+2r+1)}{(r+1)(n+r+1)} = 4\,;$$

whence the radius of convergence of the given series is $1/4$, by the Ratio Test.

Let y_n denote the sum of the series. First, we consider the case $n \geq 0$. For $n = 0$, we have

$$y_0 = \sum_{r\geq 0} \binom{2r}{r} x^r = \sum_{r=0}^{\infty} \binom{-\frac{1}{2}}{r} (-4x)^r = \frac{1}{\sqrt{1-4x}} .$$

For $n = 1$,

$$\begin{aligned}
y_1 &= \sum_{r\geq 0} \binom{2r+1}{r} x^r = \sum_{r\geq 0} \left(\frac{2r+1}{r+1}\right) \binom{2r}{r} x^r \\
&= \sum_{r\geq 0} \left(2 - \frac{1}{r+1}\right) \binom{2r}{r} x^r = 2\sum_{r\geq 0} \binom{2r}{r} x^r - \sum_{r\geq 0} \frac{1}{r+1} \binom{2r}{r} x^r .
\end{aligned}$$

The last summation above corresponds to the generating function of the sequence of Catalan numbers and is known to equal $\left(1-\sqrt{1-4x}\right)/(2x)$. Hence,

$$\begin{aligned} y_1 &= \frac{2}{\sqrt{1-4x}} - \frac{1-\sqrt{1-4x}}{2x} = \frac{4x-\sqrt{1-4x}+(1-4x)}{2x\sqrt{1-4x}} \\ &= \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right). \end{aligned}$$

Furthermore, for $n \geq 1$, we have

$$\begin{aligned} y_n - y_{n-1} &= \sum_{r\geq 0}\left[\binom{n+2r}{r} - \binom{n-1+2r}{r}\right]x^r \\ &= \sum_{r\geq 1}\binom{n-1+2r}{r-1}x^r = \sum_{r\geq 0}\binom{n+1+2r}{r}x^{r+1} = xy_{n+1}. \end{aligned}$$

Changing n to $n-1$, we obtain, for $n \geq 2$, the recurrence relation

$$xy_n - y_{n-1} + y_{n-2} = 0.$$

The characteristic equation is $x\lambda^2 - \lambda + 1 = 0$ and the characteristic roots are $\lambda = \left(1 \pm \sqrt{1-4x}\right)/(2x)$. Hence,

$$y_n = A\left(\frac{1+\sqrt{1-4x}}{2x}\right)^n + B\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n,$$

where A and B are functions of x yet to be determined.

Setting $n = 0$, we obtain

$$A + B = y_0 = \frac{1}{\sqrt{1-4x}},$$

and setting $n = 1$, we get

$$A\left(\frac{1+\sqrt{1-4x}}{2x}\right) + B\left(\frac{1-\sqrt{1-4x}}{2x}\right) = y_1 = \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right).$$

Solving (or simply by inspection), we find that $A = 0$ and $B = \dfrac{1}{\sqrt{1-4x}}$. Hence, for all $n \geq 0$,

$$y_n = \frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n.$$

To make the right side meaningful at $x = 0$, we rewrite y_n as

$$y_n = \frac{1}{\sqrt{1-4x}}\left(\frac{2}{1+\sqrt{1-4x}}\right)^n.$$

[*Ed.:* The case where $n < 0$ was handled as in Solution I above.]

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2953. *Proposed by Titu Zvonaru, Bucharest, Romania.*

Let m, n be positive integers with $n > 1$, and let a, b, c be positive real numbers satisfying $a^{m+1} + b^{m+1} + c^{m+1} = 1$. Prove that

$$\frac{a}{1 - ma^n} + \frac{b}{1 - mb^n} + \frac{c}{1 - mc^n} \geq \frac{(m+n)^{1+\frac{m}{n}}}{n}.$$

Comments by the editor: The given inequality, which is an obvious attempt to generalize ***CRUX with MAYHEM*** Problem 2935 [2004 : 174; 2005 : 188] by the same proposer, is clearly false as stated. This was pointed out by several readers. For one thing, some of the denominators of the terms on the left side could easily be 0. OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA noted that if $m = n = 2$, $a = b = 1/\sqrt{2}$, and $c = (1 - 1/\sqrt{2})^{1/3}$, then $1 - ma^n = 0$. WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria and D. KIPP JOHNSON, Beaverton, OR, USA both gave examples showing that the left side could be negative; while LI ZHOU, Polk Community College, Winter Haven, FL, USA gave a counterexample in which the left side is positive.

There were four incorrect solutions, all of which proved the "validity" of the given inequality under the "implicit" assumption that the quantities $1 - ma^n$, $1 - mb^n$, and $1 - mc^n$ are all positive. This was apparently the intention of the proposer. The published proof for Problem 2935 [2005 : 188] can be modified easily with little change to yield a proof for the revised version of the current inequality.

2954. [2004 : 297, 299] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let Γ be the circumcircle of $\triangle ABC$. The tangents to Γ at B and C intersect at M. The line through M parallel to AB intersects Γ at D and E, and intersects AC at F.

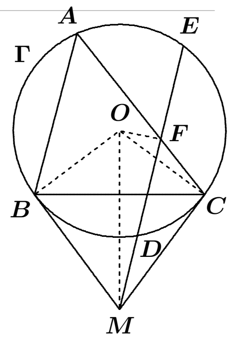
Prove that F is the mid-point of DE.

I. *Solution by Michel Bataille, Rouen, France.*

We use the notation $[\cdot, \cdot, \cdot, \cdot]$ for the cross ratio (of four points on a line or a circle, or four concurrent lines).

Let K be the point of intersection of ME and BC. Then K is on the polar of M with respect to Γ. Therefore, we have $[K, M, D, E] = -1$, which implies that $[CB, CM, CD, CE] = -1$. Thus, $[B, C, D, E] = -1$, and hence $[AB, AC, AD, AE] = -1$. Cutting by a transversal DE, we get $[\infty, F, D, E] = -1$. This means that F is the mid-point of DE.

II. *Composite of similar solutions by Toshio Seimiya, Kawasaki, Japan; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.*



Let O be the centre of Γ. Since MB and MC are tangent to Γ at B and at C, respectively, we have $\angle OBM = \angle OCM = 90^\circ$. Therefore, the points O, B, M and C lie on a circle with diameter OM.

We have $\angle MBC = \angle BAC$, because MB is tangent to Γ at B. Also, $\angle BAC = \angle MFC$, because $MF \parallel AB$. Thus, $\angle MBC = \angle MFC$, which implies that B, M, C, and F are cyclic. But we already know that B, M, and C lie on the circle with diameter OM. Therefore, F lies on this same circle. Hence, $\angle OFM = 90^\circ$. Then F is the mid-point of the chord DE.

III. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Seydewitz's Theorem states: If a triangle is inscribed in a conic section, then any line conjugate to one side meets the other two sides in conjugate points (see [1]). Here we have DM conjugate to BC with respect to Γ, since the pole of BC is M. Thus, DM intersects AB and AC at points which are harmonic conjugates with respect to the segment DE. Since DM meets AB at ∞, F must be the mid-point of DE.

Reference:

[1] H.S.M. Coxeter, *Projective Geometry*, 2nd edition, Springer, 1987.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; BOB SERKEY, Leonia, NJ, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; BABIS STERGIOU, Chalkida, Greece; MIHAELA VÂJIAC, Chapman University, Orange, CA, USA, and BOGDAN SUCEAVĂ, California State University, Fullerton, CA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania.

2955. [2003 : 297, 299] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer. For each positive integer k, let f_k be the k^{th} Fibonacci number; that is, $f_1 = 1$, $f_2 = 1$, and $f_{k+2} = f_{k+1} + f_k$ for all $k \geq 1$. Prove that

$$\left(\sum_{k=1}^{n} f_{k+1}^2\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) \;\geq\; n^2\,.$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC, USA.

The result follows immediately from the observations that $f_{n+1} \geq n$ for every positive integer n and $\sum\limits_{k=1}^{n} \frac{1}{f_{2k}} \geq \frac{1}{f_2} = 1$.

The following claim gives a sharper inequality for $n \neq 3$:

Claim. For every positive integer n except $n = 3$,

$$\left(\sum_{k=1}^{n} f_{k+1}\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) \;\geq\; n^2\,.$$

Proof: It is straightforward to check that the claim holds for $n = 1$, $n = 2$, and $n = 4$ (but, alas, not $n = 3$).

To apply induction, we assume that the claim holds for some integer $n \geq 4$. We have $f_{n+2} \geq 2n$ and $\sum\limits_{k=1}^{n} \frac{1}{f_{2k}} \geq \frac{4}{3}$ (since $n \geq 4$), and hence,

$$\begin{aligned}
&\left(\sum_{k=1}^{n+1} f_{k+1}\right)\left(\sum_{k=1}^{n+1} \frac{1}{f_{2k}}\right) \\
&\quad= \;\left(\sum_{k=1}^{n} f_{k+1}\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) + f_{n+2}\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) + \left(\sum_{k=1}^{n+1} f_{k+1}\right)\left(\frac{1}{f_{2n+2}}\right) \\
&\quad\geq \;n^2 + 2n(4/3) \;\geq\; (n+1)^2\,.
\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2956. [2004 : 297, 299] *Proposed by David Loeffler, student, Trinity College, Cambridge, UK.*

Let A, B, C be the angles of a triangle. Prove that

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \;<\; 2$$

if and only if

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) \;<\; 2\,.$$

Solution by YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON.

Since $A+B+C=\pi$, we have

$$\tan\left(\frac{A}{2}\right) \;=\; \cot\left(\frac{B+C}{2}\right) \;=\; \frac{1-\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}{\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)}\,,$$

and thus,

$$\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)+\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)+\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right) \;=\; 1\,.$$

Hence, $\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right) \;<\; 2$ is equivalent to

$$\left(\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)\right)^2 \;<\; 4\,,$$

which can be expressed as

$$\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right) \;<\; 2\,,$$

since the left side of the last inequality is clearly positive.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; VEDULA N. MURTY, Dover, PA, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2957. [2004 : 297] *Proposed by K.R.S. Sastry, Bangalore, India.*

Let ABC and $A'B'C'$ be two triangles having $BC=a$, $B'C'=s-a$, etc., where $s=\frac{1}{2}(a+b+c)$. Prove that the triangles are isosceles if and only if $\tan\left(\frac{B}{2}\right)$ is the geometric mean of $\tan\left(\frac{A'}{2}\right)$ and $\tan\left(\frac{B'}{2}\right)$.

Solution by D. Kipp Johnson, Beaverton, OR, USA.

The statement of the problem is not sufficiently precise. We shall prove that the given tangent condition is true if and only if the triangles are isosceles with respective apex angles at A and A'.

By the Triangle Inequality in $\triangle A'B'C'$, we must have

$$(s-a)+(s-b) \ > \ s-c\,,$$

with similar inequalities obtained by permuting a, b, c. The above inequality simplifies to $c > s/2$. Thus, each side of $\triangle ABC$ must exceed one quarter of the perimeter $2s$. Equivalently, three times the shortest side must exceed the sum of the other two sides.

Note that the semiperimeter of $\triangle A'B'C'$ is half that of the $\triangle ABC$:

$$s' \;=\; \frac{(s-a)+(s-b)+(s-c)}{2} \;=\; \frac{3s-(a+b+c)}{2} \;=\; \frac{s}{2}\,.$$

We will need the following known identity: $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$. [Briefly, we have $\tan\left(\frac{B}{2}\right) = \frac{r}{s-b} = \frac{[ABC]}{s(s-b)} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$.] Using this identity, we get

$$\begin{aligned}
&\tan\left(\frac{A'}{2}\right)\tan\left(\frac{B'}{2}\right) \\
&\quad= \sqrt{\frac{(s'-(s-b))(s'-(s-c))}{s'(s'-(s-a))}}\cdot\sqrt{\frac{(s'-(s-a))(s'-(s-c))}{s'(s'-(s-b))}} \\
&\quad= \sqrt{\frac{(s'-s+c)^2}{(s')^2}} \;=\; \sqrt{\frac{(c\,-\,s/2)^2}{(s')^2}} \;=\; \frac{c\,-\,s/2}{s'} \qquad \text{since } c > s/2 \\
&\quad= \frac{2c}{s}-1\,,
\end{aligned}$$

and

$$\begin{aligned}
\tan^2\left(\frac{B}{2}\right) &= \frac{(s-a)(s-c)}{s(s-b)} \;=\; \frac{s^2-(a+b+c)s+bs+ac}{s(s-b)} \\
&= \frac{-s^2+bs+ac}{s(s-b)} \;=\; \frac{ac}{s(s-b)}-1\,.
\end{aligned}$$

The condition that $\tan\left(\frac{B}{2}\right)$ is the geometric mean of $\tan\left(\frac{A'}{2}\right)$ and $\tan\left(\frac{B'}{2}\right)$ can therefore be written in the following equivalent forms:

$$\begin{aligned}
\frac{ac}{s(s-b)}-1 &= \frac{2c}{s}-1\,, \\
\frac{c(b-c)}{s(s-b)} &= 0\,, \\
b &= c\,, \\
b' &= c'\,.
\end{aligned}$$

Comment. It is clear that when we interchange A and B, we will have $\tan^2\left(\frac{A}{2}\right) = \tan\left(\frac{A'}{2}\right)\tan\left(\frac{B'}{2}\right)$ if and only if $a = c$ (equivalently, $a' = c'$). Interestingly, replacing B by C produces a somewhat different conclusion. In this case, a similar argument shows that $\tan^2\left(\frac{C}{2}\right) = \tan\left(\frac{A'}{2}\right)\tan\left(\frac{B'}{2}\right)$ if and only if $a = c$ *or* $b = c$ (or equivalently, $a' = c'$ *or* $b' = c'$).

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

2960. [2004 : 298, 300] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let m_a, w_a, and h_a be the lengths of the median, the angle-bisector, and the altitude, respectively, from the right-angled vertex A of triangle ABC to the hypotenuse. Suppose that the sides a and c are fixed in length, while the length of side b varies subject to $a > b \geq c$.

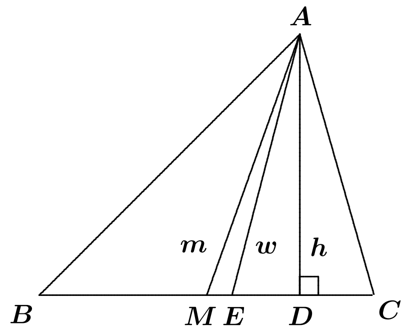
Evaluate $\lim\limits_{b\to c}\dfrac{m_a - h_a}{w_a - h_a}$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

First we note that the problem as stated is incorrect. If sides a and c are fixed in length, then so is side b, by the Pythagorean Theorem. The problem will be correct if we assume that only c is fixed. We will drop the condition $\angle A = 90^\circ$ and solve the problem for any triangle with $a > b > c$, assuming that c is fixed.

There will be no confusion if we drop the subscripts and write m, w, and h instead of m_a, w_a, and h_a. Let the median, the angle-bisector, and the altitude from vertex A intersect the side BC at points M, E, and D, respectively. By the Pythagorean Theorem, we have $m^2 - h^2 = MD^2$. Thus,

$$m - h = \frac{MD^2}{m+h}.$$



Similarly, $w^2 - h^2 = ED^2$, which implies that $w - h = \dfrac{ED^2}{w+h}$.

We also have
$$MB = \frac{a}{2}, \quad EB = \frac{ac}{b+c},$$
and
$$BD = c\cos B = \frac{a^2+c^2-b^2}{2a},$$
where the last expression follows from the Law of Cosines. Using these results, we obtain
$$\begin{aligned} MD &= MB - BD = \frac{(b-c)(b+c)}{2a} \\ \text{and} \quad ED &= EB - BD = \frac{(b-c)\big((b+c)^2-a^2\big)}{2a(b+c)}. \end{aligned}$$
Therefore,
$$\frac{m-h}{w-h} = \frac{MD^2(w+h)}{ED^2(m+h)} = \frac{(b+c)^4(w+h)}{\big((b+c)^2-a^2\big)^2(m+h)}.$$
Now, $\lim\limits_{b\to c} m = \lim\limits_{b\to c} w = h$. Hence,
$$\lim_{b\to c}\frac{m-h}{w-h} = \frac{(2c)^4}{\big((2c)^2-a^2\big)^2}.$$
In the case when $\angle A = 90^\circ$, we have $\lim\limits_{b\to c} a = c\sqrt{2}$; whence,
$$\lim_{b\to c}\frac{m-h}{w-h} = 4.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect solution submitted.

Many solvers pointed out the error in the condition and then proceeded to assume that c is fixed or that a is fixed. Both approaches produce the same result. Zvonaru was the only other solver who initially solved the problem for any triangle and then deduced the answer for the case $\angle A = 90^\circ$, as in the featured solution above.

2961. [2004 : 298, 300] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC and $A'B'C'$ be two right triangles with right angles at A and A'. If w_a and $w_{a'}$ are the interior angle bisectors of angles A and A', respectively, prove that $aw_a a'w_{a'} \geq bcb'c'$, with equality if and only if both ABC and $A'B'C'$ are isosceles.

Solution by Titu Zvonaru, Bucharest, Romania.

Let h_a be the altitude from A. Because the shortest distance from A to BC is along the perpendicular, we have

$$aw_a \;\geq\; ah_a \;=\; 2[ABC] \;=\; bc\,.$$

Equality holds if and only if $h_a = w_a$; that is, if and only if $\triangle ABC$ is isosceles. This is true for as many triangles as we wish—we can have one or many triangles instead of two. Since the quantities involved are all positive, one can multiply the respective sides of each inequality together while maintaining the inequality.

Comment. The conclusion continues to hold if the bisectors are replaced by any other cevians.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2962. [2003 : 298, 300] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC and $A'B'C'$ be two triangles satisfying $a \geq b \geq c$ and $a' \geq b' \geq c'$. If h_a, $h_{a'}$ are the altitudes from the vertices A, A', respectively, to the opposite sides, prove that

(i) $bb' + cc' \geq ah_{a'} + a'h_a$, (ii) $bc' + b'c \geq ah_{a'} + a'h_a$.

Comment by Michel Bataille, Rouen, France.

This is the same problem as 2860 [2003 : 318; 2004 : 315]. [The editor apologizes for this embarassment: the solution to 2860 appeared in the same issue in which problem 2962 was posed.]

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2963. [2004 : 367, 370] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any acute-angled triangle. Let r and R be the inradius and circumradius, respectively, and let s be the semiperimeter; that is, $s = \frac{1}{2}(a+b+c)$. Let m_a be the length of the median from A to BC, and let w_a be the length of the internal bisector of $\angle A$ from A to the side BC. We define m_b, m_c, w_b and w_c similarly. Prove that

(a) $\dfrac{3s^2 - r^2 - 4Rr}{8sRr} \le \sum\limits_{\text{cyclic}} \dfrac{m_a}{aw_a} \le \dfrac{s^2 - r^2 - 4Rr}{7sRr}$;

(b) $\dfrac{3}{4} \le \sum\limits_{\text{cyclic}} \dfrac{m_a^2}{b^2+c^2} \le \dfrac{4R+r}{4R}$.

Solution to part (a) by Arkady Alt, San Jose, CA, USA.

First we note that the right inequality is incorrect. For example, if ABC is equilateral, then

$$\sum_{\text{cyclic}} \frac{m_a}{aw_a} = \frac{3}{a} > \frac{6}{7a} = \frac{s^2 - r^2 - 4Rr}{7sRr}.$$

We will instead prove that

$$\frac{3s^2 - r^2 - 4Rr}{8sRr} \le \sum_{\text{cyclic}} \frac{m_a}{aw_a} \le \frac{s^2 - r^2 - 4Rr}{2sRr}.$$

We use the following well-known identities:

$$\begin{aligned} 4m_a^2 &= 2(b^2+c^2) - a^2, \\ w_a^2 &= \frac{bc\big((b+c)^2 - a^2\big)}{(b+c)^2}, \\ abc &= 4sRr, \\ a^2+b^2+c^2 &= 2s^2 - 2r^2 - 8Rr, \\ ab+bc+ca &= s^2 + r^2 + 4Rr. \end{aligned}$$

From the first two identities above, we get

$$\frac{m_a^2}{w_a^2} = \frac{(b+c)^2}{4bc} \cdot \frac{2(b^2+c^2) - a^2}{(b+c)^2 - a^2}. \tag{1}$$

Now we observe that

$$2bc \le (b+c)^2 - a^2 \le 4bc. \tag{2}$$

The left inequality is true because it is equivalent to $b^2 + c^2 \ge a^2$, which is true for any acute triangle, and the right inequality is true because it is equivalent to $|b-c| \leq a$, which is true for any triangle. From (2), we get

$$\frac{(b-c)^2}{4bc}+1 \;\leq\; \frac{(b-c)^2}{(b+c)^2-a^2}+1 \;\leq\; \frac{(b-c)^2}{2bc}+1\,;$$

that is,

$$\frac{(b+c)^2}{4bc} \;\leq\; \frac{2(b^2+c^2)-a^2}{(b+c)^2-a^2} \;\leq\; \frac{b^2+c^2}{2bc}\,.$$

Recalling (1), we get

$$\frac{(b+c)^4}{16b^2c^2} \;\leq\; \frac{m_a^2}{w_a^2} \;\leq\; \frac{(b+c)^2(b^2+c^2)}{8b^2c^2}\,.$$

Now, using the easy-to-prove inequality $(b+c)^2 \leq 2(b^2+c^2)$, we obtain

$$\frac{(b+c)^4}{16b^2c^2} \;\leq\; \frac{m_a^2}{w_a^2} \;\leq\; \frac{(b^2+c^2)^2}{4b^2c^2}\,.$$

Taking square roots throughout and dividing by a gives

$$\frac{(b+c)^2}{4abc} \;\leq\; \frac{m_a}{aw_a} \;\leq\; \frac{b^2+c^2}{2abc}\,,$$

where equality occurs if and only if $b=c$.

Using similar inequalities for $\frac{m_b}{bw_b}$ and $\frac{m_c}{cw_c}$, we obtain

$$\sum_{\text{cyclic}} \frac{m_a}{aw_a} \;\leq\; \sum_{\text{cyclic}} \frac{b^2+c^2}{2abc} \;=\; \frac{a^2+b^2+c^2}{abc} \;=\; \frac{s^2-r^2-4Rr}{2sRr}$$

and

$$\begin{aligned}\sum_{\text{cyclic}} \frac{m_a}{aw_a} \;&\geq\; \sum_{\text{cyclic}} \frac{(b+c)^2}{4abc} \;=\; \frac{1}{2abc}\left(\sum_{\text{cyclic}} a^2+\sum_{\text{cyclic}} bc\right)\\ &=\; \frac{1}{8sRr}(2s^2-2r^2-8Rr+s^2+r^2+4Rr)\\ &=\; \frac{3s^2-r^2-4Rr}{8sRr}\,,\end{aligned}$$

as claimed. Equality occurs in both inequalities if and only if $a=b=c$.

Solution to part (b) by Michel Bataille, Rouen, France.

We prove that

$$\frac{3}{4} \;<\; \sum_{\text{cyclic}} \frac{m_a^2}{b^2+c^2} \;\leq\; \frac{4R+r}{4R}\,.$$

Since $4m_a^2 = 2(b^2 + c^2) - a^2$, it is easily seen that our inequality is equivalent to

$$2 - \frac{r}{R} \le \sum_{\text{cyclic}} \frac{a^2}{b^2 + c^2} < 3 .$$

Since the triangle is acute, the cosines of all angles are positive. Using the Cosine Law, we obtain

$$\sum_{\text{cyclic}} \frac{a^2}{b^2 + c^2} = \sum_{\text{cyclic}} \frac{b^2 + c^2 - 2bc\cos A}{b^2 + c^2} = 3 - \sum_{\text{cyclic}} \frac{2bc\cos A}{b^2 + c^2} < 3 .$$

On the other hand, using the well-known identity

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

and the easy inequality $2bc \le b^2 + c^2$, we obtain

$$\begin{aligned} \sum_{\text{cyclic}} \frac{a^2}{b^2 + c^2} &= 3 - \sum_{\text{cyclic}} \frac{2bc\cos A}{b^2 + c^2} \ge 3 - \sum_{\text{cyclic}} \cos A \\ &= 3 - \left(1 + \frac{r}{R}\right) = 2 - \frac{r}{R} . \end{aligned}$$

Equality holds if and only if $a = b = c$.

Also solved by ARKADY ALT, San Jose, CA, USA (part (b)); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (part (b)); MICHEL BATAILLE, Rouen, France (part (a)); JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (b)); VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Janous believes that the lower bound of $3/4$ in inequality (b) can be increased to 1, but he does not have a proof. We encourage our readers to try to find a bound better than $3/4$.

The editors apologize for the typo in the right side of the inequality of part (a). The proposer's version was the correct one (found also by Alt and Bataille). Several other solvers either gave a counterexample or suggested a correct version and solved it.