Mayhem Solutions

M122. Proposed by the Mayhem Staff.

In a certain province, vehicle licence plates each have exactly three letters followed by three digits. We are told that to produce such a licence plate, it costs $n$ for each digit $n > 0$ and $10$ for each digit $0$. For letters, the costs are proportional to the position of the letter in the alphabet, namely, $1$ for A, $2$ for B, and so on, up to $26$ for Z.

(a) Find the cost of producing an entire set of licence plates (that is, from AAA 000 to ZZZ 999).

(b) Determine how many plates would cost exactly $100$.

Solution by Geneviève Lalonde, Massey, ON.

(a) In a set of licence plates, each digit occurs the same number of times, as does each letter. Therefore, we can work with averages. The average cost of a digit is $\frac{1}{10}(1 + 2 + \ldots + 10) = \frac{1}{10} \left(\frac{10 \times 11}{2}\right) = 5.50$, and the average cost of a letter is $\frac{1}{26}(1+2+\ldots+26) = \frac{1}{26} \left(\frac{26 \times 27}{2}\right) = 13.50$.

Thus, the average cost of a licence plate is $5.50 \times 3 + 13.50 \times 3 = 57$.

Since the total number of licence plates is $26^3 \times 10^3 = 17 576 000$, the total cost of all of them is $17 576 000 \times 57 = 1 001 832 000$.

(b) Consider any licence plate that costs exactly $100$. Since a letter costs at most $26$, the maximum cost of 3 letters is $78$, implying that the numbers cost at least $22$. Similarly, the maximum cost of the numbers is $30$, which means that the letters cost at least $70$. For $k \in \{0, 1, \ldots, 8\}$, if the letters cost $(70 + k)$, then the numbers cost $(30 - k)$. These are the only possibilities. We now need to determine the number of ways to choose the 3 letters and 3 numbers for each $k \in \{0, 1, \ldots, 8\}$.

Since both sets of costs above are of the form $3n - 8, 3n - 7, \ldots, 3n$ ($n = 26$ for the letters and $n = 10$ for the numbers), let us next examine all the possible ways to sum three numbers to $3n - \ell$, when the numbers can all range from $1$ to $n$. We claim that the number of ways is $\frac{1}{2}(\ell + 1)(\ell + 2)$. The following table demonstrates this for $\ell = 0, 1, 2, 3$. [Ed: We leave it to the reader to prove this in general.]

<table>
<thead>
<tr>
<th>Sum</th>
<th>Possibilities</th>
<th>#</th>
<th>Sum</th>
<th>Possibilities</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3n$</td>
<td>$n, n, n$</td>
<td>1</td>
<td>$3n - 1$</td>
<td>$n, n, n - 1$</td>
<td>3</td>
</tr>
<tr>
<td>$3n - 2$</td>
<td>$n, n, n - 2$</td>
<td>3</td>
<td>$3n - 3$</td>
<td>$n, n, n - 3$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$n, n - 1, n - 1$</td>
<td>3</td>
<td></td>
<td>$n, n - 1, n - 2$</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>$n - 1, n - 1, n - 1$</td>
<td>1</td>
<td></td>
<td>$n - 1, n - 1, n - 1$</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>6</td>
<td></td>
<td>Total</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
Now, we simply calculate the number of ways to achieve the allowable costs for letters and numbers.

<table>
<thead>
<tr>
<th>Letters</th>
<th># Ways</th>
<th>Numbers</th>
<th># Ways</th>
<th>Total # Ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>$70</td>
<td>45</td>
<td>$30</td>
<td>1</td>
<td>45</td>
</tr>
<tr>
<td>$71</td>
<td>36</td>
<td>$29</td>
<td>3</td>
<td>108</td>
</tr>
<tr>
<td>$72</td>
<td>28</td>
<td>$28</td>
<td>6</td>
<td>168</td>
</tr>
<tr>
<td>$73</td>
<td>21</td>
<td>$27</td>
<td>10</td>
<td>210</td>
</tr>
<tr>
<td>$74</td>
<td>15</td>
<td>$26</td>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>$75</td>
<td>10</td>
<td>$25</td>
<td>21</td>
<td>210</td>
</tr>
<tr>
<td>$76</td>
<td>6</td>
<td>$24</td>
<td>28</td>
<td>168</td>
</tr>
<tr>
<td>$77</td>
<td>3</td>
<td>$23</td>
<td>36</td>
<td>108</td>
</tr>
<tr>
<td>$78</td>
<td>1</td>
<td>$22</td>
<td>45</td>
<td>45</td>
</tr>
</tbody>
</table>

Thus, there are 1287 licence plates that cost exactly $100.

**M123. Proposed by the Mayhem Staff.**

In what base is 221 a factor of 1215?

*Solution by Mikel Díez, 3º ESO student, IES Sagasta, Logroño, Spain.*

The number 1215 in base $a$ has the value $a^3 + 2a^2 + a + 5$ and 221 in base $a$ has the value $2a^2 + 2a + 1$. Now,

$$a^3 + 2a^2 + a + 5 = (2a^2 + 2a + 1)\left(\frac{1}{2}a + \frac{1}{2}\right) + \left(-\frac{1}{2}a + \frac{9}{2}\right).$$

Since $1215(a)$ must be a multiple of $221(a)$, the remainder, $-\frac{1}{2}a + \frac{9}{2}$, must be 0 and $\frac{1}{2}a + \frac{1}{2}$ must be an integer. Therefore, $a = 9$.

*Also solved by Robert Bilinski, Outremont, QC.*

**M124. Proposé par l'Équipe de Mayhem.**

Sans l'aide d'une table, trouver si 2003 est un nombre premier.

*Solution par Robert Bilinski, Outremont, QC.*

Il suffit d'essayer la division par tous les entiers premiers de 2 jusqu'à $\sqrt{2003} \approx 44.75$, soit 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43. Si on ne s'en souvient pas (la mémoire est-elle une liste ?), on peut se faire un crible d'Ératosthène rapidement. Puisqu'il n'y a aucun nombre dans cette liste qui donne un quotient entier, on conclut que 2003 est premier.

**M125. Proposed by the Mayhem Staff.**

List all of the positive integers less than 122003 that are both perfect squares and perfect cubes.
Solution by Mikel Díez, 3º ESO student, IES Sagasta, Logroño, Spain.

In order to be both perfect squares and perfect cubes they must be perfect sixth powers (and, conversely, every perfect sixth power is both a perfect square and a perfect cube). The only solutions are:

\[ 1^6 = 1, \quad 2^6 = 64, \quad 3^6 = 729, \quad 4^6 = 4096, \]
\[ 5^6 = 15625, \quad 6^6 = 46656, \quad 7^6 = 117649. \]

Also solved by Robert Bilinski, Outremont, QC.

M126. Proposed by the Mayhem Staff.

Given that the letters \( A, B, C, D, E, F \) represent distinct decimal digits, find the values of these letters so that

\[ ABC \times DEF = 232323. \]

Here, \( ABC \) and \( DEF \) represent 3-digit numbers.

Combination of solutions by Robert Bilinski, Outremont, QC; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

The numbers are 851 and 273. They are obtained by factoring 232323:

\[ 232323 = 3 \cdot 7 \cdot 13 \cdot 23 \cdot 37, \]

and then examining all the possible groupings of the factors that yield two 3-digit numbers.

Also solved by Doug Newman, Lancaster, CA, USA.


Prove that if \( a, b \in \mathbb{R} \) and \( a - b = 1 \), then \( a^3 - b^3 \geq \frac{1}{4} \).

Solution by the Austrian IMO Team.

We know that \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \). Using \( a - b = 1 \), the given inequality is equivalent to

\[ a^2 + ab + b^2 \geq \frac{1}{4}. \]

Furthermore, we can write \( a \) as \( a = 1 + b \), yielding

\[ (1 + b)^2 + (1 + b)b + b^2 \geq \frac{1}{4}. \]

This inequality is equivalent to each of the following

\[ (1 + 2b + b^2) + (b + b^2) + b^2 \geq \frac{1}{4}, \]
\[ 3b^2 + 3b + \frac{3}{4} \geq 0, \]
\[ 3 \left( b + \frac{1}{2} \right)^2 \geq 0. \]
Because a square number is always positive, the last line is true. Thus, the given inequality is true.

Also solved by Gustavo Krémker, Universidad CAECE, Buenos Aires, Argentina; Robert Bilinski, Outremont, QC; and Doug Newman, Lancaster, CA, USA.

M128. Proposed by Lobzang Dorji, Paro, Bhutan.
(a) Nine dots are uniformly spaced in a $3 \times 3$ square array as shown. Verify that 8 non-congruent triangles can be formed using three of the dots as vertices.

(b) Suppose that a $4 \times 4$ square array of dots is employed. How many non-congruent triangles could be formed using three of the dots as vertices?

Solution to part (a) by Doug Newman, Lancaster, CA, USA.
The 8 non-congruent triangles that can be formed are as follows:

Also solved by Robert Bilinski, Outremont, QC; and David Wagner, University of New Brunswick, Fredericton, NB.

Solution to part (b) by Robert Bilinski, Outremont, QC; Doug Newman, Lancaster, CA, USA; and David Wagner, University of New Brunswick, Fredericton, NB.

We list below the 29 non-congruent triangles.

Ed: Note that by using reflections, rotations and translations, we may always position one vertex in the upper left corner of the grid. Also, we can always find a second vertex on or below the main diagonal (from upper left to lower right). The first nine triangles above have the second vertex 1 unit below the first; the next eight have the second vertex 2 units below the first; and the following four have the second vertex 3 units below the first. These are the only triangles with either a horizontal or vertical side. The remaining eight triangles systematically move the second vertex across and down the grid, keeping on or below the main diagonal.

Wagner used another systematic approach to determine that the number of non-congruent triangles in a $5 \times 5$ grid is 79. Comments are welcomed concerning either a general term or a verification/proof that a solution is complete for an $n \times n$ grid.
M129. Proposed by the Mayhem Staff.

A die is tossed. If the die lands on ‘1’ or ‘2’, then one coin is tossed. If the die lands on ‘3’, then two coins are tossed. Otherwise, three coins are tossed. Given that the resulting coin tosses produced no ‘heads’, what is the probability that the die landed on ‘1’ or ‘2’.

Solution by Robert Bilinski, Outremont, QC.

Let \( A \) be the event that the number 1 or 2 appears on the die, \( B \) the event that the number 3 appears, and \( C \) the event that a number more than 3 appears. Thus, \( P(A) = \frac{1}{3}, P(B) = \frac{1}{6}, \) and \( P(C) = \frac{1}{2}. \) Let \( NH \) be the event that no heads appear when the coins are tossed. Then we have

\[
P(NH \mid A) = \frac{1}{2}, \quad P(NH \mid B) = \frac{1}{4}, \quad \text{and} \quad P(NH \mid C) = \frac{1}{8}.
\]

Hence,

\[
P(NH) = P(NH \mid A) \cdot P(A) + P(NH \mid B) \cdot P(B) + P(NH \mid C) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{8} \cdot \frac{1}{2} = \frac{13}{48}.
\]

But we want \( P(A \mid NH). \)

\[
P(A \mid NH) = \frac{P(NH \mid A) \cdot P(A)}{P(NH)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{13}{48}} = \frac{8}{13}.
\]

One incorrect solution was also received.