Problem of the Month

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Problem (2003/4 British Mathematical Olympiad, Round 1) A set of positive integers is defined to be \textit{wicked} if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set. Find the number of wicked subsets of the set \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\).

The first step in a problem which defines some new terminology is to try to understand the terminology. We write down a few examples of subsets which are wicked and subsets which are not wicked. Wicked subsets include \(\{\}\) (the empty set), \(\{1, 2, 4, 5, 8\}\), \(\{6, 7\}\), and so on. Subsets which are not wicked include \(\{2, 3, 4, 5, 7, 8, 9\}\), \(\{1, 2, 3\}\), and others.

At this stage, we also want to recall that if a set \(S\) has \(n\) elements, then the total number of subsets of \(S\), including \(S\) itself and the empty set, is \(2^n\) (since each element of \(S\) is either in or not in a particular subset).

After a bit of fiddling around, we see that there are quite a few wicked subsets of the set given in the problem. Trying to write them all down is probably not a good idea (unless, of course, we were stuck in that snowstorm that we mentioned in the December issue). Furthermore, there does not seem to be a quick way to characterize these subsets and count them directly.

A good next step is to try finding the number of wicked subsets of a smaller set. Let \(S_n = \{1, 2, 3, \ldots, n\}\), and let \(W_n\) be the number of wicked subsets of \(S_n\). We want to calculate \(W_{10}\). We first start with something easier.

(a) Consider \(S_0 = \{\}\). There is only one subset—the empty set itself—and this subset is wicked; hence, \(W_0 = 1\).

(b) Consider \(S_1 = \{1\}\). There are only two subsets, \(\{\}\) and \(\{1\}\), both of which are wicked; hence, \(W_1 = 2\).

(c) Consider \(S_2\). There are 4 subsets, each wicked; hence, \(W_2 = 4\).

(d) Consider \(S_3\). There are 8 subsets, only one of which is not wicked (namely, \(\{1, 2, 3\}\)); hence \(W_3 = 7\).

(e) Consider \(S_4\). There are 16 subsets, only 3 of which are not wicked (namely, \(\{1, 2, 3, 4\}\), \(\{1, 2, 3\}\), and \(\{2, 3, 4\}\)); hence, \(W_4 = 13\).

We can now proceed in two ways, both of which involve looking at smaller wicked subsets and trying to construct larger ones. This approach seems sensible because we are able to handle the smaller cases.

Solution 1. We start by trying to use the wicked subsets of \(S_9\) to construct the wicked subsets of \(S_{10}\). If \(A\) is a wicked subset of \(S_9\), then \(A\) is also a subset of \(S_{10}\) and is still wicked. Thus, all \(W_9\) wicked subsets of \(S_9\) are also wicked subsets of \(S_{10}\).

Which wicked subsets of \(S_9\) turn into wicked subsets of \(S_{10}\) when we put the additional number 10 into them? Those which remain wicked are those
that do not already contain both 8 and 9. Now, how many wicked subsets of \(S_9\) do not contain both 8 and 9? If a wicked subset \(A\) contains 8 and 9, then it cannot contain 7, since it is wicked, and the rest of \(A\) must be a wicked subset of \(S_6\). Thus, the number of wicked subsets of \(S_9\) containing 8 and 9 equals \(W_0\), the number of wicked subsets of \(S_6\). Therefore, the number of wicked subsets of \(S_9\) which do not contain both 8 and 9 is \(W_9 - W_6\).

Now, every wicked subset of \(S_{10}\) either does not contain 10 (in which case it is a wicked subset of \(S_9\)) or does contain 10 (in which case the set we get by removing 10 is a wicked subset of \(S_9\)). Thus,

\[
W_{10} = W_9 + (W_0 - W_6) = 2W_9 - W_6.
\]

In the same way, we see that \(W_9 = 2W_8 - W_5\), and \(W_8 = 2W_7 - W_4\), and so on. Hence,

\[
W_{10} = 2W_0 - W_6 = 2(2W_8 - W_5) - W_6 = 4W_8 - W_6 - 2W_5 = 4(2W_7 - W_4) - W_6 - 2W_5 = 8W_7 - W_6 - 2W_5 - 4W_4 = 8(2W_6 - W_3) - W_6 - 2W_5 - 4W_4 = 15W_6 - 2W_5 - 4W_4 - 8W_3 = 15(2W_5 - W_2) - 2W_5 - 4W_4 - 8W_3 = 28W_5 - 4W_4 - 8W_3 - 15W_2 = 28(2W_4 - W_1) - 4W_4 - 8W_3 - 15W_2 = 52W_4 - 8W_3 - 15W_2 - 28W_1 = 52(13) - 8(7) - 15(4) - 28(2) = 504.
\]

Thus, the number of wicked subsets of \(S_{10}\) is 504.

Well, that was certainly better than trying to write out all of the wicked subsets of \(S_{10}\), but it was still a bit painful. There must be a better way.

**Solution 2.** We look at this in a more general way. Let \(A\) be any wicked subset of \(S_n\). How many such sets \(A\) do not contain \(n\)? If \(A\) does not contain \(n\), then \(A\) is itself a wicked subset of \(S_{n-1}\). There are \(W_{n-1}\) such sets \(A\).

Assume now that \(A\) contains \(n\). If \(A\) does not contain \(n-1\), then the part of \(A\) that is left after removing \(n\) is a wicked subset of \(S_{n-2}\). Thus, there are \(W_{n-2}\) wicked subsets of \(S_n\) which contain \(n\) and do not contain \(n-1\). If \(A\) does contain \(n-1\), then \(A\) cannot contain \(n-2\), since it cannot contain 3 consecutive integers. Hence, the part of \(A\) that we get after removing \(n\) and \(n-1\) is a wicked subset of \(S_{n-3}\). There are \(W_{n-3}\) such sets \(A\).

These are all of the possibilities for \(A\). Thus,

\[
W_n = W_{n-1} + W_{n-2} + W_{n-3}.
\]

(We can check that \(W_3 = W_2 + W_1 + W_0\) and \(W_4 = W_3 + W_2 + W_1\).) In other words, each term after the third is the sum of the three previous terms. Now we can write out the sequence starting at \(W_0\) to get 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, ... . Thus, \(W_{10} = 504\).