MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Amy Cameron (Carleton University), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

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Mayhem Problems

Please send your solutions to the problems in this edition by 1 July 2005. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

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\textbf{M176. Proposed by the Mayhem Staff.}

How many points \((x, y)\) with positive integral coordinates lie on the curve defined by \(x^2 + y^2 + 2xy - 2005x - 2005y - 2006 = 0\)?

\textbf{M177. Proposed by Babis Stergiou. Chalkida, Greece.}

Let \(ABC\) be a triangle with angle \(A\) acute. Let \(BD\) and \(CE\) be two of its altitudes, with \(D\) on \(AC\) and \(E\) on \(AB\). On diameters \(AB\) and \(AC\) construct circles \(\Gamma_1\) and \(\Gamma_2\), respectively. Let \(M\) be the point of intersection of \(\Gamma_1\) and \(CE\), and let \(N\) be the point of intersection of \(\Gamma_2\) and \(BD\), where \(M\) and \(N\) are interior points of \(\triangle ABC\). Prove that \(AM = AN\).

\textbf{M178. Proposed by the Mayhem Staff.}

Show that, if \(10a + b\) is a multiple of 7, then \(a - 2b\) must also be a multiple of 7.

\textbf{M179. Proposed by the Mayhem Staff.}

(a) Find all two-digit numbers that increase by 75\% when their digits are reversed.

(b) Show that no three-digit numbers increase by 75\% when their digits are reversed.
M180. Proposed by the Mayhem Staff.

(a) Show that 121(b) is a perfect square in any base \( b > 2 \).
(b) Determine the smallest value of \( b \) for which \( 232(b) \) is a perfect square.

M181. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

A 3–4–5 right triangle has squares drawn outward on each of its sides. Lines are drawn through the outer corners of these squares, as shown. These lines form a triangle \( ABC \). Determine whether \( \triangle ABC \) is also a right triangle.

M176. Proposé par l'Équipe de Mayhem.

Combien de points \((x, y)\) de coordonnées entières positives se trouvent sur la courbe définie par \( x^2 + y^2 + 2xy - 2005x - 2005y - 2006 = 0 \)?

M177. Proposé par Babis Stergiou, Chalkida, Grèce.

Soit \( ABC \) un triangle avec un angle aigu en \( A \). Soit \( BD \) et \( CE \) deux de ses hauteurs, avec \( D \) sur \( AC \) et \( E \) sur \( AB \). On construit deux cercles \( \Gamma_1 \) et \( \Gamma_2 \) avec respectivement \( AB \) et \( AC \) comme diamètres. Soit \( M \) le point d'intersection de \( \Gamma_1 \) avec \( CE \), \( N \) celui de \( \Gamma_2 \) avec \( BD \), \( M \) et \( N \) étant à l'intérieur du triangle \( ABC \). Montrer que \( AM = AN \).

M178. Proposé par l'Équipe de Mayhem.

Si \( 10a + b \) est un multiple de 7, montrer qu'il doit en être de même pour \( a - 2b \).

M179. Proposé par l'Équipe de Mayhem.

(a) Trouver tous les nombres de deux chiffres qui augmentent de 75% lorsqu'on permuter leurs deux chiffres.
(b) Montrer qu'aucun nombre de trois chiffres n'augmente de 75% lorsqu'on renverse l'ordre de ses chiffres.

M180. Proposé par l'Équipe de Mayhem.

(a) Montrer que \( 121(b) \) est un carré parfait dans toute base \( b > 2 \).
(b) Déterminer la plus petite valeur de \( b \) telle que \( 232(b) \) soit un carré parfait.

M181. Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.

On donne un triangle rectangle de côtés 3–4–5, ainsi que les carrés extérieurs dessinés sur ses côté respectifs. On dessine des droites passant par les coins extérieurs de ces carrés, comme le montre la figure. Ces droites forment un triangle \( ABC \). Déterminer si ce dernier est aussi un triangle rectangle.
Mayhem Solutions

M119. Proposed by the Mayhem Staff.

Andrew, Bernard, and Charles play a game in which the loser has to triple the money of each other player. Three games are played, and the successive losers are Andrew, then Bernard, and finally, Charles. Each player ends with $27. How much money did each person have at the outset?

I. Solution by Robert Bilinski. Outremont, QC.

Let \( a, b, \) and \( c \) be the initial number of dollars that Andrew, Bernard, and Charles held, respectively. The following table represents the evolution of the holdings after each of the three rounds. We take for granted that the total amount of money will always be \( a + b + c \).

<table>
<thead>
<tr>
<th>Start</th>
<th>First Round</th>
<th>Second Round</th>
<th>Third Round</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( a - 2b - 2c )</td>
<td>( 3a - 6b - 6c )</td>
<td>( 9a - 18b - 18c )</td>
</tr>
<tr>
<td>( b )</td>
<td>( 3b )</td>
<td>( 7b - 2c - 2a )</td>
<td>( 21b - 6c - 6a )</td>
</tr>
<tr>
<td>( c )</td>
<td>( 3c )</td>
<td>( 9c )</td>
<td>( 25c - 2a - 2b )</td>
</tr>
</tbody>
</table>

Thus, we need to solve the system

\[
\begin{align*}
9a - 18b - 18c &= 27, \\
21b - 6c - 6a &= 27, \\
25c - 2a - 2b &= 27,
\end{align*}
\]

which yields the solution \( a = 55, \) \( b = 19, \) and \( c = 7. \)

II. Solution by Geneviève Lalonde. Massey, ON.

Let \( A_n, B_n, \) and \( C_n \) be the respective amounts that Andrew, Bernard, and Charles hold after round \( n. \) We are given that \( A_3 = B_3 = C_3 = 27. \) Working backwards, we get

\[
\begin{align*}
A_2 &= \frac{27}{3} = 9, \\
B_2 &= \frac{27}{3} = 9, \\
C_2 &= 27 + 18 + 18 = 63,
\end{align*}
\]

then,

\[
\begin{align*}
A_1 &= \frac{9}{3} = 3, \\
B_1 &= 9 + 6 + 42 = 57, \\
C_1 &= \frac{63}{3} = 21, \\
\end{align*}
\]

and finally,

\[
\begin{align*}
A_0 &= 3 + 38 + 14 = 55, \\
B_0 &= \frac{57}{3} = 19, \\
C_0 &= \frac{21}{3} = 7, \\
\end{align*}
\]

Therefore, Andrew started with $55, Bernard started with $19, and Charles started with $7.
\textbf{M120. Proposed by the Mayhem Staff.}

Solve for \( x \), where \( 0 \leq x < 2\pi \):

\[ 2^{1+3\cos x} - 10 \times 2^{-1+2\cos x} + 2^{2+\cos x} - 1 = 0. \]

\textit{Solution by Robert Bilinski. Outremont, QC.}

The given equation is equivalent to \( 2X^3 - 5X^2 + 4X - 1 = 0 \), where \( X = 2^{\cos x} \). We see that \( X = 1 \) is a solution; whence, the new equation factors into \( (X - 1)(2X^2 - 3X + 1) = 0 \). But \( X = 1 \) is also a root of the second factor. Hence, we have \( (X - 1)^2(2X - 1) = 0 \), which gives \( X = \frac{1}{2} \) as a second solution for \( X \). This means that either \( 2^{\cos x} = 0 \) or \( 2^{\cos x} = \frac{1}{2} \), which is equivalent to \( \cos x = 0 \) or \( \cos x = -1 \). In the range \( 0 \leq x \leq 2\pi \), the solutions are \( x = \frac{\pi}{2} \), \( x = \pi \), \( x = \frac{3\pi}{2} \).

\textbf{M121. Proposed by the Mayhem Staff.}

Use each of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to form prime numbers whose sum is as small as possible.

\textit{Solution by the editors.}

The even digits 4, 6, and 8 cannot be in the units position of any prime.

Let us first restrict ourselves to primes having one or two digits. Then the leading digits of three of the primes must be the digits 4, 6, and 8. Among the two-digit primes there are two primes with leading digit 8 (83 and 89), two with 6 (61 and 67), and three with 4 (41, 43, and 47). Since we are looking for the smallest sum, we should try to have the remaining primes be single-digit primes, which means that 9 cannot be used. Thus, we must have 89 as one of our two-digit primes. Since 2 and 5 cannot be the units digit of any prime with at least two digits, we should keep them as two of our three single-digit primes. Hence, our set should include the primes 2, 5, and 89.

If 3 is the remaining single-digit prime, then the remaining two-digit primes are 41 and 67; if 7 is the remaining single-digit prime, then the remaining primes are 43 and 61. Since the same digits appear in the units column in both sets, the sums must be the same, namely 207. Clearly, this sum is the smallest possible under the restriction that we only consider primes having one or two digits. The two sets which yield this sum are \{2, 3, 5, 41, 67, 89\} and \{2, 5, 7, 43, 61, 89\}.

If we consider allowing primes with more than two digits, then we can surely have only one such, since two would sum to more than 300, which is too large. If we are to keep the sum below 207, then the leading digit of this prime must be 1. Furthermore, the tens digits of three primes must include the digits 4, 6, and 8, but this gives us a sum of more than 280, which is again too large. Therefore, the smallest sum is 207, and the above two sets provide the only solutions.

Also solved by Robert Bilinski, Outremont, QC.
Problem of the Month

Ian VanderBurgh, University of Waterloo

Problem (2003/4 British Mathematical Olympiad, Round 1) A set of positive integers is defined to be *wicked* if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set. Find the number of wicked subsets of the set \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.

The first step in a problem which defines some new terminology is to try to understand the terminology. We write down a few examples of subsets which are wicked and subsets which are not wicked. Wicked subsets include \{\} (the empty set), \{1, 2, 4, 5, 8\}, \{6, 7\}, and so on. Subsets which are not wicked include \{2, 3, 4, 5, 7, 8, 9\}, \{1, 2, 3\}, and others.

At this stage, we also want to recall that if a set \(S\) has \(n\) elements, then the total number of subsets of \(S\), including \(S\) itself and the empty set, is \(2^n\) (since each element of \(S\) is either in or not in a particular subset).

After a bit of fiddling around, we see that there are quite a few wicked subsets of the set given in the problem. Trying to write them all down is probably not a good idea (unless, of course, we were stuck in that snowstorm that we mentioned in the December issue). Furthermore, there does not seem to be a quick way to characterize these subsets and count them directly.

A good next step is to try finding the number of wicked subsets of a smaller set. Let \(S_n = \{1, 2, 3, \ldots, n\}\), and let \(W_n\) be the number of wicked subsets of \(S_n\). We want to calculate \(W_{10}\). We first start with something easier.

(a) Consider \(S_0 = \{\}\). There is only one subset—the empty set itself—and this subset is wicked; hence, \(W_0 = 1\).

(b) Consider \(S_1 = \{1\}\). There are only two subsets, \{\} and \{1\}, both of which are wicked; hence, \(W_1 = 2\).

(c) Consider \(S_2\). There are 4 subsets, each wicked; hence, \(W_2 = 4\).

(d) Consider \(S_3\). There are 8 subsets, only one of which is not wicked (namely, \{1, 2, 3\}); hence \(W_3 = 7\).

(e) Consider \(S_4\). There are 16 subsets, only 3 of which are not wicked (namely, \{1, 2, 3, 4\}, \{1, 2, 3\}, and \{2, 3, 4\}); hence, \(W_4 = 13\).

We can now proceed in two ways, both of which involve looking at smaller wicked subsets and trying to construct larger ones. This approach seems sensible because we are able to handle the smaller cases.

**Solution 1.** We start by trying to use the wicked subsets of \(S_9\) to construct the wicked subsets of \(S_{10}\). If \(A\) is a wicked subset of \(S_9\), then \(A\) is also a subset of \(S_{10}\) and is still wicked. Thus, all \(W_9\) wicked subsets of \(S_9\) are also wicked subsets of \(S_{10}\).

Which wicked subsets of \(S_9\) turn into wicked subsets of \(S_{10}\) when we put the additional number 10 into them? Those which remain wicked are those
that do not already contain both 8 and 9. Now, how many wicked subsets of $S_9$ do not contain both 8 and 9? If a wicked subset $A$ contains 8 and 9, then it cannot contain 7, since it is wicked, and the rest of $A$ must be a wicked subset of $S_6$. Thus, the number of wicked subsets of $S_9$ containing 8 and 9 equals $W_9$, the number of wicked subsets of $S_6$. Therefore, the number of wicked subsets of $S_9$ which do not contain both 8 and 9 is $W_9 - W_6$.

Now, every wicked subset of $S_{10}$ either does not contain 10 (in which case it is a wicked subset of $S_9$) or does contain 10 (in which case the set we get by removing 10 is a wicked subset of $S_9$). Thus,

$$W_{10} = W_9 + (W_9 - W_6) = 2W_9 - W_6.$$  

In the same way, we see that $W_9 = 2W_8 - W_5$, and $W_8 = 2W_7 - W_4$, and so on. Hence,

$$W_{10} = 2W_9 - W_6 = 2(2W_8 - W_5) - W_6 = 4W_8 - W_6 - 2W_5$$

$$= 4(2W_7 - W_4) - W_6 - 2W_5 = 8W_7 - W_6 - 2W_5 - 4W_4$$

$$= 8(2W_6 - W_3) - W_6 - 2W_5 - 4W_4 = 15W_6 - 2W_5 - 4W_4 - 8W_3$$

$$= 15(2W_5 - W_2) - 2W_5 - 4W_4 - 8W_3$$

$$= 28W_5 - 4W_4 - 8W_3 - 15W_2$$

$$= 28(2W_4 - W_1) - 4W_4 - 8W_3 - 15W_2$$

$$= 52W_4 - 8W_3 - 15W_2 - 28W_1$$

$$= 52(13) - 8(7) - 15(4) - 28(2) = 504.$$  

Thus, the number of wicked subsets of $S_{10}$ is 504.

Well, that was certainly better than trying to write out all of the wicked subsets of $S_{10}$, but it was still a bit painful. There must be a better way.

**Solution 2.** We look at this in a more general way. Let $A$ be any wicked subset of $S_n$. How many such sets $A$ do not contain $n$? If $A$ does not contain $n$, then $A$ is itself a wicked subset of $S_{n-1}$. There are $W_{n-1}$ such sets $A$.

Assume now that $A$ contains $n$. If $A$ does not contain $n - 1$, then the part of $A$ that is left after removing $n$ is a wicked subset of $S_{n-2}$. Thus, there are $W_{n-2}$ wicked subsets of $S_n$ which contain $n$ and do not contain $n - 1$. If $A$ does contain $n - 1$, then $A$ cannot contain $n - 2$, since it cannot contain 3 consecutive integers. Hence, the part of $A$ that we get after removing $n$ and $n - 1$ is a wicked subset of $S_{n-3}$. There are $W_{n-3}$ such sets $A$.

These are all of the possibilities for $A$. Thus,

$$W_n = W_{n-1} + W_{n-2} + W_{n-3}.$$  

(We can check that $W_3 = W_2 + W_1 + W_0$ and $W_4 = W_3 + W_2 + W_1$.) In other words, each term after the third is the sum of the three previous terms. Now we can write out the sequence starting at $W_0$ to get 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, . . . . Thus, $W_{10} = 504$.  

Pólya’s Paragon

Fun With Numbers (Part 1)

Shawn Godin

Carl Friedrich Gauss, one of the greatest mathematicians of all time, said “Mathematics is the Queen of the Sciences, and number theory is the Queen of Mathematics”. With that thought in mind, we will focus on numbers—the numbers 1, 2, 3, 4, … that are familiar to most people over the age of three.

Our first stop is in an area that may have given rise to the notion of number in the first place. Suppose we are given a number of counters, such as coins. In how many ways can they be arranged in a rectangular pattern of rows and columns? If we had six counters, for example, we could arrange them in a rectangle in four possible ways, as shown below left. (We consider a $1 \times 6$ rectangle to be different from a $6 \times 1$ rectangle, and a $2 \times 3$ rectangle to be different from a $3 \times 2$ rectangle, for reasons to be seen later.) In the table on the right below, we have filled in the number 4 under “Number of Rectangles” corresponding to the case where the number of counters is 6.

![Diagram of arrangements](image.png)

<table>
<thead>
<tr>
<th>Number of Counters</th>
<th>Number of Rectangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>4</td>
<td>?</td>
</tr>
<tr>
<td>5</td>
<td>?</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Rather than my having all the fun, you should construct your own table like the one above. Try to consider up to 20 counters, at least, and look for patterns in your “Number of Rectangles” column before you read on.

I hope you’re not cheating! What did you notice in your table? One thing that jumps out is that 1 is the only number that can be represented by only 1 rectangle. When we classify counting numbers, we put the number 1 in a class of its own. We call it a unit.

You should have noticed also that some numbers produce only boring rectangles. The number 3, for example, can only be a column or row of 3. This number and others like it, such as 2, 5, 7, 11, 13, 17, and 19, do not break down. These are the prime numbers. The remaining numbers (which are not prime and not 1) can each be represented as rectangles in more than two ways. These are the composite numbers.
We have come up with a classification of numbers: every counting number greater than 1 is either prime or composite. This classification is central in number theory, especially, in multiplicative number theory, which examines how numbers multiply together to give other numbers.

By now you may have realized that the role of the rectangles in all this discussion is just to help us visualize multiplication. When we form a $3 \times 2$ rectangle out of 6 counters, we are factoring the number 6 as $3 \times 2$, showing that the numbers 3 and 2 divide evenly into 6.

Mathematicians have some language and notation for talking about such things. When one number divides evenly into another number, the first is called a divisor of the second. For example, 3 is a divisor of 6. To write this more briefly, we just write $3 | 6$. Similarly, we write $5 \nmid 6$ to say that 5 does not divide 6 evenly, or that 5 is not a divisor of 6. (Mathematicians, being notoriously lazy, do not like to write more than they have to!)

Going back to our rectangles, we look at the number of rows that can appear in the rectangles for a particular number, say 6. We see that the possibilities are 1, 2, 3, and 6. These are all the (positive) divisors of 6. Thus, the number of rectangles is just the number of divisors (now you see why we counted both the $2 \times 3$ and $3 \times 2$ rectangles).

Again, since mathematicians love to make things shorter and easier, they came up with a notation: $d(n)$ is the number of divisors of $n$. For example, $d(1) = 1$, $d(6) = 4$, $d(19) = 2$, and $d(25) = 3$. The function $d(n)$ is called the divisor function.

We have been breaking numbers into pairs of divisors. But suppose we carry this idea further by breaking down the divisors as much as possible. We end up with an underlying principal in number theory:

**Fundamental Theorem of Arithmetic.** Every positive number greater than 1 can be written as the product of prime numbers in a unique way (up to the order of the primes).

If we agree to arrange the primes from smallest to largest, then there is only one way to write a number as a product of primes. This is the prime factorization of the number. For example, the prime factorization of 12 is $12 = 2 \times 2 \times 3 = 2^2 \times 3$.

Notice that prime factorizations would not be unique if we considered 1 to be prime. For example, we could write 6 as $2 \times 3$, or as $1 \times 2 \times 3$, or as $1 \times 1 \times 2 \times 3$, and so on.

For homework, construct a fairly lengthy table, showing the prime factorization and the number of divisors of each number. Your table should start out as shown. Try to see how the value of $d(n)$ is related to the prime factorization $n$, and look for relationships among the specific values of $d(n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Factorization</th>
<th>$d(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$2^2$</td>
<td>3</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>12</td>
<td>$2^2 \times 3$</td>
<td>6</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Savage Numbers

Ian VanderBurgh

In the final problem of the 2003 Euclid Contest, a positive integer $n$ is defined to be "savage" if the integers \{1, 2, \ldots, n\} can be partitioned into three sets $A$, $B$, and $C$ such that

(i) the sum of the elements in each of $A$, $B$, and $C$ is the same,

(ii) $A$ contains only odd numbers,

(iii) $B$ contains only even numbers, and

(iv) $C$ contains every multiple of 3 (and possibly other numbers).

The problem asks solvers to show that 8 is a savage integer, to show that if $n$ is an even savage integer then $\frac{1}{2}(n + 4)$ is an integer, and to find all even savage numbers less than 100.

This was a difficult problem, but it is actually well within our reach to find all savage integers.

We will use the notation $|T|$ to represent the sum of the elements in the set $T$, and we will define $S = \{1, 2, \ldots, n\}$. There are two summation formulas that we will use:

$1 + 2 + \cdots + k = \frac{1}{2}k(k + 1), \quad 1 + 3 + \cdots + (2\ell - 1) = \ell^2.$

If you have never seen the proofs of these formulas, try them out!

Before we determine which positive integers are savage, let us look at two problems which seem to be only slightly related to the original problem, but which will turn out to be key in determining all savage integers.

Let $A_1 = \{1, 5, 7, 11, 13, 17, \ldots\}$ and $B_1 = \{2, 4, 8, 10, 14, 16, \ldots\}$. That is, $A_1$ contains all odd positive integers which are not multiples of 3, and $B_1$ contains all even positive integers which are not multiples of 3.

**Problem 1:** Show that every even positive integer can be expressed as the sum of one or more elements of $B_1$.

The solution to this problem and each of the following numbered problems below can be found at the end—this way you will not be tempted to peak!

Now, which positive integers $m$ can be expressed as the sum of one or more elements of $A_1$? The best way to start is by making a list:

$1, 5, 6 = 1 + 5, 7, 8 = 1 + 7, 11, 12 = 5 + 7, 13 = 1 + 5 + 7,$

$14 = 1 + 13, 16 = 5 + 11, 17, 18 = 1 + 17, 19, 20 = 1 + 19,$

$21 = 1 + 7 + 13, 22 = 5 + 17, 23, 24 = 1 + 23, \ldots$

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Problem 2: It is clear that we have found all possible sums less than 16. Show now that every positive integer greater than or equal to 16 can be expressed as the sum of one or more elements of $A_1$.

Having played with these problems, we can see that the sets $A_1$ and $B_1$ should be somehow connected to the sets $A$ and $B$ in the original problem. Let us return to the original problem.

We can first use condition (i) from the definition of savage integers to narrow down our hunt for savage integers. Since we are partitioning $S$ into three sets $A$, $B$, and $C$ which all have the same sum, then

$$|S| = |A| + |B| + |C| = 3|A|.$$ 

Thus, $|S| = \frac{1}{3}n(n+1)$ must be divisible by 3. Then $n(n+1)$ must be divisible by 3, which implies that either $n$ is divisible by 3 or $n + 1$ is divisible by 3.

Problem 3: Show that if $n$ is divisible by 3, then $|C| > \frac{1}{3}|S|$ (that is, the set $C$ is "too big"), which means that $n$ cannot be savage. (Try doing this both algebraically and in a more intuitive way.)

From the result of Problem 3, we see that $n+1$ must be divisible by 3 in order for $n$ to be savage. If $n$ is even, then $n$ must be of the form $n = 6k + 2$ (the general form of an even integer $n$ for which $n + 1$ is divisible by 3); if $n$ is odd, then $n = 6k + 5$ (the general form of an odd integer $n$ for which $n + 1$ is divisible by 3).

We next define three new sets which are subsets of $S = \{1, 2, \ldots, n\}$.

$$A_0 = \{\text{all odd numbers in } S \text{ which are not multiples of 3}\},$$

$$B_0 = \{\text{all even numbers in } S \text{ which are not multiples of 3}\},$$

$$C_0 = \{\text{all numbers in } S \text{ which are multiples of 3}\}.$$ 

Based on the conditions (ii), (iii), and (iv), the partitioning of $S$ into three sets $A$, $B$, and $C$ can only be accomplished by moving integers out of $A_0$ and $B_0$ into $C_0$, since $C$ must contain every element that is initially in $C_0$ and $A$ and $B$ cannot contain any of the elements that are initially in $B_0$ and $A_0$, respectively.

We now consider the two cases for $n$ to see if it is possible to transform $A_0$, $B_0$, and $C_0$ into $A$, $B$, and $C$.

Case 1: $n = 6k + 2$.

In this case,

$$S = \{1, 2, 3, \ldots, 6k, 6k + 1, 6k + 2\},$$

$$A_0 = \{1, 5, 7, 11, \ldots, 6k - 1, 6k + 1\},$$

$$B_0 = \{2, 4, 8, 10, \ldots, 6k - 2, 6k + 2\},$$

$$C_0 = \{3, 6, 9, \ldots, 6k - 3, 6k\}.$$ 

From this we deduce that

$$\frac{1}{3}|S| = \frac{1}{3}(1 + 2 + 3 + \cdots + (6k + 2)) = \frac{1}{3}\left(\frac{1}{2}(6k + 2)(6k + 3)\right) = (3k + 1)(2k + 1) = 6k^2 + 5k + 1,$$
\[ |A_0| = 1 + 5 + 7 + 11 + \cdots + (6k - 1) + (6k + 1) \\
= 1 + 3 + 5 + 7 + \cdots + (6k - 1) + (6k + 1) \\
- (3 + 9 + \cdots + (6k - 3)) \\
= (3k + 1)^2 - 3(1 + 3 + \cdots + (2k - 1)) \\
= (3k + 1)^2 - 3k^2 = 6k^2 + 6k + 1, \\
\]
\[ |B_0| = 2 + 4 + 8 + 10 + \cdots + (6k - 2) + (6k + 2) \\
= 2 + 4 + 6 + 8 + 10 + \cdots + (6k - 2) + 6k + (6k + 2) \\
- (6 + 12 + \cdots + 6k) \\
= 2(1 + 2 + 3 + \cdots + (3k + 1)) - 6(1 + 2 + \cdots + k) \\
= (3k + 1)(3k + 2) - 3k(k + 1) = 6k^2 + 6k + 2, \\
\]
\[ |C_0| = 3 + 6 + 9 + \cdots + 6k = 3(1 + 2 + 3 + \cdots + 2k) \\
= 3(\frac{1}{2}(2k)(2k + 1)) = 6k^2 + 3k. \\
\]

Thus, in order to create three sets \( A, B, \) and \( C \) each of whose elements sum to \( \frac{1}{3}|S| \), we must move elements of sum \( k \) from \( A_0 \) into \( C_0 \) and elements of sum \( k + 1 \) from \( B_0 \) to \( C_0 \). We will first examine the case of \( n = 6k + 5 \) in the above way before checking to see if this is possible.

**Case 2.** \( n = 6k + 5 \).

In this case,

\[
S = \{1, 2, 3, \ldots, 6k + 3, 6k + 4, 6k + 5\}, \\
A_0 = \{1, 5, 7, 11, \ldots, 6k + 1, 6k + 5\}, \\
B_0 = \{2, 4, 8, 10, \ldots, 6k + 2, 6k + 4\}, \\
C_0 = \{3, 6, 9, \ldots, 6k, 6k + 3\}. \\
\]

**Problem 4:** Using the same methods as in the previous case, show that

\[
\frac{1}{3}|S| = 6k^2 + 11k + 5, \quad |A_0| = 6k^2 + 12k + 6, \\
|B_0| = 6k^2 + 12k + 6, \quad |C_0| = 6k^2 + 9k + 3. \\
\]

Using the results of Problem 4, if we wish to create three sets \( A, B, \) and \( C \) each of whose elements sum to \( \frac{1}{3}|S| \), we must move elements of sum \( k + 1 \) from \( A_0 \) into \( C_0 \) and elements of sum \( k + 1 \) from \( B_0 \) to \( C_0 \).

Where do we go now? In the first case, \( n = 6k + 2 \) will be savage if we can find elements of \( A_0 \) which sum to \( k \) and elements of \( B_0 \) which sum to \( k + 1 \). In the second case, \( n = 6k + 5 \) will be savage if we can find elements of \( A_0 \) which sum to \( k + 1 \) and elements of \( B_0 \) which sum to \( k + 1 \).

In order to do this we can actually go back to our results from Problems 1 and 2 above. To justify this, let us look at \( A_0 \) in the first case. When \( n = 6k + 2 \), the largest element of \( A_0 \) is \( 6k + 1 \). Since we are trying to determine if there are elements in \( A_0 \) which sum to \( k \), we certainly cannot
use the largest element of $A_0$. On the other hand, we can add even larger elements to $A_0$ and no effect will be made on whether there are elements that sum to $k$. In other words, asking whether or not there are elements in $A_0 = \{1, 5, 7, 11, \ldots, 6k - 1, 6k + 1\}$ which sum to $k$ is the same as asking if there are elements which sum to $k$ in $A_1 = \{1, 5, 7, 11, 13, 17, \ldots\}$. We can make the same argument for extending $A_0$ to $A_1$ and $B_0$ to $B_1$ in both cases.

In case 1, where $n = 6k + 2$, using the results of Problems 1 and 2, we see that there are elements in $A_0$ which sum to $k$ if and only if $k$ equals 1, 5, 6, 7, 8, 11, 12, 13, 14, or if $k \geq 16$. Also, there are elements in $B_0$ which sum to $k + 1$ if and only if $k + 1$ is even, or $k$ is odd. Since we require both conditions to hold in order for $n$ to be savage, it follows that $n = 6k + 2$ is savage if and only if $k$ equals 1, 5, 7, 11, 13, or any odd number greater than or equal to 17.

**Problem 5:** Show that $n = 6k + 5$ is savage if and only if $k$ equals 5, 7 or any odd number greater than or equal to 11.

Since in each of the two cases, we have seen that $k$ must be odd and at least 1, we can tidy things up a bit by replacing $k$ with $2\ell + 1$ and stipulating that $\ell$ must be at least 0. Therefore, the savage integers are

\[ n = 12\ell + 8 \quad \text{where } \ell \geq 0, \ell \neq 1, \ell \neq 4, \text{ and } \ell \neq 7 \]

and \[ n = 12\ell + 11 \quad \text{where } \ell \geq 2 \text{ and } \ell \neq 4. \]

Coincidentally, of course, $2003 = 12(166) + 11$, which means that 2003 is a savage integer.

**Solutions to the Problems**

**Problem 1:** $B_1$ contains the number 2, every even number of the form $6j - 2$ with $j \geq 1$, and every even number of the form $6j + 2$ with $j \geq 1$. Hence, the only even positive integers which are not in $B_1$ already (and thus not obviously the sum of elements from $B_1$) are those of the form $6j$, which can be obtained by adding 2 + ($6j - 2$). Therefore, every even positive integer is the sum of elements from $B_1$.

**Problem 2:** The key here is that we can use the numbers 1, 5, and 7 to get the integers 5, 6, 7, and 8, and that every integer of the forms $6j - 1$ and $6j + 1$ is in $A_1$ for $j \geq 2$. (That is, 11, 13, 17, 19, and so on.) Then, adding our representations for 5, 6, 7, and 8, we get

\[ 6j + 4 = 5 + (6j - 1), \quad 6j + 5 = 6 + (6j - 1) = 1 + 5 + (6j - 1), \]
\[ 6j + 6 = 7 + (6j - 1), \quad 6j + 7 = 8 + (6j - 1) = 1 + 7 + (6j - 1), \]
\[ 6j + 8 = 7 + (6j + 1), \quad 6j + 9 = 8 + (6j + 1) = 1 + 7 + (6j + 1), \]

which give us 6 consecutive integers expressed as the sum of elements of $A_1$. Using the smallest value of $j = 2$ generates 16, 17, 18, 19, 20, 21; increasing the value of $j$ by 1 gives us the next 6 consecutive integers, and so on, giving us all positive integers.
Problem 3: If \( n \) is divisible by 3, then \( n = 3k \) for some positive integer \( k \). Thus, \( S = \{1, 2, \ldots, 3k\} \) (with \( |S| = \frac{1}{3}(3k)(3k + 1) = \frac{1}{2}(9k^2 + 3k) \)), and \( C \) contains at least \( \{3, 6, 9, \ldots, 3k\} \), whose sum is

\[
3 + 6 + \cdots + 3k = 3(1 + 2 + \cdots + k) = 3 \left( \frac{1}{2} k(k + 1) \right) = \frac{3}{2}(3k^2 + 3k),
\]

which is greater than \( \frac{1}{3}|S| = \frac{1}{2}(3k^2 + k) \). Since \( |C| \) must be exactly equal to \( \frac{1}{3}|S| \) if \( n \) is savage, we conclude that \( n \) cannot be divisible by 3. (More intuitively, we can divide the integers from 1 to \( n \) into groups of 3 starting with 1. The largest integer in each group (the multiple of 3) must go into the set \( C \), which means that the sum of all of the elements of \( C \) must be more than \( \frac{1}{3} \) of the total. Therefore, \( n \) cannot be savage.)

Problem 4:

\[
\frac{1}{3}|S| = \frac{1}{3}(1 + 2 + 3 + \cdots + (6k + 5)) = \frac{1}{3}\left(\frac{1}{2}(6k + 5)(6k + 6)\right)
\]

\[
|A_0| = 1 + 5 + 7 + 11 + \cdots + (6k - 1) + (6k + 1) - 3 + 9 + \cdots + (6k + 3)
\]

\[
= (3k + 3)^2 - 3(1 + 3 + \cdots + (2k + 1))
\]

\[
= (3k + 3)^2 - 3(k + 1)^2 = 6k^2 + 12k + 6,
\]

\[
|B_0| = 2 + 4 + 8 + 10 + \cdots + (6k + 2) + (6k + 4) - (6 + 12 + \cdots + 6k)
\]

\[
= 2(1 + 2 + 3 + \cdots + (3k + 2)) - 6(1 + 2 + \cdots + k)
\]

\[
= (3k + 2)(3k + 3) - 3k(k + 1) = 6k^2 + 12k + 6,
\]

\[
|C_0| = 3 + 6 + \cdots + 6k + (6k + 3) = 3(1 + 2 + \cdots + (2k + 1))
\]

\[
= 3\left(\frac{1}{2}(2k + 1)(2k + 2)\right) = 6k^2 + 9k + 3.
\]

Problem 5: In the case where \( n = 6k + 5 \), there are elements in \( A_0 \) which sum to \( k + 1 \) if and only if \( k + 1 \) is equal to 1, 5, 6, 7, 8, 11, 12, 13, 14, or if \( k \geq 16 \). Also, there are elements in \( B_0 \) which sum to \( k + 1 \) if and only if \( k + 1 \) is even, or \( k \) is odd. Therefore, \( n = 6k + 5 \) is savage if and only if \( k \) equals 5, 7, or any odd number greater than or equal to 11.

Ian VanderBurgh
Canadian Mathematics Competition
Centre for Education in Mathematics and Computing
University of Waterloo
Waterloo, Ontario
N2L 3G1
iwvanderburgh@math.uwaterloo.ca