Pólya's Paragon

Triangular Tidbits (Part 2)

Shawn Godin

When last we met, we reviewed the definitions of the trigonometric ratios sine, cosine, and tangent, we looked at the Law of Sines, and we saw how the radius of the circumcircle of a triangle is related to the sides and angles of the triangle. In this issue we will continue looking at triangles, but we will save the trigonometry for the homework.

Sometimes in mathematical problem-solving, a technique or concept turns out to be useful even though it seems totally unrelated to the problem at hand. We will see an example of this. But first we need a definition.

Definition In a triangle, a line segment drawn from a vertex to any point on the opposite side is called a cevian. (Thus, for example, medians are just special cevians.)

Ceva's Theorem In \( \triangle ABC \), if the three cevians \( AX, BY, \) and \( CZ \) are concurrent, then

\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.
\]

Now let’s prove Ceva’s Theorem. Since it involves all kinds of ratios, one might be tempted to attack the problem with vectors. You may do that if you wish, for fun, but we will attack it by bringing in an unexpected concept, namely area. Before that, however, we need a couple of lemmas. (These are essentially theorems, but we call them lemmas to indicate that they are just steps on the way to our main theorem.)

We will write \([ABC]\) to denote the area of a triangle \(ABC\).

Lemma 1 If two triangles have the same base, the ratio of their areas is equal to the ratio of their respective heights.

Proof: Consider the two triangles \(AXY\) and \(BXY\) in the diagram. We have \([AXY] = \frac{1}{2}bh_a\) and \([BXY] = \frac{1}{2}bh_b\).

Thus,

\[
\frac{[AXY]}{[BXY]} = \frac{\frac{1}{2}bh_a}{\frac{1}{2}bh_b} = \frac{h_a}{h_b}.
\]

Lemma 2 If two triangles have the same height, the ratio of their areas is equal to the ratio of their respective bases.

The proof of this is left as homework.
Now we will prove Ceva's Theorem. Let the point of concurrency of the three cevians be \( P \), as in the diagram on the right.

Clearly, \( \triangle ABX \) and \( \triangle AXC \) have a common height. Therefore,

\[
\frac{[ABX]}{[AXC]} = \frac{BX}{XC}.
\]

Similarly, since \( \triangle PBX \) and \( \triangle PXC \) have a common height, we get

\[
\frac{[PBX]}{[PXC]} = \frac{BX}{XC}.
\]

Since we have common ratios, the ratio of the differences is also the same; that is,

\[
\frac{[ABX] - [PBX]}{[AXC] - [PXC]} = \frac{BX}{XC}.
\]

When we look at the diagram, we also see that \([ABX] - [PBX] = [ABP]\) and \([AXC] - [PXC] = [APC]\). Hence,

\[
\frac{[ABP]}{[APC]} = \frac{BX}{XC}.
\]  \hspace{1cm} (1)

By similar arguments, we get

\[
\frac{[BCP]}{[ABP]} = \frac{CY}{YA} \quad \text{and} \quad \frac{[APC]}{[BCP]} = \frac{AZ}{ZB}.
\]  \hspace{1cm} (2)

Using (1) and (2), we have

\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{[ABP]}{[APC]} \cdot \frac{[BCP]}{[APC]} \cdot \frac{[APC]}{[BCP]} = 1.
\]

The converse of Ceva's Theorem is also true; that is, if three cevians \( AX, BY, CZ \) satisfy the equation

\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,
\]

then the cevians are concurrent. Note that this equation is certainly satisfied when the three cevians are medians, because then \( \frac{BX}{XC} = \frac{CY}{YA} = \frac{AZ}{ZB} = 1 \). Thus, we have an easy proof that the medians of a triangle are concurrent.

For homework, use Ceva's Theorem to show that the three altitudes of a triangle are concurrent and that the three angle bisectors are concurrent (don't forget your trigonometry!)