MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le premier mai 2005. Les solutions reçues après cette date ne seront prises en compte que s’il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l’anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l’anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l’Université de Montréal, d’avoir traduit les problèmes.

M151. (Reconsidéré) Proposé par Babis Stergiou, Chalkida, Grèce.

Soit $a$, $b$ et $c$ des nombres réels avec $abc = 1$. Montrer que

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \geq 2(a^2b + b^2c + c^2a).$$

[Ed. Vedula N. Murty, Dover, PA, USA a noté que l’inégalité n’est pas correcte. Son contre-exemple est $a = 2$, $b = -1/2$, $c = -1$. Pour établir cette inégalité, il faut supposer de plus que $a$, $b$ et $c$ sont positifs.]

M169. Proposé par Équipe de Mayhem.

Montrer que

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2003} - \frac{1}{2004} = \frac{1}{1003} + \frac{1}{1004} + \frac{1}{1005} + \cdots + \frac{1}{2003} + \frac{1}{2004}.$$ 

M170. Proposé par Équipe de Mayhem.

Evaluer $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 90^\circ$. 

M171. **Proposé par Neven Jurić. Zagreb, Croatie.**

Il y a 12 pentominos distincts (non-congruents), dont 3 apparaissent dans la figure à droite. Chaque pentomino couvre une surface de 5 carrés unités. (À noter que **pentominoes** est une marque enregistrée de Solomon W. Golomb.)

1. Trouver les 9 pentominos restants.
2. Répartir les 12 pentominos sur les 60 cases numérotées du diagramme à droite, de sorte que chaque pentomino couvre des chiffres dont la somme soit égale à 10.

M172. **Proposé par Mihály Bencze. Brașov, Roumanie.**

Soit $I$ le centre du cercle inscrit d'un triangle $ABC$. Montrer que si l'un des triangles $AIB$, $BIC$, ou $CIA$ est semblable au triangle $ABC$, alors les angles du triangle $ABC$ sont en progression géométrique.

M173. **Proposé par K.R.S. Sastry. Bangalore, Inde.**

On suppose que les diagonales $AC$ et $BD$ d'un parallélogramme $ABCD$ déterminent les angles $\alpha$ et $\beta$ comme indiqué dans la figure ci-dessous.

1. Montrer qu'un tel arrangement des angles est possible si et seulement si les diagonales sont proportionnelles aux côtés.
2. Utiliser la trigonométrie pour exprimer $\beta$ en fonction de $\alpha$.

M174. **Proposé par K.R.S. Sastry. Bangalore, Inde.**

On désigne par $x$ la mesure d'un angle d'un triangle non dégénéré. Déterminer $x$, sachant que

$$
\frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x}.
$$
M175. *Proposé par l’Équipe de Mayhem.*

Un ensemble $S$ est formé de cinq entiers positifs. Montrer qu’il est toujours possible de trouver un sous-ensemble de $S$, contenant trois éléments, de telle sorte que la somme des éléments de ce sous-ensemble soit un multiple de 3.

M151. (Revisited) *Proposed by Babis Stergiou, Chalkida, Greece.*

Let $a$, $b$, $c$ be real numbers with $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \geq 2(a^2b + b^2c + c^2a).$$

[Ed. Vedula N. Murty, Dover, PA, USA has observed that the inequality is incorrect. His counter-example is $a = 2$, $b = -1/2$, $c = -1$. To prove the inequality, it must be assumed that $a$, $b$, and $c$ are positive.]

M169. *Proposed by the Mayhem Staff.*

Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2003} - \frac{1}{2004} = \frac{1}{1003} + \frac{1}{1004} + \frac{1}{1005} + \cdots + \frac{1}{2003} + \frac{1}{2004}.$$ 

M170. *Proposed by the Mayhem Staff.*

Evaluate $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 90^\circ$.

M171. *Proposed by Neven Jurić, Zagreb, Croatia.*

There are 12 distinct (non-congruent) pentominoes, 3 of which are shown to the right. Each pentomino covers an area of 5 square units. (Note: *Pentominoes* is a registered trademark of Solomon W. Golomb.)

1. Find the remaining 9 pentominoes.

2. Arrange all 12 pentominoes on the 60 numbered cells in the diagram to the right, so that each pentomino covers numbers whose sum is 10.


Let $I$ denote the centre of the inscribed circle in triangle $ABC$. Prove that if one of the triangles $AIB$, $BIC$, or $CIA$ is similar to triangle $ABC$, then the angles of triangle $ABC$ are in geometric progression.

Suppose that the diagonals \( AC \) and \( BD \) of a parallelogram \( ABCD \) determine angles \( \alpha \) and \( \beta \) as shown in the diagram below.

1. Prove that such an arrangement of angles is possible if and only if the diagonals are proportional to the sides.
2. Use trigonometry to express \( \beta \) in terms of \( \alpha \).

\[ M174. \text{ Proposed by K.R.S. Sastry, Bangalore, India.} \]

Let \( x \) denote the measure of an angle of a non-degenerate triangle. Determine \( x \), given that

\[
\frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x}.
\]

\[ M175. \text{ Proposed by the Mayhem Staff.} \]

A set \( S \) consists of five positive integers. Show that it is always possible to find a subset of \( S \) containing three elements such that the sum of the elements in the subset is a multiple of 3.

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**Mayhem Solutions**

\[ M106. \text{ Proposed by the Mayhem Staff.} \]

A 4 by 4 square has an area of 16 square units and a perimeter of 16 units. That is, the area and perimeter are numerically equivalent (ignoring units of measurement). Are there any other rectangles with integral dimensions that share this property? If possible, show that you have found all such examples.

*Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

If a rectangle with sides \( x \) and \( y \) has the given property, then we must have \( 2(x + y) = xy \); that is,

\[
y = \frac{2x}{x - 2} = 2 + \frac{4}{x - 2}.
\]
Therefore, the only positive integers \( x \) for which \( y \) is a positive integer are 3, 4, and 6. It follows that a \( 3 \times 6 \) rectangle is the only other one with the given property.

Also solved by Alfans, grade 11 student, SMU Methodist, Palembang, Indonesia; Robert Bilinski, Outremont, QC; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

**M107. Proposed by the Mayhem Staff.**

A right-angled triangle has legs of length \( a \) and \( b \). A circle of radius \( r \) touches the two legs and has its centre on the hypotenuse. Show that

\[
\frac{1}{a} + \frac{1}{b} = \frac{1}{r}.
\]

**Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.**

In \( \triangle ABC \), let \( a \) and \( b \) denote the sides \( BC \) and \( AC \), respectively. Let \( T \) be the point where the circle touches the side \( AC \). Since \( \triangle ABC \) and \( \triangle AOT \) are similar, we have \( \frac{a}{b} = \frac{r}{b-r} \). Then

\[
ab - ar = rb
\]

\[
ab = r(b + a)
\]

\[
\frac{1}{r} = \frac{b + a}{ab} = \frac{1}{a} + \frac{1}{b}.
\]

Also solved by Alfans, grade 11 student, SMU Methodist, Palembang, Indonesia.

**M108. Proposed by the Mayhem Team.**

Given a cube with its eight corners cut off by planes, how many diagonals joining the 24 new 'corners' lie completely inside the cube?

**Solution by Geneviève Lalonde, Massey, ON.**

A diagonal will lie inside the new figure if it joins two vertices that are not on the same face. Each vertex is on 3 faces—a triangular face where the corner of the cube used to be, and two octagonal faces which are remnants of the original square faces. Thus, each vertex is on the same face as 7 vertices from one of the octagonal faces and 6 additional vertices from the other octagonal face (since the two octagonal faces share a side and 2 vertices). The vertices from the triangular face have already been counted among the vertices from the two octagonal faces. Therefore, each vertex is connected by internal diagonals to \( 24 - 7 - 6 - 1 = 10 \) other vertices.

Hence, the total number of internal diagonals of the new figure is \( \frac{1}{2} (24 \cdot 10) = 120 \).
**M109. Proposed by the Mayhem Staff.**

If all plinks are plonks and some plunks are plinks, which of the statements X, Y, Z must be true?

X: All plinks are plonks.
Y: Some plonks are plinks.
Z: Some plunks are not plunks.

*Solution by Geneviève Lalonde, Massey, ON.*

Consider the most general way that these three sets can be related in the diagram on the right. We will use $n(k)$ to represent the number of members in region $k$.

Since all plinks are plonks, we have $n(1) = n(6) = 0$. Similarly, since some plunks are plinks, then $n(6) + n(7) \neq 0$. Putting this together with the first condition, we get $n(7) \neq 0$. Now we look at each of the statements X, Y, Z.

Statement X is equivalent to $n(1) = n(4) = 0$, which may or may not be true since we have no information on region 4.
Statement Y is equivalent to $n(5) + n(7) \neq 0$. Since we know that $n(7) \neq 0$, this statement is true.
Statement Z is equivalent to $n(1) + n(4) \neq 0$. We know that $n(1) = 0$, but we have no information on region 4. This means that Z may or may not be true.

Thus, the only statement that must be true is Y.

*Also solved by Alisan, grade 11 student, SMU Methodist, Palenbaug, Indonesia; and Robert Bilinski, Outremont, QC.*

**M110. Proposed by the Mayhem Staff.**

Given any starting number (other than 1), get a new number by dividing the number 1 larger than your starting number by the number 1 smaller than your starting number. Then do the same with this new number. What happens? Explain!

*Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.*

We obtain the starting number.

Indeed, let $x \neq 1$ be the starting number. In the second step we get the number 

$$
\frac{x + 1}{x - 1} + \frac{1}{x + 1} - 1 = \frac{2x}{x - 1} = \frac{2x}{2} = x,
$$

as claimed.

*Also solved by Alisan, grade 11 student, SMU Methodist, Palenbaug, Indonesia; Robert Bilinski, Outremont, QC; and Laura Steil, student, Samford University, Birmingham, Alabama, USA.*
M111. Proposed by the Mayhem Staff.

A crossword is like a crossword except that the answers are numbers with one digit in each square. What is the sum of all the digits in the solution to this crossword?

CLUES

Across
1. See 3 Down
3. A cube
4. Five times 3 Down

Down
2. A Square
3. Four times 1 Across

Solution by Laura Steil, student, Samford University, Birmingham, Alabama, USA.

Because 1-across, 3-down, and 4-across are related, we can write expressions for these values using a variable $x$. Thus, if we let 3-down be equal to $x$, then we know that 1-across is $x/4$ and 4-across is $5x$. This also tells us that $x$ must be divisible by 4, because 1-across must be an integer.

Since 3-across is a cube and is only one digit, it must be 1 or 8. Also, since 3-down is divisible by 4, the only possibilities are 12, 16, 84, or 88. But we know that 4-across must be 5 times whatever value we use for 3-down. The only case that fits is 84, which makes 4-across equal to 240. Also, since we now know that 3-down is 84, we also know that 1-across is $\frac{1}{4}(84) = 21$.

Now the only value we need is 2-down, and we know that it must be a three-digit square, starting with 1 and ending with 0. The only value that will work is 100.

In summary, we get the solution on the right.

Thus, the sum of the digits that solve the crossword is

$$2 + 1 + 8 + 4 + 2 + 0 + 0 = 17.$$ 

[Ed: Note that all numbers in the puzzle could be zero—a much less interesting solution!]

Also solved by Alfan, grade 11 student, SMU Methodist, Palembang, Indonesia; and Robert Bilinski, Outremont, QC.
M112. Proposed by the Mayhem Staff.

Given that $ABCDEF$ is a regular hexagon and $G$ is the mid-point of $AB$, determine the ratio of the total area of hexagon $ABCDEF$ to the area of triangle $GDE$.

I. Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.

Let $r$ and $a$ be the radius and the apothem of $ABCDEF$ respectively. Applying the Theorem of Pythagoras, we obtain $a = \frac{\sqrt{3}}{2} r$. Then the area of $ABCDEF$ is

$$[ABCDEF] = \frac{6r \cdot a}{2} = \frac{3\sqrt{3} r^2}{2}. \quad (1)$$

Since the altitude of $\triangle GDE$ is $2a$, the area of $\triangle GDE$ is

$$[GDE] = \frac{r \cdot 2a}{2} = \frac{\sqrt{3}}{2} r^2. \quad (2)$$

From (1) and (2), we have

$$\frac{[ABCDEF]}{[GDE]} = \frac{\frac{3\sqrt{3} r^2}{2}}{\frac{\sqrt{3}}{2} r^2} = 3.$$

II. Solution by Robert Bilinski, Outremont, QC.

We give a proof without words showing that $\frac{[ABCDEF]}{[GDE]} = 3$. (Shaded areas in the diagrams are equal.)

Also solved by Alfian, grade 11 student, SMU Methodist, Palembang, Indonesia.

M113. Proposed by Neven Jurjić, Zagreb, Croatia.

The king is on an open $m \times n$ chessboard. On each of its $mn$ cells the total number of possible moves by the king from that cell is written. Find the sum of all these $mn$ numbers.

Solution by Geneviève Lalonde, Massey, ON.

A king can move one space in any direction (horizontally, vertically, or diagonally). We get 4 cases: if the king is in the interior of the board, he has 8 possible moves (a move to each of the 8 surrounding squares); if he is on an
edge but not in a corner, he has 5 possible moves; and if he is in a corner, he
has 3 possible moves. Thus, if we place the number of possible moves from
each position into that position, our board looks like this:

$$\begin{array}{|c|c|c|}
\hline
n \text{ columns} & 3 & 5 \\
\hline
\hline
m \text{ rows} & 3 & 5 \\
\hline
\hline
3 & 5 \\
\hline
5 & 8 \\
\hline
\vdots & \vdots \\
\hline
5 & 8 \\
\hline
3 & 5 \\
\hline
\end{array}$$

The total of all the numbers on the board is
$$8(m - 2)(n - 2) + 5 \left(2(m - 2) + 2(n - 2)\right) + 3(4) = 8mn - 6m - 6n + 4.$$ 

One incorrect solution was received.

**M114. Proposed by Seyamack Jafari, Bandar Imam, Khozestan, Iran.**

In the spiral below prove that
$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \cdot X_1^2 \cdot X_2^2 \cdots X_n^2,$$
where the height of each triangle indicated in the diagram is 1 unit.

**Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina, modified by the editors.**

For \( n = 1, 2, 3, \ldots \), the \( n \)th triangle in the spiral is a right triangle in which one leg has length \( X_n \). Let \( Y_n \) be the length of the other leg. The hypotenuse then has length \( Y_{n+1} \). Note that \( Y_1 = X_0 \).

By the Pythagorean Theorem,
$$Y_{n+1}^2 = X_n^2 + Y_n^2.$$

\[ (1) \]
On the other hand, by writing the area of the triangle in two different ways (using two different bases and corresponding heights), we get
\[ Y_{n+1} = X_n Y_n. \] (2)

Applying mathematical induction to (1), with \( Y_1 = X_0 \), it can be shown that
\[ Y_{n+1}^2 = X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2. \]
Similarly, applying induction to (2), we get
\[ Y_{n+1} = X_0 \cdot X_1 \cdot X_2 \cdots X_n. \]
Therefore,
\[ X_0^2 + X_1^2 + \cdots + X_n^2 = Y_{n+1}^2 = X_0^2 \cdot X_1^2 \cdot X_2^2 \cdots X_n^2. \]

Also solved by Robert Bilinski, Outremont, QC.

**M115. Proposed by the Mayhem Staff.**

The twenty-third term of an arithmetic sequence is three times the value of the fifth term. Find the ratio of the twenty-third term to the first term of the sequence. Express the ratio in the form \( p : q \) where \( p \) and \( q \) are integers.

*Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.*

Let \( \{a_n\} \) be the sequence in the problem, where \( a_n = a_1 + (n - 1)d \). Since \( a_{23} = 3 \cdot a_5 \), we have
\[
22d + a_1 &= 3(4d + a_1) = 12d + 3a_1, \\
10d &= 2a_1, \\
5d &= a_1.
\]

Then
\[
\frac{a_{23}}{a_1} = \frac{22d + a_1}{a_1} = \frac{22d + 5d}{5d} = \frac{27}{5}.
\]

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia; and Robert Bilinski, Outremont, QC.

**M116. Proposed by the Mayhem Staff.**

A polynomial \( f(x) \) satisfies the condition that \( f(5 - x) = f(5 + x) \) for all real numbers \( x \). If \( f(x) = 0 \) has 4 distinct real roots, find the sum of these roots.

*Solution by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.*

Since \( f(x) = 0 \) has 4 distinct real roots and we are only looking at the sum of these roots, we may assume that \( f(x) \) is a quartic polynomial. Thus,
\[ f(x) = ax^4 + bx^3 + cx^2 + dx + e, \]
and the sum of the roots of $f(x)$ is $-b/a$. Now

$$f(5 - x) = ax^4 - x^3(20a + b) + x^2(150a + 15b + c)$$

$$- x(500a + 75b + 10c + d) + 625a + 125b + 25c + 5d + e,$$

$$f(5 + x) = ax^4 + x^3(20a + b) + x^2(150a + 15b + c)$$

$$+ x(500a + 75b + 10c + d) + 625a + 125b + 25c + 5d + e.$$ 

Hence, $20a + b = 0$; that is, $b/a = -20$. Therefore, the sum of the roots of $f(x)$ is 20.

[Ed. Note that $f$ is symmetric about $x = 5$; therefore, the sum of the four real roots must be $4 \times 5 = 20$.]

Also solved by Gustavo Krimer, Universidad CAECE, Buenos Aires, Argentina.

M117. Proposed by the Mayhem Staff.

A person cashes a cheque at the bank. By mistake, the teller pays the number of cents as dollars and the number of dollars as cents. The person spends $3.50 before noticing the mistake, then, on counting the money, finds that the remaining money is exactly double the amount of the cheque. For what amount was the cheque made out?

Solution by Geneviève Lalonde, Massey, ON.

Let $d$ and $c$ represent the number of dollars and cents on the original cheque. Thus, the amount of the cheque, in cents, is $100d + c$. Since we always have less than 100 cents on a cheque, we must have $c < 100$. The information in the problem yields

$$100c + d - 350 = 2(100d + c),$$

$$98c - 350 = 199d,$$

$$14(7c - 25) = 199d.$$ 

Since 199 is prime, we must have $d = 14k$ for some positive integer $k$. Then

$$7c - 25 = 199k.$$

If $k = 1$, we get $c = 32$ and $d = 14$. When we try $k = 2$ and $k = 3$, we do not get an integer solution for $c$. Higher values of $k$ will force $c > 100$, which is not allowed. Thus, the amount on the cheque was $14.32.


You are given a sheet of paper of size $2003 \times 2004$. You are allowed to cut it either horizontally or vertically (that is, parallel to an edge). You wish to obtain $2003 \times 2004$ unit squares. You are not allowed to fold or stack pieces of the paper. Determine the minimum number of cuts required.

Ed.: No solutions have been received. The problem remains open.
Problem of the Month
Ian VanderBurgh, University of Waterloo

This month, we have a “two-for-one” holiday special—two related problems together in one article! Both problems come from the 2002 Australian Mathematics Competition, and both deal with the sum

$$1 + 11 + 111 + \cdots + \underbrace{111 \ldots 111}_{2002 \text{ digits}}.$$  

Let $S$ be the number obtained by performing this summation.

Problem 1. What are the last five digits of $S$?

We can answer this question with some careful accounting work. 

Solution. If we had a very large sheet of paper on which to write down all 2002 numbers in the above sum, we could do the addition just like we were taught in elementary school. In fact, we can do this calculation without the large sheet of paper. We start by adding up the units column. Since this column consists of 2002 digits all of which are 1, our result is 2002. We write down the 2 and carry 200 to the tens column. Adding up the tens column, we get 200 + 2001 = 2201. We write down the 2 and carry 220 to the hundreds column. We could continue in this way, but let’s turn instead to a more interesting method.

We start by splitting up each term in the sum into powers of 10:

$$S = 1 + (1 + 10) + (1 + 10 + 100) + \cdots + (1 + 10 + \cdots + 10^{2001}).$$

Now we note that each term in this sum includes a 1, each term after the first includes a 10, each term after the second includes a 100, and so on. Collecting like powers of 10, we get

$$S = 2002(1) + 2001(10) + 2000(100) + \cdots + 2(10^{2000}) + 1(10^{2001}).$$

Since we are interested in determining only the last five digits of the sum, we can ignore terms that end in at least five zeroes. This leaves us with only four terms:


We could calculate this directly, but we can simplify our task further. We can replace the term 1999(1000) by just 99(1000), since the difference between these is 1900(1000), which ends in five zeroes. Similarly, we can replace 1998(10000) by 8(10000). Thus, the last five digits of $S$ are the same as the last five digits of

$$2002(1) + 2001(10) + 99(1000) + 8(10000) = 2002 + 20010 + 99000 + 80000 = 201012.$$  

Therefore, the last five digits of $S$ are 01012.
That wasn't so bad. However, if we had been asked for the last 100
digits of $S$, neither of the above methods would have been very appealing
(unless we were stuck in a blizzard with nothing to do).

Let us now look at the second problem.

**Problem 2.** How many times does the digit 1 occur in $S$?

Here we must try to be a bit more clever.

Sometimes, when a number consisting of a sequence of 1s appears, it
is useful to recognize that the number is one-ninth of a number consisting
of a sequence of 9s. Well, that doesn't seem totally useful until we recognize
that a number consisting of a sequence of 9s is 1 less than a power of 10...

**Solution.** Using the above idea, we have

\[
S = 1 + 11 + 111 + \cdots + \underbrace{111\ldots111}_{\text{2002 digits}}
\]

\[
= \frac{1}{9}(10 - 1) + \frac{1}{9}(10^2 - 1) + \frac{1}{9}(10^3 - 1) + \cdots + \frac{1}{9}(10^{2002} - 1)
\]

\[
= \frac{1}{9}\left((10 + 10^2 + 10^3 + \cdots + 10^{2002}) - 2002\right)
\]

\[
= \frac{1}{9}\left(\underbrace{111\ldots111000000}_{\text{2002 digits}} + 11110 - 2002\right)
\]

\[
= \frac{1}{9}\left(\underbrace{111\ldots111000000}_{\text{1998 digits}} + 9108\right) = \frac{1}{9}\left(\underbrace{111\ldots11109108}_{\text{1998 digits}}\right).
\]

We have reduced the problem so that now we just have to divide a very large
number by 9. (At this stage, it is worth checking that this very large number
is actually divisible by 9. The sum of its digits is $1998 + 9 + 1 + 8 = 2016$,
which is divisible by 9. Therefore, the number itself is divisible by 9. That's
a relief!)

We could start doing long division and hope to find a pattern, or we
could notice that the integer $11111111$ is divisible by 9. Using a calculator
(or a napkin), we get $11111111 = 9(12345679)$. In our very large number
above, we group the 1998 leading 1s into blocks of nine 1s. Thus, we get

\[
S = \frac{1}{9}\left(111111111(10^5 + 10^{14} + \cdots + 10^{1994}) + 9108\right)
\]

\[
= \frac{1}{9}(111111111)(10^5 + 10^{14} + \cdots + 10^{1994}) + 1012
\]

\[
= 12345679(10^5 + 10^{14} + \cdots + 10^{1994}) + 1012
\]

\[
= 12345679012345679\cdots01234567901012,
\]

where the block "12345679" occurs 222 times (once without a leading 0, 221
times with a leading 0). Therefore, the digit 1 occurs 224 times in $S$. 
Pólya’s Paragon

Triangular Tidbits (Part 2)

Shawn Godin

When last we met, we reviewed the definitions of the trigonometric ratios sine, cosine, and tangent, we looked at the Law of Sines, and we saw how the radius of the circumcircle of a triangle is related to the sides and angles of the triangle. In this issue we will continue looking at triangles, but we will save the trigonometry for the homework.

Sometimes in mathematical problem-solving, a technique or concept turns out to be useful even though it seems totally unrelated to the problem at hand. We will see an example of this. But first we need a definition.

**Definition** In a triangle, a line segment drawn from a vertex to any point on the opposite side is called a cevian. (Thus, for example, medians are just special cevians.)

**Ceva’s Theorem** In \( \triangle ABC \), if the three cevians \( AX \), \( BY \), and \( CZ \) are concurrent, then

\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.
\]

Now let’s prove Ceva’s Theorem. Since it involves all kinds of ratios, one might be tempted to attack the problem with vectors. You may do that if you wish, for fun, but we will attack it by bringing in an unexpected concept, namely area. Before that, however, we need a couple of lemmas. (These are essentially theorems, but we call them lemmas to indicate that they are just steps on the way to our main theorem.)

We will write \([ABC]\) to denote the area of a triangle \( ABC \).

**Lemma 1** If two triangles have the same base, the ratio of their areas is equal to the ratio of their respective heights.

**Proof:** Consider the two triangles \( AXY \) and \( BXY \) in the diagram. We have

\[
[AXY] = \frac{1}{2}bh_a \quad \text{and} \quad [BXY] = \frac{1}{2}bh_b.
\]

Thus,

\[
\frac{[AXY]}{[BXY]} = \frac{\frac{1}{2}bh_a}{\frac{1}{2}bh_b} = \frac{h_a}{h_b}.
\]

**Lemma 2** If two triangles have the same height, the ratio of their areas is equal to the ratio of their respective bases.

The proof of this is left as homework.
Now we will prove Ceva’s Theorem. Let the point of concurrency of the three cevians be \( P \), as in the diagram on the right.

Clearly, \( \triangle ABX \) and \( \triangle AXC \) have a common height. Therefore,
\[
\frac{[ABX]}{[AXC]} = \frac{BX}{XC}.
\]

Similarly, since \( \triangle PBX \) and \( \triangle PXC \) have a common height, we get
\[
\frac{[PBX]}{[PXC]} = \frac{BX}{XC}.
\]
Since we have common ratios, the ratio of the differences is also the same; that is,
\[
\frac{[ABX] - [PBX]}{[AXC] - [PXC]} = \frac{BX}{XC}.
\]

When we look at the diagram, we also see that \( [ABX] - [PBX] = [ABP] \) and \( [AXC] - [PXC] = [APC] \). Hence,
\[
\frac{[ABP]}{[APC]} = \frac{BX}{XC} \tag{1}
\]

By similar arguments, we get
\[
\frac{[BCP]}{[ABP]} = \frac{CY}{YA} \quad \text{and} \quad \frac{[APC]}{[BCP]} = \frac{AZ}{ZB} \tag{2}
\]

Using (1) and (2), we have
\[
\frac{BX \cdot CY \cdot AZ}{XC \cdot YA \cdot ZB} = \frac{[ABP]}{[APC]} \cdot \frac{[BCP]}{[APC]} \cdot \frac{[APC]}{[BCP]} = 1. \]

The converse of Ceva’s Theorem is also true; that is, if three cevians \( AX, BY, CZ \) satisfy the equation
\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,
\]
then the cevians are concurrent. Note that this equation is certainly satisfied when the three cevians are medians, because then \( \frac{BX}{XC} = \frac{CY}{YA} = \frac{AZ}{ZB} = 1 \).
Thus, we have an easy proof that the medians of a triangle are concurrent.

For homework, use Ceva’s Theorem to show that the three altitudes of a triangle are concurrent and that the three angle bisectors are concurrent (don’t forget your trigonometry!)
Mayhem Year End Wrap Up

Shawn Godin

December already! How the time has flown. We have continued to see increased contribution to Mayhem by the readers, and for that I say a heartfelt “Thank you”. Mayhem is really a function of its readers, and their contribution is invaluable to making the journal what it is. We hope that in 2005 we will continue to gain more readers (and contributors), and still hear from our regulars.

At this point I must thank the people without whom I would have had a nervous breakdown long ago! First, and foremost, I would like to thank Mayhem Assistant Editor, JOHN GRANT McLoughlin. John has continued to supply us with very interesting problems and has provided me with guidance and support at every turn.

Next I must thank those who are leaving us and won’t be returning in 2005. The first is PAUL OTTAWAY, who started the feature we call Pólya’s Paragon. His informal articles were a great addition to Mayhem. Paul is continuing his graduate studies. Paul, your contributions will be missed. Also leaving us is LARRY RICE, who assisted with the Mayhem solutions. Thanks, Larry. We wish all the best to Paul and Larry in their future endeavours.

We have a new face: IAN VANDERBURGH joined us in September and has resurrected the Problem of the Month. Ian’s columns have really added to Mayhem. I look forward to more of his columns and to working with him in 2005.

I also need to thank those people who have been so helpful behind the scenes: RICHARD HOSHINO, DAN MACKINNON, BRUCE SHAWYER, and GRAHAM WRIGHT. They were always there when I was in need, and they always came through. Thanks everyone!

All the best of the season to all our readers and contributors! I hope that you have a great year in 2005 and that you will help us continue to make CRUX with MAYHEM grow and improve. Happy problem solving! We’ll see you in 2005.