SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


Let $Q$ be a square of side length 1, and let $S$ be a set consisting of a finite number of squares such that the sum of their areas is $\frac{1}{2}$.

Prove that the set $S$ can be packed inside the square $Q$.

Editor: The theorem in the featured solution in [2004 : 252–253] is flawed. We present here a different solution.

Solution by Li Zhou, Polk Community College. Winter Haven. FL. USA, adapted by the editors.

Lemma. Let $S = \{S_i\}_{i=1}^n$ be a set of squares with respective side lengths $x_1 \geq x_2 \geq \cdots \geq x_n$. Let $R$ be an $a \times x_1$ rectangle such that $x_1 \leq a$. If $S$ cannot be packed inside $R$, then there exists an integer $k$ with $1 \leq k < n$ such that $\{S_i\}_{i=1}^k$ can be packed inside $R$ to cover at least half the area of $R$.

Proof. The proof is by induction on the number of squares $n$. If $n = 1$, then $S = \{S_1\}$ can be packed inside $R$ and the lemma is vacuously true. Fix $n \geq 2$, and assume that the lemma is true whenever the number of squares is less than $n$. Let $S = \{S_i\}_{i=1}^n$ be a set of squares satisfying the hypothesis of the lemma. Pack $S_1$ into one end of $R$ (as shown in the figure on the next page).

Now we give a recursive construction that defines a positive integer $k$. Let $k_0 = 1$. If $x_2 > a - x_1$, then we take $k = k_0$. Otherwise, we consider the rectangle $R_1$ of dimensions $x_2 \times x_1$ that is immediately beside $S_1$ in $R$. By the induction hypothesis, there exists an integer $k_1$ with $2 \leq k_1 < n$ such that $\{S_i\}_{i=2}^{k_1}$ can be packed inside $R_1$ to cover at least half the area of $R_1$. If $x_{k_1+1} > a - (x_1 + x_2)$, then we take $k = k_1$. Otherwise, we consider the rectangle $R_2$ of dimensions $x_{k_1+1} \times x_1$ that is uncovered and immediately beside $R_1$ in $R$. By the induction hypothesis, there exists an integer $k_2$ with $k_1 + 1 < k_2 < n$ such that $\{S_i\}_{i=k_1+1}^{k_2}$ can be packed inside $R_2$ to cover at least half the area of $R_2$.

We continue in this manner until the process terminates, which must happen after a finite number of iterations, say $m$ iterations. We then have $k = k_m$, and the set of squares $\{S_i\}_{i=1}^k$ is packed inside $R$. We claim that these squares cover at least half the area of $R$. 

Let $U$ be the uncovered rectangle that is immediately beside $R_m$ in $R$, and let $w$ be the width of $U$ (as shown in the figure). By our construction, $w < x_{k_m+1}$, and hence, $w < x_1$. Therefore, the area of square $S_1$ is greater than the area of $U$. Moreover, by our construction, each of the rectangles $R_1, R_2, \ldots, R_m$ is at least half covered by squares. It follows that our claim is true, and the lemma is proved.

**Theorem.** Let $S = \{S_i\}_{i=1}^n$ be a set of squares with respective side lengths $x_1 \geq x_2 \geq \cdots \geq x_n$. Let $Q$ be an $a \times b$ rectangle such that $x_1 \leq a \leq b$ and $\sum_{i=1}^n x_i^2 \leq \frac{1}{2} ab$. Then $S$ can be packed inside $Q$.

**Proof.** The proof is by induction on the number of squares $n$. The theorem is clearly true for $n = 1$. Fix $n \geq 2$, and assume that the theorem is true whenever the number of squares is less than $n$. Let $S = \{S_i\}_{i=1}^n$ be a set of squares satisfying the hypothesis of the theorem. Embed $Q$ in the $xy$-plane as $[0, a] \times [0, b]$. If $S$ can be packed inside the rectangle $R = [0, a] \times [0, x_1]$ (a subset of $Q$), then we are done. Otherwise, the lemma yields an integer $k$ with $1 \leq k < n$ such that $\{S_i\}_{i=1}^k$ can be packed inside $R$ to cover at least half the area of $R$. Hence, $\sum_{i=k+1}^n x_i^2 \leq \frac{1}{2} a(b - x_1)$. Also,

$$(x_1 + x_{k+1})^2 \leq 2(x_1^2 + x_{k+1}^2) \leq ab \leq b^2.$$  

Thus, $x_{k+1} \leq b - x_1$. Also, $x_{k+1} \leq x_1 \leq a$. By the induction hypothesis, we see that $\{S_i\}_{i=k+1}^n$ can be packed inside $[0, a] \times [x_1, b]$, completing the proof.

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**2881.** [2003 : 466] Proposed by Christopher J. Bradley, Bristol, UK.

A set of four non-negative integers $a$, $b$, $c$, $d$ are said to have the property $P$ if all of $bc + db + cb$, $ac + cd + da$, $ab + bd + da$, $ab + bc + ca$ are perfect squares.

The sequence $\{u_n\}$ is defined by $u_1 = 0$, $u_2 = 1$, $u_3 = 1$, $u_4 = 4$ and, for $n \geq 1$,

$$u_{n+4} = 2u_{n+3} + 2u_{n+2} + 2u_{n+1} - u_n. $$

Prove that the set $\{u_n, u_{n+1}, u_{n+2}, u_{n+3}\}$ has the property $P$ for all $n \geq 1$. 
Solution by Kathleen E. Lewis. SUNY Oswego. Oswego. NY. USA.

This problem can be reduced to Problem 2802 [2003: 44; 2004: 50] by showing that the sequence defined here also satisfies the recurrence relation

\[ u_{n+3} = u_n + u_{n+1} + u_{n+2} + 2\sqrt{u_n u_{n+1} + u_n u_{n+2} + u_{n+1} u_n + 2} \]

for all \( n \geq 1 \). In this case, \( u_n, u_{n+1}, u_{n+2}, \) and \( u_{n+3} \) play the roles of the integers \( p, q, r, \) and \( s, \) respectively, of Problem 2802.

We will show by induction that the sequence satisfies this alternative recurrence relation. First note that

\[ u_4 = 4 = 0 + 1 + 1 + 2\sqrt{0 + 0 + 1}; \]

Thus, the relation is satisfied at the first stage. Now suppose that

\[ u_{k+3} = u_k + u_{k+1} + u_{k+2} + 2\sqrt{u_k u_{k+1} + u_k u_{k+2} + u_{k+1} u_k + 2} \]

for some \( k \geq 1 \). In the notation of problem 2802, we will use \( m \) for the quantity \( \sqrt{u_k u_{k+1} + u_k u_{k+2} + u_{k+1} u_k + 2} \). We want to show that

\[ u_{k+4} = u_k + u_{k+1} + u_{k+2} + u_{k+3} + 2\sqrt{u_k u_{k+1} + u_k u_{k+2} + u_{k+1} u_k + 2} \]

\[ + 2\sqrt{u_k u_{k+1} + u_k u_{k+2} + u_{k+1} u_k + 2} = u_k + u_{k+1} + u_{k+2} + u_{k+3} + 2u_{k+4} \]  

(1)

Consider the quantity under the radical. By the induction hypothesis,

\[ u_{k+1} u_{k+2} + u_{k+1} u_{k+3} + u_{k+2} u_{k+3} \]

\[ = u_k + u_{k+1} + u_{k+2} + u_{k+1} u_{k+2} + u_{k+2} u_{k+3} \]

\[ = m^2 + 2m(u_k + u_{k+1} + u_{k+2} + u_{k+1} u_k + 2) \]

\[ = (m + u_k + u_{k+1} + u_{k+2})^2. \]

Therefore, the quantity on the right side of (1) is equal to

\[ u_{k+1} u_{k+2} + u_{k+1} u_{k+3} + 2(u_k + u_{k+1} + u_{k+2}) = u_{k+3} + 3u_{k+2} + 3u_{k+1} + 2u_k. \]

But, by the induction hypothesis, \( 2m = u_{k+3} - u_k = u_{k+2} - u_{k+1} \); whence, the right side of (1) is equal to \( 2u_{k+3} + 2u_{k+2} + 2u_{k+1} - u_k \), which is the recursive definition of \( u_{k+4} \).

This shows that the sequence satisfies the alternative recursive relation of problem 2802 for all \( n \geq 1 \); hence, the set \( \{u_n, u_{n+1}, u_{n+2}, u_{n+3}\} \) has the property \( P \) for all \( n \geq 1 \).

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Bataille points out that “a solution can be found in the interesting article by the [proposer] of the problem: Heron Triangles and Touching Circles, Math. Gazette 87(2003) No 508, pp 36–41.”

If \( x \in (0, \frac{\pi}{2}) \), \( 0 \leq a \leq b \), and \( 0 \leq c \leq 1 \), prove that

\[
\left( \frac{c + \cos x}{c + 1} \right)^b < \left( \frac{\sin x}{x} \right)^a.
\]

Solution by Michel Bataille, Rouen, France.

We add the hypothesis \((a, b) \neq (0, 0)\), since the inequality is false for \(a = b = 0\). If \(a = 0\), then \(b > 0\), and the function \(f(x) = t^b\) is strictly increasing on \((0, \infty)\). Since \(0 < \frac{c + \cos x}{c + 1} < 1\), we have

\[
\left( \frac{c + \cos x}{c + 1} \right)^b < 1 = \left( \frac{\sin x}{x} \right)^0.
\]

Suppose now that \(0 < a \leq b\). Letting \(r = \frac{b}{a}\), the proposed inequality becomes \(\left( \frac{c + \cos x}{c + 1} \right)^r < \frac{\sin x}{x}\). Since \(r \geq 1\), we see that

\[
\left( \frac{c + \cos x}{c + 1} \right)^r \leq \frac{c + \cos x}{c + 1}.
\]

Hence, it suffices to prove that \(\frac{c + \cos x}{c + 1} < \frac{\sin x}{x}\). But for a fixed \(\alpha \in (0, 1)\), the function \(f(t) = \frac{t + \alpha}{t + 1} = 1 - \frac{1 - \alpha}{t + 1}\) is clearly increasing on \([0, 1]\). Thus, it suffices to show that \(\frac{1 + \cos x}{2} < \frac{\sin x}{x}\), which is equivalent to

\[
x \cos^2 \left( \frac{x}{2} \right) < 2 \sin \left( \frac{x}{2} \right) \cos \left( \frac{x}{2} \right), \text{ or } \frac{x}{2} < \tan \left( \frac{x}{2} \right).
\]

The last inequality is well known to be true for \(x \in (0, \pi)\) and our proof is complete.

Also solved by CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD L. HESS, Rancho Palos Verdes, CA, USA; WALTER JANOUS, Ussulinengymnasium, Innsbruck, Austria; JUAN-BOSCO ROMERO MARQUEZ, Universidad de Valladolid, Valladolid, Spain; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Besides Bataille, four other solvers noticed and stated that the proposed inequality is not valid without additional constraints on \(a\) and \(b\). Guersenzaig assumed that \(b \neq 0\). Hess and Janous excluded the case \(a = b = 0\). Zhou assumed that \(a > 0\) or \(a < b\). It is easy to see that all these hypotheses are equivalent to that used by Bataille.

2883. [2003 : 466] Proposed by Šefket Arslanagić and Faruk Zejnullahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Suppose that \(x, y, z \in [0, 1)\) and that \(x + y + z = 1\). Prove that

\[
\sqrt{\frac{xy}{z + xy}} + \sqrt{\frac{yz}{x + yz}} + \sqrt{\frac{zx}{y + zx}} \leq \frac{3}{2}.
\]
Essentially the same solution by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Vasile Cîrtoaje, University of Ploiești, Romania; Titu Zvonaru, Bucharest, Romania; and the proposers.

Since \( z + xy = z(x + y + z) + xy = (x + z)(y + z) \), the AM–GM Inequality yields

\[
\sqrt{\frac{xy}{z + xy}} = \sqrt{\frac{xy}{(x + z)(y + z)}} \leq \frac{1}{2} \left( \frac{x}{x + z} + \frac{y}{y + z} \right).
\]

Hence,

\[
\sum_{cyc} \sqrt{\frac{xy}{z + xy}} \leq \frac{1}{2} \sum_{cyc} \left( \frac{x}{x + z} + \frac{y}{y + z} \right) = \frac{1}{2} \left( \frac{x + y}{x + y} + \frac{y + z}{y + z} + \frac{z + x}{z + x} \right) = \frac{3}{2}.
\]

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; VASILE CÎRTOAJE, University of Ploiești, Romania (a second solution); CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALThER JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA (two solutions); ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUEFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers (a second solution).

Though it is easy to show that the equality holds if and only if \( x = y = z = \frac{1}{3} \), only Bornsztein, Specht, Woo, Zvonaru, and the proposer explicitly mentioned this, with Zvonaru being the only one who actually gave a full proof.

Zhao commented that if we replace \( x, y, z \) with \( \frac{1}{\sqrt{a^2}}, \frac{1}{\sqrt{b^2}}, \text{ and } \frac{1}{\sqrt{c^2}} \), respectively (assuming \( x, y, z > 0 \) since the inequality is trivial if any of \( x, y, z \) is zero), then the constraint becomes \( a + b + c = abc \) and the inequality becomes

\[
\frac{1}{\sqrt{1 + a^2}} + \frac{1}{\sqrt{1 + b^2}} + \frac{1}{\sqrt{1 + c^2}} \leq \frac{3}{2},
\]

which is well known and appeared in the 1998 Korean Math Olympiad.

\[2884. \quad [2003 : 467] \text{ Proposed by Niels Bejegaard, Copenhagen, Denmark.} \]

Suppose that \( a, b, c \) are the sides of a non-obtuse triangle. Give a geometric proof and hence, a geometric interpretation of the inequality

\[ a + b + c \geq \sum_{cyc} \sqrt{a^2 + b^2 - c^2}. \]

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Lemma. Suppose that \( A \) and \( B \) are two points outside a circle centred at \( O \) such that \( AB \) intersects the circle. If \( X \) and \( Y \) are two points on the circle such that \( AX \) and \( BY \) are tangents, then \( AB \geq AX + BY \).
Proof: Since the two tangents from a point outside the circle have the same length, we may assume that the points \( X \) and \( Y \) are in the half-plane (with respect to the line \( AB \)) which does not contain the point \( O \) (the point \( O \) can lie on \( AB \)). Let the lines \( OX \) and \( OY \) intersect the line \( AB \) at points \( P \) and \( Q \), respectively. Then the points \( A, P, Q, \) and \( B \) are on the line \( AB \), in that order. We know that the length of any side of a right triangle does not exceed the length of the hypotenuse. Since \( AXP \) and \( BYQ \) are right triangles, we have

\[
AX + BY \leq AP + BQ \leq AB.
\]

Equality occurs if and only if the line \( AB \) is tangent to the circle. This completes the proof of the lemma.

Now, let \( ABC \) be an acute triangle, let \( M \) be the mid-point of \( BC \), and let \( D \) be the foot of the altitude from the vertex \( A \). Let \( Y \) and \( Z \) be points on the nine-point circle such that \( BY \) and \( CZ \) are tangents. Using the Cosine Law, we obtain

\[
\sqrt{a^2 + b^2 - c^2} + \sqrt{a^2 - b^2 + c^2} = \sqrt{2ab \cos C + \sqrt{2ac \cos B}} = \sqrt{2 \cdot CB \cdot CD + \sqrt{2 \cdot BC \cdot BD}} = 2 \sqrt{CM \cdot CD + 2BD \cdot BM} = 2 \cdot CZ + 2 \cdot BY \leq 2a,
\]

where the inequality follows from our lemma. Thus,

\[
\sqrt{a^2 + b^2 - c^2} + \sqrt{a^2 - b^2 + c^2} \leq 2a.
\]

Similarly,

\[
\sqrt{b^2 + c^2 - a^2} + \sqrt{b^2 - c^2 + a^2} \leq 2b
\]

and

\[
\sqrt{c^2 + a^2 - b^2} + \sqrt{c^2 - a^2 + b^2} \leq 2c.
\]

The proposed inequality follows by adding the last three inequalities. Equality holds if and only if the triangle is equilateral.

We note that the proposed inequality is also true for a right triangle. If, say, \( c^2 = a^2 + b^2 \), then the inequality becomes \( a^2 + b^2 \geq (\sqrt{2} - 1)ab \), which is true, since \( a^2 + b^2 \geq 2ab > (\sqrt{2} - 1)ab \).

A geometric interpretation is as follows: the perimeter of a non-obtuse triangle is always greater than or equal to the sum of the lengths of all six tangents from the vertices of the triangle to its nine-point circle, with equality if and only if the triangle is equilateral.

The same geometric interpretation was found by MICHEL BATAILLE, Rouen, France; and the proposer, PETER Y. WOO, Biola University, La Mirada, CA, USA found a different geometric interpretation (it is lengthier and requires additional constructions; thus, we do not include it here). Each of these three solvers has also submitted a proof of the inequality. The following solvers have only proved the inequality: ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania (3 solutions).

The moderator of this problem does not believe that there is a good definition of "geometric proof"; hence, the list of solvers includes those who have submitted any valid proof.

Let $O$ and $I$ be the circumcentre and the incentre, respectively, of triangle $ABC$. Denote the cevians through $O$ by $AA', BB'$, and $CC'$, and those through $I$ by $AD, BE, CF$. The sides of the triangle are $a, b,$ and $c$.

1. If \( \frac{AA'}{a} = \frac{BB'}{b} = \frac{CC'}{c} \), prove that \( \triangle ABC \) is equilateral.

2. If \( \frac{AD}{a} = \frac{BE}{b} = \frac{CF}{c} \), prove that \( \triangle ABC \) is equilateral.

3. Give an answer to Sastry's question [1998: 280]: For an internal point $P$ and its corresponding cevians $AD, BE, CF$, with \( \frac{AD}{a} = \frac{BE}{b} = \frac{CF}{c} \),
prove or disprove that \( \triangle ABC \) is equilateral.

Solution by Toshio Seimiya, Kawasaki, Japan.

1. Assume first that $b > c$. Then $\angle ABC < \angle ACB$.
Since $OB = OC$, we have $\angle OBC = \angle OCB$. Let $S$ be a point on the side $AC$ such that $\angle SBC = \angle ACB$, and let $T$ be the point of intersection of $BS$ and $OC'$. Since $\angle B'BC = \angle TCB$ and $\angle B'CB = \angle TBC$, we have $\triangle B'BC \cong \triangle TCB$. Thus, $BB' = CT < CC'$.

Since $AB < AC$ and $BB' < CC'$, it follows that $\frac{BB'}{AC} < \frac{CC'}{AB}$; that is, $\frac{BB'}{b} < \frac{CC'}{c}$. This contradicts $\frac{BB'}{b} = \frac{CC'}{c}$.

If $b < c$, we have a similar contradiction. Therefore, $b = c$.

Similarly, from $\frac{AA'}{a} = \frac{BB'}{b}$, it follows that $a = b$. Hence, $a = b = c$, so that \( \triangle ABC \) is equilateral.

2. Assume first that $b > c$. Then $\angle ABC > \angle ACB$.
Hence, $\angle ABE = \angle EBC > \angle ACF = \angle FCB$.
Let $G$ be the point on the segment $AE$ such that $\angle GBE = \angle ACF$, and let $H$ be the point where $BG$ intersects $CF$. Then $H$ is a point on the segment $FI$, and hence, $CH < CF$. Since $\triangle GBE \sim \triangle GCH$, we have $\frac{BE}{CH} = \frac{BG}{CG}$.

Since $\angle GBE = \angle ACH$ and $\angle EBC > \angle HCB$, it follows that $\angle GBC > \angle GCB$. Then $BG < CG$; that is $\frac{BG}{CG} < 1$. Hence, $\frac{BE}{CH} < 1$, and thus, $BE < CH < CF$.

Since $AB < AC$ and $BE < CF$, it follows that $\frac{BE}{AC} < \frac{CF}{AB}$; that is $\frac{BE}{b} < \frac{CF}{c}$. This contradicts $\frac{BE}{c} = \frac{CF}{c}$.

If $b < c$, we have a similar contradiction. Therefore, $b = c$.

\[ \triangle ABC \]
Similarly, from \( \frac{AD}{a} = \frac{BE}{b} \), it follows that \( a = b \). Hence, \( a = b = c \), which shows that \( \triangle ABC \) is equilateral.

3. The conclusion that \( \triangle ABC \) is equilateral is not true. We give a counterexample as follows.

Let acute angle \( \theta \) be such that \( \frac{1}{2} < \tan \theta < \frac{1}{\sqrt{3}} \). Then \( 0 < \theta < \frac{\pi}{2} \), and \( \sin \theta < \frac{1}{2} \). Construct isosceles triangle \( \triangle ABC \) with \( AB = AC \) and \( \angle BAC = 2\theta \). We have \( BC = a, \ C A = b \) and \( AB = C \), where \( b = c \).

Let \( D \) be the mid-point of \( BC \). Then, \( \angle BAD = \angle CAD = \theta \) and \( AD \perp BC \). Thus,

\[
\frac{AD}{BC} = \frac{AD}{2DC} = \frac{1}{2 \tan \theta} < 1.
\]

Hence, \( AD < BC \). Since

\[
2 \tan \theta \sin 2\theta = \frac{2 \sin \theta}{\cos \theta} \times 2 \sin \theta \cos \theta = 4 \sin^2 \theta < 1,
\]

we have

\[
\sin 2\theta < \frac{1}{2 \tan \theta} = \frac{AD}{BC} < 1.
\]

Let \( H \) be the foot of the perpendicular from \( B \) to \( AC \). Then

\[
BH = AB \sin 2\theta < AB \times \frac{AD}{BC} < AB.
\]

There must be a point \( E \) on the segment \( AH \) such that \( BE = AB \times \frac{AD}{BC} \).

Now

\[
\frac{AD}{BC} = \frac{BE}{AB} = \frac{BE}{AC}.
\]

Let \( P \) be the point of intersection of \( BE \) and \( AD \), and let \( F \) be the point of intersection of \( CP \) with \( AB \). Since \( AD \) is the perpendicular bisector of \( BC \), we have \( CF = BE \). Thus, \( \frac{AD}{AC} = \frac{AD}{BC} = \frac{CF}{AB} \); that is,

\[
\frac{AD}{a} = \frac{BE}{b} = \frac{CF}{c}.
\]

This relation holds for the three concurrent cevians \( AD, BE, \) and \( CF \), but \( \triangle ABC \) is not equilateral.

Remark: The following theorem can easily be proved.

**Theorem.** Suppose that \( \triangle ABC \) is equilateral and that \( P \) is an interior point. Suppose that \( AD, BE, \) and \( CF \) are three cevians through \( P \).

If \( \frac{AD}{BC} = \frac{BE}{CA} = \frac{CF}{AB} \), then \( P \) is the circumcentre ( incentre) of \( \triangle ABC \).

Parts 1 and 2 above are the converses of this theorem.
Proposed by Panos E. Tsaoussoglou, Athens, Greece.

If \( a, b, c \) are positive real numbers such that \( abc = 1 \), prove that

\[
ab^2 + bc^2 + ca^2 \geq ab + bc + ca.
\]

I. Nearly identical solutions Chip Curtis, Missouri Southern State College, Joplin, MO, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON, and Li Zhou, Polk Community College, Winter Haven, FL, USA.

By the AM–GM Inequality, \( \frac{ab^2}{3} + \frac{2bc^2}{3} \geq \sqrt[3]{(ab^2)(bc^2)} = bc \), and similarly, \( \frac{bc^2}{3} + \frac{2ca^2}{3} \geq ca \) and \( \frac{ca^2}{3} + \frac{2ab^2}{3} \geq ab \). Adding the three inequalities completes the proof.

II. Solution by Christopher J. Bradley, Bristol, UK.

Since \( abc = 1 \), the inequality is equivalent to

\[
\frac{b}{c} + \frac{c}{a} + \frac{a}{b} \geq \frac{1}{c} + \frac{1}{a} + \frac{1}{b}
\]

Applying the Cauchy–Schwarz Inequality to the vectors \( \left[ \sqrt[3]{\frac{b}{c}}, \sqrt[3]{\frac{c}{a}}, \sqrt[3]{\frac{a}{b}} \right] \) and \( \left[ \frac{1}{\sqrt[3]{b}}, \frac{1}{\sqrt[3]{c}}, \frac{1}{\sqrt[3]{a}} \right] \), we have \( \left( \frac{b}{c} + \frac{c}{a} + \frac{a}{b} \right) \left( \frac{1}{b} + \frac{1}{c} + \frac{1}{a} \right) \geq \left( \frac{1}{c} + \frac{1}{a} + \frac{1}{b} \right)^2 \), from which (1) follows.

III. Similar solutions by Arkady Alt. San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Joe Howard, Portales, NM, USA; and Titu Zvonaru, Bucharest, Romania.

Since \( abc = 1 \), there are positive real numbers \( x, y, z \) such that \( a = \frac{x}{y} \), \( b = \frac{y}{z} \), and \( c = \frac{z}{x} \). The given inequality is then equivalent to

\[
\frac{xy}{z^2} + \frac{yz}{x^2} + \frac{zx}{y^2} \geq \frac{x}{z} + \frac{y}{x} + \frac{z}{y},
\]

or

\[
x^3y^3 + y^3z^3 + z^3x^3 \geq x^3y^2z + xyz^2 + x^2yz^2.
\]

Inequality (2) follows from Muirhead’s Theorem on majorization since the vector \([3, 3, 0]\) majorizes the vector \([3, 2, 1]\). Note that equality holds if and only if \( x = y = z \); that is, if and only if \( a = b = c = 1 \). Alternately, the AM–GM Inequality could be applied to obtain

\[
x^3y^3 + 2y^3z^3 \geq 3\sqrt[3]{(x^3y^3)(y^3z^3)} = 3xyz^2.
\]

Similarly, \( y^3z^3 + 2z^3x^3 \geq 3x^2yz^3 \) and \( z^3x^3 + 2x^3y^3 \geq 3x^3y^2z \). Adding these three inequalities, (2) follows.
Also solved by MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAE, University of Ploiești, Romania; NATALIO H. GUERENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Urnsulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; MARCELO RUFINO de OLIVEIRA, Belém, Brazil; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2887. [2003 : 468; corrected 2004 : 38] Proposed by Vedula N. Murty, Dover, PA, USA.

If $a$, $b$, $c$ are the sides of $\triangle ABC$ in which at most one angle exceeds $\frac{\pi}{3}$, and if $R$ is its circumradius, prove that

$$a^2 + b^2 + c^2 \leq 6R^2 \sum_{\text{cyclic}} \cos A.$$  

Solution by Joe Howard, Portales, NM, USA.

We use the following well-known facts (see [1]):

$$\prod_{\text{cyclic}} \cos A = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2},$$  \hspace{1cm} (1)

$$\sum_{\text{cyclic}} \cos A = \frac{R + r}{R},$$  \hspace{1cm} (2)

$$\sum_{\text{cyclic}} \cos B \cos C = \frac{r^2 + s^2 - 4R^2}{4R^2},$$  \hspace{1cm} (3)

$$a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr,$$  \hspace{1cm} (4)

$$R \geq 2r \quad \text{(Euler’s Inequality)}.$$  \hspace{1cm} (5)

Under the given condition, we must have $\sum_{\text{cyclic}} (2 \cos A - 1) \leq 0$, which expands to

$$8 \prod_{\text{cyclic}} \cos A + 2 \sum_{\text{cyclic}} \cos A \leq 1 + 4 \sum_{\text{cyclic}} \cos B \cos C.$$

Using equations (1), (2) and (3), we obtain

$$8 \left( \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2} \right) + 2 \left( \frac{R + r}{R} \right) \leq 1 + 4 \left( \frac{r^2 + s^2 - 4R^2}{4R^2} \right),$$

which simplifies to

$$s^2 \leq 3r^2 + 3R^2 + 6Rr = r^2 + 3R^2 + 7rR + r(2r - R)$$

$$\leq r^2 + 3R^2 + 7rR,$$
using (5). This last inequality is easily seen to be equivalent to the proposed inequality (by using (2) and (4)). Equality holds if and only if the triangle is equilateral.

Reference


The corrected version of the problem was also solved by VASILE CÎRTIAJOGE, University of Ploesti, Romania; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinegyatnasium, Innsbruck, Austria; LI ZHOU, Polk Communtiy College, Winter Haven, FL, USA; and the proposer. The following solvers submitted counterexamples to the original statement of the problem and suggested alternatives to correct the inequality: VASILE CÎRTIAJOGE, University of Ploesti, Romania; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany. Counterexamples to the original statement of the problem only were found by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA. There were also two incorrect solutions and one incorrect counterexample submitted.

Klamkin and Specht suggested and proved the inequality

$$a^2 + b^2 + c^2 \leq 8R^2 \sum_{\text{cyclic}} \cos A,$$

while Cîrtoaje suggested and proved

$$18R^2(-1 + \sum_{\text{cyclic}} \cos A) \leq a^2 + b^2 + c^2 \leq 2R^2(3 + \sum_{\text{cyclic}} \cos A),$$

instead of the original inequality. Janous gave a proof similar to Howard's proof; he also commented that triangles such as the ones considered in the present problem are referred to (in [1]) as Triangles of Bager's Type II.

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2888★ [2003: 468; corrected 2004: 38] Proposed by Vedula N. Murty, Dover, PA, USA.

Let $a, b, c$ be the sides of $\triangle ABC$, in which at most one angle exceeds $\frac{\pi}{3}$. Give an algebraic proof of

$$8a^2b^2c^2 + \prod_{\text{cyclic}} (b^2 + c^2 - a^2) \leq 3abc \sum_{\text{cyclic}} a(b^2 + c^2 - a^2).$$

Comment: The proposer was looking for a proof not involving trigonometry. Since this point was not stated clearly enough, all trigonometric proofs were also deemed acceptable.

1. Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA.

Let $F(a, b, c)$ be the expression obtained by subtracting the left side of the proposed inequality from the right side. Then
\[ F(a, b, c) = \sum_{\text{cyclic}} a^6 + 3abc \sum_{\text{cyclic}} (a^2b + ab^2) - 3abc \sum_{\text{cyclic}} a^3 \]
\[ \quad - \sum_{\text{cyclic}} (a^4b^2 + a^2b^4) - 6a^2b^2c^2. \]

We need to show that \( F(a, b, c) \geq 0. \)
Without loss of generality, we may assume that \( A \leq B \leq C. \) Then \( \cos C \leq \frac{1}{2} \leq \cos B \leq \cos A. \) Then, applying the Law of Cosines to angle \( B, \) we have
\[ c^2 + a^2 - b^2 - ca \geq 0. \]  
We now make the change of variables: \( x = a, \ y = b - a, \) and \( z = c - b. \) Clearly, \( x, y, z \geq 0. \) Then (1) is equivalent to
\[ (x + y + z)^2 + x^2 - (x + y)^2 - x(x + y + z) \geq 0, \]
which in turn is equivalent to
\[ z(x + 2y + z) \geq xy. \]  
With further help from a computer algebra system, we obtain
\[ F(a, b, c) = F(x, x + y, x + y + z) \]
\[ = z^6 + 6xz^5 + 6yz^5 + 25xyz^4 + 14y^2z^4 + x^4yz + 10x^2z^4 \]
\[ + 6x^3z^3 + 16y^3z^3 + 39xy^2z^3 + 29x^2yz^3 + x^4z^2 \]
\[ + 8y^4z^2 + 26xy^3z^2 + 11x^3yz^2 + 27x^2y^2z^2 + x^4y^2 \]
\[ - (xy)^2(2y^2 + 2xy + 3xz) - (xy)(4xy^2z). \]

The inequality (2) implies that
\[ -(xy)^2(2y^2 + 2xy + 3xz) \geq -z^2(x + 2y + z)^2(2y^2 + 2xy + 3xz) \]
and
\[ -(xy)(4xy^2z) \geq -(x + 2y + z)(4xy^2z). \]
Substituting into the expression for \( F(a, b, c) \) above, we obtain
\[ F(a, b, c) \geq z^6 + 6xz^5 + 6yz^5 + 25xyz^4 + 14y^2z^4 + x^4yz + 10x^2z^4 \]
\[ + 6x^3z^3 + 16y^3z^3 + 39xy^2z^3 + 29x^2yz^3 + x^4z^2 \]
\[ + 8y^4z^2 + 26xy^3z^2 + 11x^3yz^2 + 27x^2y^2z^2 + x^4y^2 \]
\[ - z^2(x + 2y + z)^2(2y^2 + 2xy + 3xz) \]
\[ - z(x + 2y + z)(4xy^2z). \]

Expanding and combining like terms yields
\[ F(a, b, c) \geq z^6 + 3(x + 2y)z^5 + (4x^2 + 11xy + 12y^2)z^4 \]
\[ + (3x^3 + 8y^3)z^3 + xy(13x + 11y)z^2 + x(x^3 + 2y^3)z^2 \]
\[ + x^2y(9x + 13y)z^2 + x^4yz + x^4y^2 \]
\[ \geq 0, \]
as desired.
II. Solution by Li Zhou. Polk Community College, Winter Haven, FL, USA.

Dividing both sides of the proposed inequality by $a^2 b^2 c^2$, we see that it is equivalent to

$$8(1 + \cos A \cos B \cos C) \leq 6 \sum_{\text{cyclic}} \cos A.$$ 

Using $1 + \cos A \cos B \cos C = \frac{1}{2} (\sin^2 A + \sin^2 B + \sin^2 C)$, the above inequality becomes

$$4(\sin^2 A + \sin^2 B + \sin^2 C) \leq 6 \sum_{\text{cyclic}} \cos A.$$ 

This is equivalent to the inequality in problem 2887 above.

Also solved by WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; and the proposer (both of whom used trigonometry). There was one incorrect solution.

ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICHARD1. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB, and PETER Y. WOO, Biola University, La Mirada, CA, USA simply pointed out that the originally posed problem was incorrect and gave counter-examples to show this. Hess demonstrated that the original inequality would hold if the summation appearing on the right side was replaced by

$$\sum_{\text{cyclic}} a(b^2 + c^2 - a^2).$$ 

Klamkin demonstrated that the original inequality would hold if the term $8a^2 b^2 c^2$ on the left side was replaced by $6a^2 b^2 c^2$.

2889. [2003 : 514] Proposed by Vedula N. Murty, Dover, PA, USA.

Suppose that $A, B, C$ are the angles of $\triangle ABC$, and that $r$ and $R$ are its inradius and circumradius, respectively. Show that

$$4 \cos(A) \cos(B) \cos(C) \leq 2 \left( \frac{r}{R} \right)^2.$$ 

Solution by Michel Bataille, Rouen, France.

Let $I$ and $H$ be the incentre and orthocentre of $\triangle ABC$, respectively. We know that

$$IH^2 = 2R^2 - 4R^2 \cos A \cos B \cos C$$

(see problem 2747 [2003 : 251]). Thus, $2R^2 - 4R^2 \cos A \cos B \cos C \geq 0$, and the proposed inequality follows immediately. Equality occurs if and only if $I = H$; that is, $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL, USA; RICHARD1. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; ANDREI SIMION, student, Cooper Union for Advancement of Science
and Art, New York, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.


Suppose that the polynomial \( A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k \) can be factored into \( A(z) = \prod_{k=1}^{n} (z - z_k) \), where the \( z_k \) are positive real numbers.

Prove that, for \( k = 1, 2, \ldots, n - 1 \),

\[
\frac{a_{n-k}}{C(n,k)} \geq \frac{a_{n-k-1}}{C(n,k+1)},
\]

where \( C(n,k) \) denotes the binomial coefficient \( \binom{n}{k} \). When does equality occur?

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

As a matter of fact, the claimed inequality is nothing less than the Macaulin–Newton Inequality, dating back at least to 1729, the year Maclaurin published his note [1]. Equality occurs if and only if all the roots of \( A(z) \) are equal.

As a very recent reference (including also a proof), consult Chapter 12, entitled “Symmetric Sums”, of the book [2]. [Ed. Janous recommends this book highly.]

References


Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER BOWEN, Halandri, Greece; NATALIO H. GUERZENZVAIG, Universidad CAECE, Buenos Aires, Argentina; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; YUEFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

Two proofreaders, Chris and Pat, were asked to read a manuscript and find the errors. Let $B$ be the number of errors which both Chris and Pat found, $C$ the number of errors found only by Chris, and $P$ the number found only by Pat; lastly, let $N$ be the number of errors found by neither of them.

Prove that $\sqrt{(B+P)(C+N)(B+C)(P+N)} \geq |BN - CP|$.

I. Solution by Kathleen E. Lewis. SUNY Oswego, Oswego, NY, USA.

The information given about numbers $B$, $C$, $P$, and $N$ actually tells us only that they are non-negative integers.

The square of the left side of the proposed inequality, namely $(B+P)(C+N)(B+C)(P+N)$, when multiplied out, is a sum of sixteen non-negative terms, two of which are $B^2N^2$ and $C^2P^2$. Therefore, this sum is greater than or equal to $B^2N^2 + C^2P^2$. The square of the right side is $B^2N^2 - 2BNCP + C^2P^2$, which is less than or equal to $B^2N^2 + C^2P^2$. Therefore, the square of the right side is less than or equal to the square of the left. Since taking the positive square roots preserves the relationship, the inequality holds.

II. Solution by the proposer.

Define two indicator variables $X$ and $Y$, where $X = 1$ if Chris catches the error (and $X = 0$ otherwise), and $Y = 1$ if Pat catches the error (and $Y = 0$ otherwise). The correlation coefficient between $X$ and $Y$ is

$$r_{x,y} = \frac{BN - CP}{\sqrt{(B+P)(C+N)(B+C)(P+N)}}.$$

The proposed inequality is immediately obtained by observing the known inequality $-1 \leq r_{x,y} \leq 1$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; YUE-FEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania.


(a) Let $A$ and $B$ be arbitrary $2 \times 2$ matrices over $\mathbb{C}$. For all complex numbers $\alpha, \beta, \gamma$, prove that

$$\det(\alpha I + \beta AB + \gamma BA) = \det(\alpha I + \gamma AB + \beta BA).$$

(Here, $I$ denotes the $2 \times 2$ identity matrix.)
(b) Is there a similar identity for $n \times n$ matrices?

[The proposer gives a “Machine” proof for (a). We want a purely algebraic proof.]
I. Solution to (a) by Richard I. Hess. Rancho Palos Verdes, CA, USA (modified slightly by the editor).

Let \( AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( BA = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \). Set

\[
P = \alpha I + \beta AB + \gamma BA \quad \text{and} \quad Q = \alpha I + \gamma AB + \beta BA.
\]

Then by direct (human) computations, we have

\[
det(P) - det(Q) = (\alpha + \beta a + \gamma e)(\alpha + \beta d + \gamma h) - (\beta b + \gamma f)(\beta c + \gamma g) - (\alpha + \gamma a + \beta e)(\alpha + \gamma d + \beta h) + (\gamma b + \beta f)(\gamma c + \beta g)
\]

\[
= \alpha(\beta - \gamma)(a + d - e - h) + (\beta^2 - \gamma^2)(ad - bc + fg - eh)
\]

\[
= \alpha(\beta - \gamma)(\text{tr}(AB) - \text{tr}(BA)) + (\beta^2 - \gamma^2)(\det(AB) - \det(BA))
\]

\[
= 0.
\]

II. Solution by Michel Bataille, Rouen, France.

(a) Let \( C = \beta AB + \gamma BA \) and \( D = \gamma AB + \beta BA \). If \( \gamma = 0 \), then clearly \( \det(C) = \det(D) \). For \( \gamma \neq 0 \), consider the polynomial

\[
P(x) = \det(x AB + \gamma BA) - \det(x BA + \gamma AB).
\]

We readily see that \( P(0) = P(\gamma) = P(-\gamma) = 0 \). Since \( P \) has degree at most two and has three distinct roots, it must be the zero polynomial. It follows that \( \det(C) = \det(D) \) for all \( \beta, \gamma \).

Now, fix \( \beta \) and \( \gamma \). Note that \( \text{tr}(C) = \text{tr}(D) = (\beta + \gamma)\text{tr}(AB) \). Hence, \( C \) and \( D \) must have the same characteristic polynomial, since they have the same determinant and the same trace. That is, \( \det(xI - C) = \det(xI - D) \). With \( x = -\alpha \), we obtain the required result.

(b) No, the identity in (a) does not hold if \( n \geq 3 \).

Let \( I_n \) denote the \( n \times n \) identity matrix, and let

\[
A_3 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B_3 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.
\]

Then straightforward calculations yield \( \det(I_3 + 2A_3B_3 + B_3A_3) = 352 \) and \( \det(I_3 + A_3B_3 + 2B_3A_3) = 348 \).

For \( n > 3 \), let \( A_n = O_{n-3} \oplus A_3 \) and \( B_n = O_{n-3} \oplus B_3 \), where \( O_{n-3} \) denotes the \((n-3) \times (n-3)\) zero matrix. Then, from the basic properties of direct sum of matrices, we have

\[
A_nB_n = O_{n-3} \oplus A_3B_3 \quad \text{and} \quad B_nA_n = O_{n-3} \oplus B_3A_3.
\]

Hence,

\[
det(I_n + 2A_nB_n + B_nA_n) = \det(I_{n-3} \oplus (I_3 + 2A_3B_3 + B_3A_3))
\]

\[
= \det(I_{n-3}) \det(I_3 + 2A_3B_3 + B_3A_3)
\]

\[
= 352
\]
and

\[
\det(I_n + A_nB_n + 2B_nA_n) = \det(I_{n-3} \oplus (I_3 + A_3B_3 + 2B_3A_3)) \\
= \det(I_{n-3}) \det(I_3 + A_3B_3 + 2B_3A_3) = 348 .
\]

Also solved (part (a) only) by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; CRISTINEL MORTICI, Valahia University of Targoviste, Romania. Both parts were also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Using the Cayley-Hamilton Theorem and more complicated arguments, Guersenzaig derived the following formula (for $2 \times 2$ matrices) from which the identity in (a) follows immediately:

\[
det(\alpha I + \beta AB + \gamma BA) = \alpha^2 + \alpha(\beta + \gamma) \text{tr}(AB) + (\beta^2 + \gamma^2) \det(AB) + \beta \gamma (\text{tr}(AB))^2 - \text{tr}(A^2B^2)) .
\]

2893. [2003 : 516] Proposed by Vedula N. Murty, Dover, PA, USA.

In [2001 : 45–47], we find three proofs of the classical inequality

\[
1 \leq \sum_{\text{cyclic}} \cos(A) \leq \frac{3}{2} .
\]

In [2002 : 85–87], we find Klamkin’s illustrations of the Majorization (or Karamata) Inequality.

Prove the above “classical inequality” using the Majorization Inequality.

[Ed. For the convenience of the reader, we review the Majorization Inequality. Let $S = (a_1, a_2, \ldots, a_n)$ and $T = (b_1, b_2, \ldots, b_n)$, where $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Suppose that $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j$ and $\sum_{j=1}^{k} a_j \geq \sum_{j=1}^{k} b_j$ for each $k = 1, 2, \ldots, n - 1$. Then we say that $S$ majorizes $T$, and we write $S \succ T$. If $S \succ T$ and $F$ is a convex function, then

\[
\sum_{j=1}^{n} F(a_j) \geq \sum_{j=1}^{n} F(b_j) .
\]

If $S \succ T$ and $F$ is a concave function, then the above inequality is reversed.]

Solution by Walther Janous, Ursulinenymgänsium, Innsbruck, Austria.

The angles in the problem are angles in a triangle $ABC$. Without loss of generality, we assume that $A \geq B \geq C$. We distinguish the case where the triangle is obtuse ($A > \pi/2$) from the case where it is non-obtuse ($A \leq \pi/2$). In both cases, we will use the fact that the function $f(x) = \cos x$ is concave on the interval $[0, \pi/2]$. 
First suppose that $A \leq \pi/2$. Then
\[
\left( \frac{\pi}{2}, \frac{\pi}{2}, 0 \right) \succ (A, B, C) \succ \left( \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right).
\]
Applying the Majorization Inequality, we have
\[
2 \cos \left( \frac{\pi}{2} \right) + \cos 0 \leq \cos A + \cos B + \cos C \leq 3 \cos \left( \frac{\pi}{3} \right).
\]
Simplifying, we obtain the required inequalities.
Now suppose that $A > \pi/2$. Then
\[
(\pi - A, 0) \succ (B, C) \succ \left( \frac{\pi - A}{2}, \frac{\pi - A}{2} \right).
\]
Applying the Majorization Inequality, we have
\[
\cos(\pi - A) + \cos 0 \leq \cos B + \cos C \leq 2 \cos \left( \frac{\pi - A}{2} \right),
\]
or
\[
- \cos A + 1 \leq \cos B + \cos C \leq 2 \sin \left( \frac{A}{2} \right).
\]
Then
\[
1 \leq \cos A + \cos B + \cos C \leq \cos A + 2 \sin \left( \frac{A}{2} \right).
\]
Finally, we note that $\cos A + 2 \sin \left( \frac{A}{2} \right) \leq \frac{3}{2}$, because
\[
\cos A + 2 \sin \left( \frac{A}{2} \right) - \frac{3}{2} = 1 - 2 \sin \left( \frac{A}{2} \right) + 2 \sin \left( \frac{A}{2} \right) - \frac{3}{2}
= - \frac{1}{2} \left( 1 - 2 \sin \left( \frac{A}{2} \right) \right) \leq 0.
\]

Also solved by Li ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incomplete solution.

**2894. [2003 : 517] Proposed by Vedula N. Murty, Dover, PA, USA.**

Suppose that $\triangle ABC$ is acute-angled. With the standard notation, prove that
\[
4abc < (a^2 + b^2 + c^2) (a \cos A + b \cos B + c \cos C) \leq \frac{9}{2} abc.
\]

**Solution by Joe Howard, Portales, NM, USA.**

Let $K = (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)$. Since $\triangle ABC$ is acute, we must have $K > 0$. Let $S$ be the area and $R$ the circumradius of $\triangle ABC$. We will use the following well-known or easy-to-prove facts.
\[
(a^2 + b^2 + c^2)(16S^2) = K + 8(abc)^2, \tag{1}
\]
\[
\sum_{\text{cyclic}} a \cos A = \frac{2S}{R}, \tag{2}
\]
\[
4SR = abc, \tag{3}
\]
\[
a^2 + b^2 + c^2 \leq 9R^2. \tag{4}
\]
(The last inequality follows from $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$, where $O$ is the circumcentre and $H$ the orthocentre of $\triangle ABC$.)

Using equations (1), (2), and (3), and the inequality $K > 0$, we obtain

\[
(a^2 + b^2 + c^2) \sum_{\text{cyclic}} a \cos A = (a^2 + b^2 + c^2) \cdot \frac{2S}{R} = (a^2 + b^2 + c^2)(16S^2) = \frac{8SR}{2abc} = \frac{K + 8(abc)^2}{2abc} = \frac{K}{2abc} + 4abc > 4abc,
\]

which proves the left-hand inequality.

Using inequality (4) and equations (2) and (3), we get

\[
(a^2 + b^2 + c^2) \sum_{\text{cyclic}} a \cos A \leq 9R^2 \cdot \frac{2S}{R} = \frac{9}{2}(4SR) = \frac{9}{2}(abc),
\]

which takes care of the right-hand inequality. Equality holds in the right-hand inequality if and only if the triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALther JANOUS, Ursulinen gymnasium, Innsbruck, Austria; JUAN-BOSCO ROMERO MARQUEZ, Universidad de Valladolid, Valladolid, Spain; MARCELO RUFINO de OLIVEIRA, Belém, Brazil; TOSHIO SEIYAMA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOLI, Athens, Greece (2 solutions); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2895. [2003: 517] Proposed by Vedula N. Murty, Dover, PA, USA.

Suppose that $A$ and $B$ are two events with probabilities $P(A)$ and $P(B)$ such that $0 < P(A) < 1$ and $0 < P(B) < 1$. Let

\[
K = \frac{2[P(A \cap B) - P(A)P(B)]}{P(A) + P(B) - 2P(A)P(B)}.
\]

Show that $|K| < 1$, and interpret the value $K = 0$.

Solution by Michel Bataille, Rouen, France.

We prove that $|K| \leq 1$.

First, note that

\[
P(A) + P(B) - 2P(A)P(B) = P(A)(1 - P(B)) + P(B)(1 - P(A)) > 0.
\]
Since $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$, we have
\[2P(A \cap B) \leq P(A) + P(B)\]
and $K \leq 1$ follows.

We remark that $K = 1$ holds if and only if $P(A) = P(B) = P(A \cap B)$. This occurs, for example, if $B = A \cup N$ where $A \cap N = \emptyset$ and $P(N) = 0$.

Now, we show that $K \geq -1$, which amounts to
\[P(A) + P(B) + 2P(A \cap B) \geq 4P(A)P(B) . \tag{1}\]

Equation (1) certainly holds if $P(A) + P(B) \leq 1$ since, in that case,
\[P(A) + P(B) + 2P(A \cap B) \geq P(A) + P(B) \geq (P(A) + P(B))^2 \geq 4P(A)P(B) . \]

Now suppose that $P(A) + P(B) > 1$. Then at least one of $P(A)$ and $P(B)$ is greater than $\frac{1}{2}$, say $P(A) > \frac{1}{2}$. Let $P(A) = \frac{1}{2} + h$ and $P(B) = \frac{1}{2} + k$, where $0 < h < \frac{1}{2}$ and $|k| < \frac{1}{2}$. Then
\[P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1 + h + k - P(A \cup B) \geq h + k , \]
and
\[4P(A)P(B) = 1 + 2(h + k) + 4hk . \]

Thus, (1) will follow if $h + k - 4hk \geq 0$. This is certainly true if $k \leq 0$, since $-4hk \geq 0$ and $h + k = P(A) + P(B) - 1 > 0$. If $k > 0$, then $\sqrt{hk} \leq \frac{1}{2}$ and
\[h + k - 4hk \geq 2\sqrt{hk} - 4hk = 2\sqrt{hk}(1 - 2\sqrt{hk}) \geq 0 . \]

Note that $K = -1$ when $P(A) = P(B) = \frac{1}{2}$ and $A \cap B = \emptyset$, and that $K = 0$ when $P(A \cap B) = P(A)P(B)$ (that is, when $A$ and $B$ are independent events).

Also solved by CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Bristol, UK; KEITH EKBLAW, Walla Walla, WA, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

\section*{2896. [2003: 517] Proposed by Mihály Bence, Brasov, Romania.}

Suppose that $0 < x_0 < x_1$ and that, for $n = 1, 2, 3, \ldots,$
\[\sqrt{1 + x_n} \left(1 + \sqrt{x_{n-1}x_{n+1}}\right) = \sqrt{1 + x_{n-1}} \left(1 + \sqrt{x_nx_{n+1}}\right) . \]

(a) Prove that the sequence $\{x_n\}$ is convergent.
(b) Find $\lim_{n \to \infty} x_n$. 

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

The given recursive formula forces each term of the sequence \( \{x_n\} \) to be non-negative. For each \( n \), let \( a_n \) be the real number in \([0, \pi/2)\) such that \( x_n = \tan^2 a_n \). The given formula becomes

\[
\sec a_n (1 + \tan a_n \tan a_{n+1}) = \sec a_{n-1} (1 + \tan a_n \tan a_{n+1}).
\]

Multiplying both sides by \( \cos a_n \cos a_{n-1} \cos a_{n+1} \), we obtain

\[
\cos(a_{n-1} - a_n) = \cos(a_n - a_{n+1}).
\]

Suppose that \( a_i = a_{i+1} \) for some \( i \geq 1 \). Then \( \cos(a_i - a_{i+1}) = 1 \), and therefore, \( \cos(a_{i-1} - a_{i+1}) = 1 \). Since \( a_n \in [0, \pi/2) \) for all \( n \), we must have \( a_{i-1} = a_{i+1} = a_i \). Then there is no least integer \( i \geq 1 \) such that \( a_i = a_{i+1} \), and therefore, there is no such integer \( i \) at all. Thus, no two consecutive terms of the sequence \( \{a_n\} \) are equal.

Since \( a_{n-1} \neq a_n \), the relation \( \cos(a_{n-1} - a_{n+1}) = \cos(a_n - a_{n+1}) \) implies that \( 2a_{n+1} = a_n + a_{n-1} \) for each \( n \). It follows by induction that

\[
a_n = \frac{a_0 + 2a_1}{3} - \frac{a_0 - a_1}{3(-2)^{n-1}},
\]

which obviously converges to \( \frac{a_0 + 2a_1}{3} \). Therefore, the sequence \( \{x_n\} \) also converges, and we have

\[
\lim_{n \to \infty} x_n = \tan^2 \left( \frac{a_0 + 2a_1}{3} \right) = \tan^2 \left( \frac{\tan^{-1} \sqrt{x_0 + 2 \tan^{-1} \sqrt{x_1}}}{3} \right).
\]

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER BOWEN, Halandri, Greece; CHRISTOPHER J. BRADLEY, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2897. [2003 : 518] Proposed by Václav Konečný, Big Rapids, MI, USA.

(a) Show that it is possible to divide a circular disc into four parts with the same area by means of three line segments of the same length.

(b) Does there exist a straight edge and compass construction (in the classical sense; that is, with a finite number of steps)?

Solution by Christopher Bowen, Halandri, Greece.

(a) Without loss of generality, we assume a unit radius so that the area of the circle is \( \pi \). Let \( AB \) be a chord that divides the area in the ratio \( 1 : 3 \).
Draw a parallel chord $CD$ of equal length. Any line $EF$ through the center $O$, with $E$ on $AB$ and $F$ on $CD$, will divide the area of the circle between these two chords into two equal parts, and, therefore, the three line segments $AB$, $CD$, and $EF$ divide the circle into four regions of equal area. It remains to insure that $EF = AB$.

Define $2\theta = \angle BOA$. Since the area of the smaller segment (of the circle) determined by $AB$ is $(2\theta - \sin 2\theta)/2 = \pi/4$, we have

$$2\theta - \sin 2\theta = \frac{\pi}{2}. \quad (1)$$

This shows that $\theta = \frac{\sin 2\theta}{2} + \frac{\pi}{4} > \frac{\pi}{4}$. Hence, if $H$ is the foot of the perpendicular to $AB$ from $O$, then

$$OH = \frac{BH}{\tan \theta} < BH.$$  

Since (by definition) $AB$ cannot be a diameter, we see that $BH < OB$; whence, $OH < BH < OB$. As line $FOE$ rotates about $O$, the length $OE$ will vary continuously from a maximum of $OB$ to a minimum of $OH$. Thus, some position of this rotating line will have length $OE = BH$, as desired.

(b) The construction of these segments must involve at least one chord of length $2\sin \theta$ such as $AB$, since, if another segment were to intersect this chord at a point interior to both segments, the two segments would by themselves create four regions, contrary to what is desired. One can construct $AB$ by ruler and compass if and only if one can construct the length $k = \sin 2\theta = 2\sin \theta \cos \theta$, with $\theta$ defined by (1). However, a length is constructible by ruler and compass if and only if the corresponding number is algebraic of degree $2^n$ over the rationals for some positive integer $n$. Since

$$\cos k = \cos(\sin 2\theta) = \cos \left(2\theta - \frac{\pi}{2}\right) = \sin 2\theta = k,$$

we obtain

$$k - \cos k = ke^0 - \frac{1}{2}e^{ik} - \frac{1}{2}e^{-ik} = 0.$$  

By the Hermite-Lindemann Theorem (which provided the first proof of the transcendence of $\pi$), if $k$ were algebraic, then the above algebraic linear combination of exponentials could not be zero. The number $k$ must therefore be transcendental, which implies that construction by ruler and compass is impossible.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D.J. SMEENK, Zalkbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LIZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.


Prove that \(\frac{(2^n)!}{1!2!4!\cdots(2^n-1)!}\) is divisible by \(\prod_{k=1}^{n} (2^k-1 + 1)\).
I. Composite of essentially the same solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Christopher Bowen, Halandri, Greece; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since

$$\frac{(2^n)!}{1!2!4!\ldots(2n-1)!} = \prod_{k=1}^{n} \frac{(2^k)!}{(2k-1)!(2k-1)!} = \prod_{k=1}^{n} \left( 2^k \right),$$

it suffices to show that \( \binom{2^k}{2k-1} \) is divisible by \( 2^{k-1} + 1 \) for all \( k \in \mathbb{N} \).

Since

$$\binom{2k+1}{2k-1+1} = \frac{2k+1}{2k-1+1} \binom{2k}{2k-1}$$

is an integer, and since the integers \( 2^k + 1 \) and \( 2^{k-1} + 1 \) are relatively prime, it follows that \( 2^{k-1} + 1 \) divides \( \binom{2^k}{2k-1} \).

II. Composite of similar solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Recall that the \( m \)th Catalan number is defined by

$$C_m = \frac{1}{m+1} \binom{2m}{m},$$

for \( m = 1, 2, 3, \ldots \). These numbers are well known to be integers.

Let \( A_n = \frac{(2^n)!}{1!2!4!\ldots(2n-1)!} \), and let \( P_n = \prod_{k=1}^{n} (2k-1) + 1 \). Then, as in Solution I above, \( A_n = \prod_{k=1}^{n} \frac{2^k}{2k-1} \).

Hence, \( A_n/P_n = \prod_{k=1}^{n} \frac{1}{2k-1+1} \left( \frac{2^k}{2k-1} \right) = \prod_{k=0}^{n-1} C_{2k} \), which is an integer.

Also solved by KEE-WAI LAU, Hong Kong, China; MIKE SPIVEY, Samford University, Birmingham, AL, USA; and the proposer.

There were actually several variations of the proof that \( 2^{k-1} + 1 \) divides \( \binom{2^k}{2k-1} \). Here are some of the identities used:

$$\frac{1}{2^n} \left( \frac{2^{n+1}}{2^n} \right) = \left( \frac{2^{n+1}}{2^n} \right) - \left( \frac{2^{n+1}}{2^n-1} \right);$$

$$2^n \left( \frac{2^{n+1}}{2^n} \right) = \left( 2^n + 1 \right) \left( \frac{2^{n+1}}{2^n-1} \right);$$

$$\left( \frac{2^k}{2k-1} \right) = \left( \frac{2^{k-1} + 1}{2k-1} \right) \left( \frac{2^k}{2k-1+1} \right).$$

Find the maximum area of a pentagon $ABCD$ inscribed in a circle such that the diagonal $AC$ is perpendicular to the diagonal $BD$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

If $ABCD$ were fixed, then since $[ABCD] = [ABCD] + [DEA]$, the area would be maximized if $E$ were placed as far away from line $AD$ as possible. We may therefore assume that $E$ is the midpoint of the arc $AD$ not containing $B$. Let $O$ be the centre of the circumscribed circle, and let $\alpha = \angle AOB$, and $\beta = \angle BOC$. Then $\angle COD = \pi - \alpha$ (because $AC \perp BD$ if and only if $\angle AOB + \angle COD = \pi$), and $\angle DOE = \angle EOA = (\pi - \beta)/2$. Thus,

$$[ABCD] = [AOB] + [BOC] + [COD] + [DOE] + [EOA]$$

$$= \frac{1}{2} \left( \sin \alpha + \sin \beta + \sin(\pi - \alpha) + 2 \sin \frac{\pi - \beta}{2} \right)$$

$$= \sin \alpha + \frac{1}{2} \sin \beta + \cos \frac{\beta}{2}.$$ 

Since $\alpha \in (0, \pi)$, we have

$$[ABCD] \leq 1 + \frac{1}{2} \sin \beta + \cos \frac{\beta}{2}.$$ 

The derivative, $\frac{d}{d\beta} \left( \frac{1}{2} \sin \beta + \cos \frac{\beta}{2} \right) = \frac{1}{2} \left( \cos \beta - \sin \frac{\beta}{2} \right)$, has its only zero in $(0, \pi)$ at $\beta = \pi/3$. It is easy to verify that the maximum occurs there. Thus,

$$[ABCD] \leq 1 + \frac{1}{2} \sin \beta + \cos \frac{\beta}{2} \leq 1 + \frac{1}{2} \sin \frac{\pi}{3} + \cos \frac{\pi}{6} = \frac{4 + 3\sqrt{3}}{4}.$$ 

The desired maximum area is therefore $\frac{4 + 3\sqrt{3}}{4}$, which is attained when $\alpha = \pi/2$ and $\beta = \pi/3$.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; G.D. CHAKERIAN, University of California, Davis, and MURRAY S. Klamkin, University of Alberta, Edmonton, AB; WALTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VAIAV KONEČNÝ, Big Rapids, MI, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; M² JESUS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution (which made the assumption that $AD$ was a diameter, turning the proposal into a simpler but still interesting problem).

Chakerian and Klamkin described our pentagon in terms of a regular 12-gon $P_1P_2\ldots P_{12}$ inscribed in a unit circle: $ABCD = P_1P_4P_6P_9$ is the pentagon of largest area inscribed in the circle having a pair of adjacent perpendicular diagonals. This pentagon has angle $120^\circ$ at $P_1$ ($= E$), and angles of $105^\circ$ at the other four vertices; the sides have lengths $\sqrt{2}$, 1, $\sqrt{2}$, 1. (Starting with $AB = P_1P_4$)

Bradley reported that in the 1980s this was training problem X5 of the late F.J. Budden, leader of the UK IMO team. It might well pre-date this. Woo likewise recalls having seen it before, perhaps a few years ago as a Putnam Competition problem.

Let \( I \) be the incentre of \( \triangle ABC \), \( r_1 \) the inradius of \( \triangle IAB \) and \( r_2 \) the inradius of \( \triangle IAC \). Computer experiments using Geometer's Sketchpad suggest that \( r_2 < \frac{5}{4}r_1 \).

(a) Prove or disprove this conjecture.
(b) Can \( \frac{5}{4} \) be replaced by a smaller constant?


(a) Let \( r \) be the inradius of \( \triangle ABC \). Let \( T \) be the point on \( AB \) such that \( \angle AIT = 90^\circ \). Let \( X \) and \( Y \) be the incentres of \( \triangle IAB \) and \( \triangle IAT \), respectively. Then \( X, Y, \) and \( A \) are collinear. Since \( \angle AIB > 90^\circ \), we have

\[
\angle AIY = \frac{1}{2} \angle AIT < \frac{1}{2} \angle AIB = \angle AI X.
\]

Thus, \( Y \) is a point on the segment \( AX \).

Let \( r_1' \) be the inradius of \( \triangle IAT \). Let \( X' \) and \( Y' \) be the feet of the perpendiculars to \( AB \) from \( X \) and \( Y \), respectively. Then \( XX' = r_1 \) and \( YY' = r_1' \). Since \( XX' \parallel YY' \), we have \( \frac{YY'}{XX'} = \frac{AY}{AX} < 1 \). That is, \( \frac{r_1'}{r_1} < 1 \), which implies that

\[
r_1' < r_1. \tag{1}
\]

Suppose that \( H \) and \( K \) are the feet of the perpendiculars from \( I \) to \( AC \) and \( AB \), respectively. Clearly, \( IK = IH = r \). In \( \triangle IAC \), it is known that \( IH > 2r_2 \). Thus,

\[
r_2 < \frac{1}{2}r. \tag{2}
\]

In \( \triangle IAT \), we see that \([IAT] = \frac{1}{2}AT \cdot IK = \frac{1}{2}AT \cdot r\), where \([IAT]\) denotes the area of \( \triangle IAT \). Moreover, \([IAT] = \frac{1}{2}(AT + AI + IT)r_1'\). Therefore,

\[
\begin{align*}
AT \cdot r & = (AT + AI + IT)r_1', \\
r & = \frac{AT + AI + IT}{AT} = 1 + \cos \alpha + \sin \alpha \\
\frac{r}{r_1'} & = 1 + \sqrt{2} \sin(\alpha + 45^\circ) \leq 1 + \sqrt{2}, \tag{3}
\end{align*}
\]

where we have set \( \alpha = \angle TAI \).

From (1), (2), and (3), we obtain

\[
\frac{r_2}{r_1} < \frac{r_2}{r_1'} < \frac{1}{2}r \leq \frac{1 + \sqrt{2}}{2} = \frac{2 + \sqrt{2}}{4} < \frac{5}{4}.
\]

Hence, \( r_2 < \frac{5}{4}r_1 \) is true.
(b) As shown in the above proof, $\frac{5}{4}$ can be replaced by $(1 + \sqrt{2})/2$, and this is the best possible.

II. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

We assert that the precise constant should be $(1 + \sqrt{2})/2$, which is slightly less than $5/4$.

Let $\alpha = \angle BAC$ be fixed, but $\beta = \angle BAC$ be variable. Then $\angle BIC = 90^\circ + \alpha$ is fixed in size. Let $\ell_1$ be the (fixed) line containing $A$ and $B$, and let $\ell_2$ be the (fixed) line containing $A$ and $C$, and be fixed. Let $k = IA$, and let $\Gamma_1$ and $\Gamma_2$ be the incircles of $\triangle IAB$ and $\triangle IAC$, respectively.

Suppose that $B$ moves along $\ell_1$ away from $A$. Then $\Gamma_1$ grows in size, and the limit of its radius $r_1$ is $\frac{1}{2}k \sin \alpha$. On the other hand, if we denote the perimeter of $\triangle IAC$ by $s$ and its area by $[AIC]$, then we see that $\angle AIC$ shrinks towards $90^\circ$ as a limit, and its radius $r_2$ shrinks towards a limit of

\[ \frac{2[AIC]}{s} = \frac{k^2 \tan \alpha}{k + k \tan \alpha + k \sec \alpha} = \frac{k \sin \alpha}{\cos \alpha + \sin \alpha + 1}. \]

Then

\[ \frac{r_1}{r_2} = \frac{1 + \sin \alpha + \cos \alpha}{2} = \frac{1}{2} (1 + \sin \alpha + \sin(90^\circ - \alpha)) \]

\[ = \frac{1}{2} (1 + \sin 45^\circ \cos(\alpha - 45^\circ)). \]

The maximum occurs when $\alpha = 45^\circ$, and the limit is $(1 + \sqrt{2})/2$.

Also solved by MANUEL BENITO, OSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; C. R. PRANESCHAR, Bangalore, India; and PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA.