MATHMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Please send your solutions to the problems in this edition by 1 May 2005. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M153. Correction. Proposed by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Two similar triangles $APB$ and $BQC$ are erected externally on a triangle $ABC$. If $R$ is a point such that $PBQR$ is a parallelogram, show that triangles $ARC$ and $APB$ are similar.

M163. Proposed by the Mayhem Staff.

Show that it is possible to put positive integer values on the faces of two dice (not necessarily the same on both dice) so that, when the dice are tossed, the outcomes 1, 2, 3, ..., 12 are equally probable.

M164. Proposed by the Mayhem Staff.

Consider the following procedure for dividing the three-digit number 375 by 8. Write down the number formed by the first two digits, namely, 37. Multiply this by 2 to get 74. Add to this the units digit of 375 (the original number), obtaining 74 + 5 = 79. Then divide by 8 to get 9 with a remainder of 7. Add this result (9, remainder 7) to the number 37 (the first two digits of 375) to get your answer: 46, remainder 7. Thus, 375 divided by 8 equals 46 with a remainder of 7.

Does this method always work for three-digit numbers? Why, or why not?
M165. Proposed by Babis Stergiou, Chalkida, Greece.

If \( a, b > 0 \), prove that

\[(a) \quad \sqrt{ab} \geq \frac{2}{1/a + 1/b}.\]

\[(b) \quad a^6 + b^6 + 8a^3 + 8b^3 + 2ab^3 + 16 \geq 36ab.\]

M166. Proposed by the Mayhem Staff.

(a) Simplify

\[(3n)^2 + (4n - 1)^2 - (5n - 1)^2, \quad (3n + 2)^2 + (4n)^2 - (5n + 1)^2.\]

(b) Using (a) or otherwise, prove that all positive integers can be represented in the form \( a^2 + b^2 - c^2 \) where \( a, b, c \) are integers and \( 0 < a < b < c \).

M167. Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.

Solve the following inequality:

\[(2 + \cos x)(1 + \sin x - \cos x) \geq \cos x(1 + 2 \sin x - \cos x).\]

M168. Proposed by Neven Jurč, Zagreb, Croatia.

How many different \( 3 \times 3 \) arrays of non-negative integers is it possible to construct so that each of the three horizontal sums and each of the three vertical sums is equal to 7, the first diagonal sum is equal to 10, and the second diagonal sum is equal to 9? (Two arrays which may be transformed into one another by rotations and/or reflections are not considered to be different.)

Here is an example of such an array:

\[
\begin{array}{ccc}
2 & 3 & 2 \\
2 & 4 & 1 \\
3 & 0 & 4
\end{array}
\]

M153. Correction. Proposé par Yufei Zhao, étudiant, Don Mills Collegiate Institute, Toronto, ON.

Extérieurement au triangle \( ABC \), on construit deux triangles semblables \( APB \) et \( BQC \). Si \( R \) est un point tel que \( PBQR \) est un paralléogramme, montrer qu'alois les triangles \( ARC \) et \( APB \) sont semblables.

M163. Proposé par Équipe de Mayhem.

Montrer qu'il est possible de mettre des entiers positifs sur les faces de deux pipés (non nécessairement les mêmes pour les deux) de sorte que tous les résultats 1, 2, 3, ..., 12 soient également probables quand on lance les pipés.
M164. *Proposé par Équipe de Mayhem.*

Pour diviser un nombre de trois chiffres par 8, on utilise une procédure dont voici un exemple, partant du nombre 375. On écrit le nombre formé des deux premiers chiffres, à savoir 37, qu’on multiplie par 2 pour obtenir 74. A quoi on ajoute le dernier chiffre de 375 (le nombre initial), donc 74 + 5 = 79. On divise ceci par 8 pour obtenir 9, reste 7. On ajoute ce résultat à 37 pour obtenir la réponse, 46 reste 7. En résumé, 375 divisé par 8 égale 46 reste 7.

Cette méthode marche-t-elle toujours pour les nombres de trois chiffres ? Pourquoi, ou pourquoi pas ?

M165. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Si $a$ et $b$ sont positifs, montrer que

(a) $\sqrt{ab} \geq \frac{2}{1/a + 1/b}$.
(b) $a^n + b^n + 8a^3 + 8b^3 + 2a^3b^3 + 16 \geq 36ab$.

M166. *Proposé par Équipe de Mayhem.*

(a) Simplifier

$$(3n)^2 + (4n - 1)^2 - (5n - 1)^2, \quad (3n + 2)^2 + (4n^2 - (5n + 1)^2).$$

(b) À l’aide de (a) ou autrement, montrer que tous les entiers positifs peuvent être représentés sous la forme $a^2 + b^2 - c^2$, où $a$, $b$ et $c$ sont des entiers et $0 < a < b < c$.

M167. *Proposé par Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Province de Guang Dong, Chine.*

Résoudre l’inéquation suivante :

$$(2 + \cos x)(1 + \sin x - \cos x) \geq \cos x(1 + 2\sin x - \cos x).$$

M168. *Proposé par Neven Jurić, Zagreb, Croatie.*

Combien de tableaux $3 \times 3$ formés d’entiers non négatifs est-il possible de construire de telle sorte que chacune des trois sommes horizontales ainsi que chacune des trois sommes verticales soit égale à 7, que la somme de la première diagonale soit égale à 10 et que la somme de la deuxième diagonale soit égale à 9 ? (Il faut noter que les tableaux formés avec les mêmes nombres mais orientés différemment par rotations et/ou réflexions ne sont pas considérés comme différents.)

Voici un exemple d’un tableau admissible :

$$
\begin{array}{ccc}
2 & 3 & 2 \\
2 & 4 & 1 \\
3 & 0 & 4
\end{array}
$$
Mayhem Solutions

M94. Proposed by J. Walter Lynch, Athens, GA, USA.

You have twelve balls which are identical in appearance. Eleven of them are in fact identical, and the other one differs slightly in weight from each of these eleven. Using a balance scale, find the odd ball in only three weighings.

Solution by Bruce Crofoot, University College of the Cariboo, Kamloops, BC.

We will not only find the odd ball in three weighings, but we will also determine whether it is heavier or lighter than the other balls.

We begin by using the scale to compare the weight of any 4 of the balls with the weight of any other 4 (the first weighing).

Case 1: The balls balance.

Then these 8 balls are all standard balls. Label the remaining 4 balls as $A$, $B$, $C$, $D$. Now use the scale to compare the weight of $A$ and $B$ with the weight of $C$ and a standard ball (the second weighing).

(a) If they balance, then the odd ball is $D$. By comparing its weight with one of the standard balls (the third weighing), we can decide whether it is heavy or light.

(b) If the balls $A$ and $B$ are heavier than the other balls, then either the odd ball is heavy and is one of $A$ and $B$, or it is light and is $C$. We now weigh $A$ against $B$ (the third weighing). If they do not balance, then the heavier ball is the odd ball; if they balance, then $C$ is the odd ball.

(c) If the balls $A$ and $B$ are lighter than the other balls, then we may proceed as in (b), except that the words "heavy" and "light" are interchanged and "heavier" becomes "lighter".

Case 2: The balls do not balance.

Label the balls on the heavier side as $A$, $B$, $C$, $D$ and those on the lighter side as $a$, $b$, $c$, $d$. Now use the scale to compare the weight of $A$, $B$, and $a$ with the weight of $C$, $D$, and $b$ (the second weighing).

(a) If they balance, then the odd ball is either $c$ or $d$, and it is light (since it was on the lighter side in the second weighing). We can determine which it is by weighing $c$ against $d$ (the third weighing). The lighter of the two is the odd ball.

(b) If the balls $A$, $B$, $a$ are heavier than $C$, $D$, $b$, then either the odd ball is heavy and is one of $A$ and $B$, or it is light and is $b$. We now weigh $A$ against $B$ (the third weighing). If they do not balance, then the heavier ball is the odd ball; if they balance, then $b$ is the odd ball;

(c) If the balls $C$, $D$, $b$ are heavier than $A$, $B$, $a$, then we may proceed by analogy with (b)
M95. **Proposed by the Mayhem Staff.**

In the figure below, given that \( h, k, m, n, \) and \( p \) are integers with \( h \neq 1 \), determine the value of \( h \).

\[
\begin{array}{c}
\text{p} \\
\text{n}
\end{array} 
\quad
\begin{array}{c}
\text{p} \\
\text{mn}
\end{array}
\]

*Ed:* No solutions have been received. The problem remains open.

M96. **Proposed by the Mayhem Staff.**

Determine the largest possible remainder that is attainable by dividing a three-digit number by the sum of its digits.

*Solution by Geneviève Lalonde, Massey, ON.*

The largest possible remainder in dividing an integer by an integer \( d \) is \( d - 1 \). Thus, if we do a systematic check starting from the largest sum of digits, we should find our result. From the table below, we see that the largest possible remainder is 24.

<table>
<thead>
<tr>
<th>Sum of Digits</th>
<th>Possibilities</th>
<th>Remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>999</td>
<td>0</td>
</tr>
<tr>
<td>26</td>
<td>998</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>989</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>899</td>
<td>15</td>
</tr>
<tr>
<td>25</td>
<td>997</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>979</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>799</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>988</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>898</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>889</td>
<td>14</td>
</tr>
</tbody>
</table>

M97. **Proposed by the Mayhem Staff.**

A standard \( 8 \times 8 \) checkerboard consists of 64 unit squares. A \( T \)-shaped tile consists of five unit squares, as shown below. The tile must be placed on the checkerboard to cover exactly five unit squares on the board.

(a) What is the maximum number of non-overlapping tiles that can be placed on the board in this manner?

(b) Assuming that overlapping is permitted, what is the minimum number of tiles required to cover the board?

*Ed:* No solutions have been received. The problem remains open.
**M98. Proposed by the Mayhem Staff.**

We say that \( N \) is an **automorphic** number if the value of \( N^2 \) ends with the string of digits forming \( N \). For example, 6 is automorphic since \( 6^2 \) ends in 6.

(a) Find all two digit automorphic numbers in base 10.

(b) Find all two digit automorphic numbers in base 6.

**Solution by Robert Bilinski, Outremont, QC, adapted by the editors.**

(a) It is a well-known fact that numbers ending in 0, 1, 5, or 6 have the same last digit in their second power.

**Case 1:** Let \( N = 10a \). Then \( N^2 = 100a^2 \) and \( N \) cannot be automorphic.

**Case 2:** Let \( N = 10a + 1 \). Then \( N^2 = 100a^2 + 20a + 1 \). The tens digit of \( N \) is given by \( a \) and the tens digit of \( a^2 \) is given by \( 2a \). The only digit which is equal to its double is 0, but that would mean that \( N \) is a one-digit number.

**Case 3:** Let \( N = 10a + 5 \). Then \( N^2 = 100a^2 + 100a + 25 = 100(a^2 + a) + 25 \). The tens digit of \( N \) is \( a \) and that of \( N^2 \) is 2. Clearly, \( N = 25 \) is the only two-digit solution.

**Case 4:** Let \( N = 10a + 6 \). Then

\[
N^2 = 100a^2 + 120a + 36 = 100(a^2 + a) + 10(2a + 3) + 6.
\]

We need \( 2a + 3 \equiv a \pmod{10} \), which gives us \( a = 7 \). Hence, a second automorphic number is \( N = 76 \).

Therefore, the only automorphic numbers in base 10 are 25 and 76.

(b) We simply consider each of the 6 possible remainders when \( N \) is divided by 6.

**Case 1:** Let \( N = 6b \). Then \( N^2 = 36b^2 \) always ends in 00 in base 6. These numbers keep their last digit but cannot be automorphic, since the second digit is never \( b \).

**Case 2:** Let \( N = 6b + 1 \). Then \( N^2 = 36b^2 + 12b + 1 \). These numbers keep their last digit, but the second digits are not equal unless \( 2b = b \); that is, \( b = 0 \). This would make \( N \) a one-digit number.

**Case 3:** Let \( N = 6b + 2 \). Then \( N^2 = 36b^2 + 24b + 4 \). These numbers don't even keep their last digit.

**Case 4:** Let \( N = 6b + 3 \). Then \( N^2 = 36b^2 + 36b + 9 = 36(b^2 + b) + 6 + 3 \) always ends in 13 in base 6. Thus, we get \( b = 1 \), and we find that an automorphic number is \( N = 13 \) (base 6).

**Case 5:** Let \( N = 6b + 4 \). Then

\[
N^2 = 36b^2 + 48b + 16 = 36(b^2 + b) + 6(2b + 2) + 4.
\]
These numbers keep their last digit. We need $2b + 2 \equiv b \pmod{6}$, which gives $b = 4$. Hence, a second automorphic number is $N = 44$ (base 6).

**Case 6:** Let $N = 6b + 5$. Then the last digit of $N^2$ is 1. The last digit doesn't even repeat.

Therefore, the only numbers written with two digits in base 6 which are automorphic are $13_{(6)} = 9_{(10)}$ and $44_{(6)} = 28_{(10)}$.

**M99. Proposed by the Mayhem Staff.**

Prove that for all positive integers $n$,

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}.$$  

I. **Similar solutions by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Siwen Sun, grade 12 student, Collège Saint-Louis, Lachine, QC.**

Denote by $S$ the left side of the identity given in the statement. Taking into account that $\binom{n}{k} = \binom{n}{n-k}$, for $k = 0, 1, \ldots, n$, we have

$$S = 0 \binom{n}{0} + 1 \binom{n}{1} + 2 \binom{n}{2} + \cdots + (n-1) \binom{n}{n} + n \binom{n}{n} = n \binom{n}{0} + (n-1) \binom{n}{1} + (n-2) \binom{n}{2} + \cdots + 1 \binom{n}{n-1} + 0 \binom{n}{n}.$$  

Adding up the preceding expressions, we get $2S = n \sum_{k=0}^{n} \binom{n}{k} = n2^n$, which gives the desired result.

II. **Solution by Robert Bilinski, Outremont, QC.**

Since $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, we have

$$\frac{d}{da} (a + b)^n = n(a + b)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} ka^{k-1} b^{n-k}.$$  

Setting $a = b = 1$ in the last equality, we get the desired result.

**M100. Proposed by the Mayhem Staff.**

Mr. and Mrs. Smith are at a party with three other married couples. Since some of the guests are not acquainted with one another, various handshakes take place. No one shakes hands with his or her spouse, and of course, no one shakes hands with himself or herself. After all of the introductions have been made, Mrs. Smith asks the other seven people how many hands they shook. Surprisingly, they all give different answers. How many hands did Mr. Smith shake?
Solution by Laura Steil, student, Samford University, Birmingham, Alabama, USA.

Since there are eight people at the party, and no person shakes his/her own hand or his/her spouse's hand, the maximum number of handshakes per person was 6. Also, because Mrs. Smith had seven different answers for the number of hands each person shook, the answers had to range from 6 down to 0.

Let A be the person who shook 6 hands, B the person who shook 5 hands, C the person who shook 4 hands, D the person who shook 3 hands, E the person who shook 2 hands, F the person who shook 1 hand, and G the person who shook 0 hands. Since A could not shake hands with G (because G shook 0 hands), then A had to shake hands with B, C, D, E, F, and Mrs. Smith. Also, since G was the only person whose hand A did not shake, then A and G must be a couple.

Since B shook hands with A, we see that B shook 4 other hands. Since B could not have shaken hands with F (because F only shook one hand, and that hand belonged to A), then B must have shaken hands with C, D, E, and Mrs. Smith. Also, B and F must be a couple, because F and G are the only possible partners for B, and G is already paired with A.

Since C shook hands with A and B, we see that C shook 2 other hands. Since C could not have shaken hands with E (because E shook only the hands of A and B), then C shook hands with D and Mrs. Smith. Also, C and E must be a couple, because the only possible partners for C are E, F, and G, and F and G are already paired with B and A, respectively.

Since all the others are paired up, this leaves D and Mrs. Smith as the last couple. Hence, D is Mr. Smith, who therefore shook 3 hands.

One incorrect solution was received.

M101. Proposed by the Mayhem Staff.

Find the smallest value of $k$ such that $k!$ ends with 100 zeroes. [Note: $k! = k(k - 1)(k - 2) \cdots (3)(2)(1)$.]

Solution by Robert Bilinski, Outremont, QC.

Since fives are rarer than twos, we will have 100 zeros at the end of $k!$ when $k$ contains the 100th factor of five among the integers from 1 to $k$. The first occurrence of the factor 5 comes when $k = 5$; the second occurs when $k = 10$; the third when $k = 15$; the fourth when $k = 20$; whereas $k = 25$ will generate the fifth and sixth. This means that $k = 50$ generates the 11th and 12th factors. And $k = 100$ generates the 23rd and 24th factors. On the other hand, $k = 125$ generates not only the 29th and 30th such factors, but also the 31st factor. Proceeding in this manner, we find that $k = 375$ generates the 93rd factor. Then, $k = 400$ has the 99th factor and $k = 405$ has the 100th factor.

Hence, 405! will be the smallest factorial ending in 100 zeroes.

One incorrect solution was received.
M102. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Suppose that \(ABCD\) is a parallelogram and that \(G_A, G_B, G_C,\) and \(G_D\) are the centroids of \(\triangle BCD, \triangle ACD, \triangle ABD,\) and \(\triangle ABC,\) respectively.

Prove that:

1. \(G_A G_B G_C G_D\) is a parallelogram;

2. \(\frac{[G_A G_B G_C G_D]}{[ABCD]} = \frac{1}{9}\), where \([ABCD]\) is the area of \(ABCD\).

I. Solution by Gustavo Krinner, Universidad CAECE, Buenos Aires, Argentina.

1. Let \(O\) be the intersection of diagonals \(BD\) and \(AC\). Since \(CO\) is the median corresponding to side \(BD\) in \(\triangle BCD\), we see that the centroid \(G_A\) lies on \(CO\) and \(OG_A = \frac{1}{3}OC\). Similarly, \(G_B\) lies on \(OD\) and \(OG_B = \frac{1}{3}OD\). Then \(\triangle COD\) and \(\triangle G_A OG_B\) are similar. Hence, \(\angle OGA = \angle OCD\), and therefore,

\[
CD \parallel G_A G_B.
\]

Similarly, we obtain

\[
AB \parallel G_C G_D, \quad (2)
\]

\[
BC \parallel G_D G_A, \quad (3)
\]

\[
DA \parallel G_B G_C. \quad (4)
\]

Finally, \(G_A G_B G_C G_D\) is a parallelogram, because, from (1) and (2), we have \(G_A G_B \parallel G_D G_C\), and from (3) and (4), we have \(G_A G_D \parallel G_B G_C\).

2. From part 1 above, \(\triangle COD\) and \(\triangle G_A OG_B\) are similar with ratio \(\frac{1}{3}\). Then, \(\frac{[\triangle G_A OG_B]}{[\triangle COD]} = \frac{1}{9}\), where \([XYZ]\) denotes the area of triangle \(XYZ\). Similarly, \(\frac{[\triangle G_B OG_C]}{[\triangle DOA]} = \frac{1}{9}\), \(\frac{[\triangle G_C OG_D]}{[\triangle AOB]} = \frac{1}{9}\), and \(\frac{[\triangle G_D OG_A]}{[\triangle BOC]} = \frac{1}{9}\). Hence, \(\frac{[G_A G_B G_C G_D]}{[ABCD]} = \frac{1}{9}\).

II. Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let \(A(a_1, b_1), B(a_2, b_2), C(a_3, b_3),\) and \(D(a_4, b_4)\) be the coordinates of the vertices of the given quadrilateral \(ABCD\). It is well known that if \(ABCD\) is a parallelogram, then (i) \(\overrightarrow{AB} \times \overrightarrow{CD} = \overrightarrow{0}\) and \(\overrightarrow{AD} \times \overrightarrow{BC} = \overrightarrow{0}\), and (ii) \([ABCD] = \|\overrightarrow{AB} \times \overrightarrow{AD}\|\), where \(\times\) is the vector cross-product, and \([ABCD]\) is the area of the plane figure \(ABCD\).

1. The coordinates of \(G_A, G_B, G_C,\) and \(G_D\) are

\[
G_A = \left(\frac{a_2 + a_3 + a_4}{3}, \frac{b_2 + b_3 + b_4}{3}\right),
\]

\[
G_B = \left(\frac{a_1 + a_3 + a_4}{3}, \frac{b_1 + b_3 + b_4}{3}\right).
\]
\[ G_C = \left( \frac{a_1 + a_2 + a_4}{3}, \frac{b_1 + b_2 + b_4}{3} \right), \]
\[ G_D = \left( \frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3} \right). \]

Taking into account (i), it follows that
\[ \overrightarrow{G_A G_B} \times \overrightarrow{G_C G_D} = \frac{1}{9}(\overrightarrow{AB} \times \overrightarrow{CD}) = 0 \]
and
\[ \overrightarrow{G_A G_D} \times \overrightarrow{G_B G_C} = \frac{1}{9}(\overrightarrow{AD} \times \overrightarrow{BC}) = 0. \]

That is, \( G_A G_B G_C G_D \) is a parallelogram.

2. Taking into account (ii), we get
\[ [G_A G_B G_D G] = \|G_A G_B \times G_A G_D\| = \frac{1}{9}\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \frac{1}{9}[ABCD]. \]

\textbf{M103. Proposed by the Mayhem Staff.}

\textbf{Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.}

Since
\[ 100^{1/n} \times 100^{2/n} \times \cdots \times 100^{2003/n} = 1000. \]

\textbf{Solution by Gabriel Krimker, grade 9 student, Buenos Aires, Argentina.}

Since
\[ 100^{1/n} \times 100^{2/n} \times \cdots \times 100^{2003/n} = (10^2)^{\frac{1+2+\cdots+2003}{n}} = 10^{\frac{2003 \times 2004}{n}} \]
we need only solve \( 10^{\frac{2003 \times 2004}{n}} = 10^3 \); whence, \( n = \frac{2003 \times 2004}{3} = 1338004. \)

Also solved by Robert Bilinski, Outremont, QC, and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

\textbf{M104. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.}

Suppose that \( ABCD \) is a parallelogram and that \( O_A, O_B, O_C, \) and \( O_D \) are the circumcentres of \( \triangle BCD, \triangle ACD, \triangle ABD, \) and \( \triangle ABC \), respectively.

Prove that:
\begin{enumerate}
  \item \( O_A O_B O_C O_D \) is a parallelogram;
  \item parallelograms \( ABCD \) and \( O_A O_B O_C O_D \) are similar;
  \item \( AO_B CO_D \) is a parallelogram;
  \item \( O_A B O_C D \) is a parallelogram;
  \item parallelograms \( AO_B CO_D \) and \( O_A B O_C D \) are similar.
\end{enumerate}
Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina, adapted by the editors.

If $ABCD$ is a rectangle, then $O_A = O_B = O_C = O_D$. Obviously we must assume that this is not the case.

1. Let $M_{CD}$, $M_{AD}$, $M_{AB}$, and $M_{BC}$ be the perpendicular bisectors of the sides $CD$, $AD$, $AB$, and $BC$, respectively. Since $AB \parallel CD$, and since $M_{CD} \perp CD$ and $M_{AB} \perp AB$, we must have $M_{CD} \parallel M_{AB}$. Similarly, $M_{BC} \parallel M_{AD}$.

Since $O_A$ and $O_B$ lie on $M_{CD}$, and $O_C$ and $O_D$ lie on $M_{AB}$, we get

$$O_AO_B \parallel O_CO_D.$$  

Similarly,

$$O_AO_D \parallel O_BO_C,$$

because $O_A$ and $O_D$ lie on $M_{BC}$, and $O_B$ and $O_C$ lie on $M_{AD}$. From (1) and (2), we see that $O_AO_BO_CO_D$ is a parallelogram.

2. To prove that parallelograms $ABCD$ and $O_AO_BO_CO_D$ are similar, it suffices to prove that $\triangle ABC$ and $\triangle O_BO_AO_D$ are similar.

We claim it is impossible to have both $O_AO_D \parallel AC$ and $O_AO_B \parallel BD$. If it were true that $O_AO_D \parallel AC$ and $O_AO_B \parallel BD$, then we would have $AC \perp BC$ (since $O_AO_D \perp BC$) and $BD \perp CD$ (since $O_AO_B \perp CD$). This would mean that the adjacent angles $BCD$ and $CDA$ of the parallelogram $ABCD$ are both obtuse, a contradiction. Hence, without loss of generality, we may assume that $O_AO_D$ meets $AC$.

Let $O$ be the intersection of the diagonals $AC$ and $BD$, let $P$ be the intersection of $O_AO_D$ with $AC$, and let $Q$ be the intersection of $O_AO_D$ with the line $AD$ (these must meet, since $O_AO_D \perp BC$ and $BC \parallel AD$). Since $O_B$ and $O_D$ lie on the perpendicular bisector of $AC$, we have $O_BO_D \perp AC$. On the other hand, $O_AO_D \perp AD$. Then, $\triangle APQ$ and $\triangle O_DPO$ are right triangles and have a common angle $\angle APQ = \angle O_DPO$. Thus, $\triangle APQ$ and $\triangle O_DPO$ are similar. Hence, $\angle PAQ = \angle PO.DO$, and since $\angle PAQ = \angle ACB$, we obtain

$$\angle O_AO_DO_B = \angle ACB.$$ (3)

Let $M_1$ be the mid-point of $AD$, and let $M_2$ be the mid-point of $AB$. Note that quadrilateral $AM_1OM_2M_2$ has two right angles, namely $\angle O_CM_1A$ and $\angle O_CM_2A$. Therefore, $\angle M_1AM_2$ and $\angle M_1OM_2$ are supplementary.

Since $\angle M_1OM_2 = O_BO_AO_D$, and $\angle BAC$ and $\angle ABC$ are supplementary, we have

$$\angle O_BO_AO_D = \angle ABC.$$ (4)

From (3) and (4), we see that $\triangle ABC$ and $\triangle O_BO_AO_D$ are similar. Hence, parallelograms $ABCD$ and $O_AO_BO_CO_D$ are similar.

3. Let $O$ be the intersection of the diagonals $AC$ and $BD$. Note that $OO_B \perp AC$, $OO_D \perp AC$, and $O$ is the mid-point of $AC$. Thus $O$, $O_B$,
$O_D$ are collinear. Similarly, $O$, $O_A$, and $O_C$ are collinear. Hence, $O$ is also the intersection of $O_B O_D$ and $O_A O_C$.

From part 2, $O_A O_B O_C O_D$ is a parallelogram, and $O$ is the intersection of its diagonals; whence, $OO_B = OO_D$. Since $O$ is the mid-point of both $AC$ and $O_B O_D$, we see that $AO_B CO_D$ is a parallelogram.

4. Since $O$ is the mid-point of $O_A O_C$ and the mid-point of $BD$, we observe that $BO_A DO_C$ is a parallelogram.

5. From part 3, we see that $AO_B CO_D$ is a parallelogram and that $O_B A = O_B C$ because $O_B$ is the circumcentre of $\triangle ACD$. Then, $AO_B CO_D$ is a rhombus. Similarly, from part 4, we see that $O_A BO_C D$ is a parallelogram and $O_A B = O_A D$ because $O_A$ is the circumcentre of $\triangle BCD$. Then $O_A BO_C D$ also is a rhombus. To prove that two rhombi are similar, it suffices to prove that their diagonals are proportional. From part 2, parallelograms $ABCD$ and $O_A O_B O_C O_D$ are similar. Hence,

$$\frac{AC}{BD} = \frac{O_B O_D}{O_A O_C}.$$

The conclusion follows.

Also solved by Robert Bilinski, Outremont, QC.

M105. Proposed by Andrew Critch, Clareville High School, Clareville, NL.

Suppose that the roots of $P(x) = x^3 - 2kx^2 - 3x^2 + hx - 4$ are distinct, and that $P(k) = P(k + 1) = 0$. Determine the value of $h$.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Since $P(k) = P(k + 1)$, we have

$$-k^3 - 4k^2 - 5k + hk + h - 6 = -k^3 - 3k^2 + hk - 4$$

$$h = k^2 + 5k + 2.$$

Then $P(x) = x^3 - 2kx^2 - 3x^2 + (k^2 + 5k + 2)x - 4$ and $P(k) = 2(k^2 + k - 2)$.

Since $k$ is a root of $P$, we conclude that $k = 1$ or $k = -2$.

For $k = 1$, we have $P(x) = (x - 1)(x - 2)$; this contradicts the fact that the roots of $P$ are distinct. Thus, $k \neq 1$.

For $k = -2$, we have $P(x) = (x - 2)(x + 1)(x + 2)$. Therefore, $k = -2$ and $h = -4$.

Also solved by Robert Bilinski, Outremont, QC and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.
Problem of the Month

Ian VanderBurgh, University of Waterloo

Some problems leap off the page and make you want to try to solve them when you first see them. For me, the following was one of those problems. Another thing that I really like about this problem is that it can be used with younger students, since the only tools necessary are logical thinking and addition.

Problem. (2002 Australian Mathematics Competition)

A $4 \times 4$ antimagic square is an arrangement of the numbers from 1 to 16 (inclusive) in a square, so that the totals of each of the four rows and four columns and two main diagonals are ten consecutive numbers in some order. The diagram below shows an incomplete antimagic square. When it is completed, what number will replace the star?

$$
\begin{array}{cccc}
4 & 5 & 7 & 14 \\
6 & 13 & 3 & * \\
11 & 12 & 9 & \\
10 & & & \\
\end{array}
$$

Solving this problem is a great example of just following your nose. I will present one way of doing this—there are certainly other approaches that can be taken. I have tried to explain the reasoning behind each step (which does make the solution a little bit longer than it needs to be).

Solution. The first question to ask is what numbers are missing from the antimagic square. Since the square is supposed to contain the numbers from 1 to 16 (inclusive), we are missing 1, 2, 8, 15, and 16.

The next logical thing to do is to add up any completed rows, columns, and diagonals to get a sense of what the sums should be:

- First row: $4 + 5 + 7 + 14 = 30$
- First column: $4 + 6 + 11 + 10 = 31$
- Diagonal: $10 + 12 + 3 + 14 = 39$

How does this help? We're told that the ten row, column, and diagonal sums are ten consecutive (and thus distinct) positive integers. Since we already have 30, 31, and 39, then these ten sums must be the integers from 30 to 39 (of which there are 10).
The next step that makes sense is to look at the sums of some of the partly completed rows, columns, and diagonals.

- Second row: \(6 + 13 + 3 = 22\)
- Third row: \(11 + 12 + 9 = 32\)
- Second column: \(5 + 13 + 12 = 30\)
- Third column: \(7 + 3 + 9 = 19\)
- Other diagonal: \(4 + 13 + 9 = 26\)

What can we conclude from this? Since the sum of the entries in the second row must be between 32 and 38 (30, 31, and 39 are already taken), the star (*) cannot be replaced by 1, 2, or 8; hence, it must be 15 or 16. (We have just narrowed down the number of possibilities for * from 5 to 2.) Similarly, the last entry in the third row must be a 1 or 2; the last entry in the second column must be 2 or 8; and the last entry in the third column must be 15 or 16.

However, it is when we reach the diagonal that we can actually reach a definitive conclusion. Since the sum of the entries on the diagonal is between 32 and 38, the last entry in the diagonal must be 8. Retracing our steps, we see that the last entry in the second column must be 2 and the last entry in the third row must be 1.

Let us regroup and see where we are:

\[
\begin{array}{cccc}
4 & 5 & 7 & 14 \\
6 & 13 & 3 & * \\
11 & 12 & 9 & 1 \\
10 & 2 & 8 &
\end{array}
\]

Now our completed rows, columns, and diagonals have sums of 30, 31, 32, 33, 34, and 39 (just add them up!), and we still have to insert 15 and 16. Looking at the third column, the sum of the first three entries is 19; thus, the remaining entry must be 16 (otherwise, the sum would be 34, which we already have elsewhere). Therefore, the completed array must be

\[
\begin{array}{cccc}
4 & 5 & 7 & 14 \\
6 & 13 & 3 & 15 \\
11 & 12 & 9 & 1 \\
10 & 2 & 16 & 8 \\
\end{array}
\]

We can check that all ten row, column, and diagonal sums are distinct. Thus, * is replaced by 15.
Pólya’s Paragon

Triangular Tidbits (Part 1)

Shawn Godin

The adventurous student who takes the time to look at some high school mathematics textbooks from years gone by will find that the amount of geometry that students are exposed to today is far less than it was 20 to 50 years ago. For the next couple of issues we will look at some properties of triangles that may not be familiar to high school students today.

Just to get us going, we will do a quick review of the definitions of the trigonometric ratios sine, cosine, and tangent. In a right triangle, when referring to one of the acute angles, we will refer to the sides as the hypotenuse (across from the right angle), the opposite side (across from the angle in question) and the adjacent side (the side which is “attached” to the angle and different from the hypotenuse). For example, in the triangle pictured below, the hypotenuse is BC, AB is the side opposite ∠C, and AC is the side adjacent to ∠C.

Keeping this terminology in mind, for an acute angle θ, we define:

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan \theta = \frac{\text{opposite}}{\text{adjacent}}
\]

We can extend these definitions by considering a circle of radius r centred at the origin O. We will start with a point P on the circle in the first quadrant. Let (x, y) be the coordinates of P. If we look at the right triangle POX, where X is the point on the positive x-axis directly below P, we get:

\[
\sin \theta = \frac{y}{r} \\
\cos \theta = \frac{x}{r} \\
\tan \theta = \frac{y}{x}
\]
We can use these to extend the definitions so that our ratios are defined for any possible angle. The original definitions can be used to “solve” a right triangle if we have a certain amount of information. Since the trigonometric ratios are well known, their values can be calculated for any angle with an inexpensive calculator (or with a set of trigonometric tables if you happened to find a textbook 20 years old or older in your original search). Unfortunately, we cannot use the original definitions to solve a non-right triangle. Let’s employ some circle geometry to develop the so-called Law of Sines.

We need two facts.

1. If $\alpha$ and $\beta$ are two angles with their vertices on the circumference of a circle and subtending the same arc (on the circumference of the circle), then $\alpha = \beta$.

2. If $\alpha$ and $\beta$ are two angles subtending the same arc and if the vertex of $\alpha$ is on the circumference of a circle and the vertex of $\beta$ is at the centre of the circle, then $\beta = 2\alpha$. A consequence of this is that the angle subtended at the circumference of a circle by a diameter is $90^\circ$.

Now consider any acute triangle $ABC$. Construct its circumcircle, and let the radius of this circumcircle be $R$. Denote by $a$, $b$, and $c$ the sides opposite angles $A$, $B$, and $C$, respectively. Construct a point $P$ on the circle so that $PC$ is a diameter. Then $\angle A = \angle BPC$ and $\angle PBC = 90^\circ$. Thus,

$$\sin A = \sin BPC = \frac{BC}{PC} = \frac{a}{2R}.$$

That is, $\frac{a}{\sin A} = 2R$. Similar arguments give $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$.

We have shown that the following result holds for any acute triangle. 
Law of Sines. For any acute triangle $ABC$ with circumradius $R$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$ 

For homework, show that the Law of Sines holds for any triangle (not necessarily acute).

Next issue we will continue looking at properties of triangles.