Polya's Paragon

What's the difference? (Part 3)

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In the last two columns ([2004 : 77–8; 2004 : 200–201]) we analyzed sequences by examining the sequences of differences, second differences, and so on. Now we wish to apply our ideas to sequences that are not generated by polynomials.

Consider the sequence \( \{a_n\} \) where \( a_n = 2^n \) (Question #1 from March's homework). We find that, for this particular sequence, each sequence of differences is equal to the original sequence.

\[
\begin{array}{c|c|c|c}
  n & 1d_n & 2d_n \\
  \hline
  1 & 1 & 1 \\
  2 & 2 & 2 \\
  4 & 4 & 4 \\
  8 & 8 & 8 \\
  16 & & \\
\end{array}
\]

We can extend the sequences easily. We have \( 1d_4 = t_4 = 16 \); then, since \( 1d_4 = t_5 - t_4 \), we see that \( t_5 = 1d_4 + t_4 = 16 + 16 = 32 \). Now that we know a new term, we can repeat the process to find as many terms as we like. The same process can be used on the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, ... (Question #4 from March's homework).

In general, if any of the sequences of differences can be extended, then we can extend the original sequence. But in some cases the table of differences doesn't yield any patterns. What do we do then?

Consider the sequence \( \{b_n\} \) where \( b_n = 5^n \) (Question #2 from March's homework). Constructing the table of differences, we get

\[
\begin{array}{c|c|c|c|c}
  n & 1d_n & 2d_n & 3d_n & 4d_n \\
  \hline
  1 & 4 & & & \\
  5 & 20 & 16 & & \\
  25 & 100 & 80 & 64 & \\
  125 & 500 & 400 & 320 & 256 \\
  625 & & & & \\
\end{array}
\]

If we don't see any patterns here, we can consider the difference table of depth 2. To construct this, we consider the sequence \( t_0, 1d_0, 2d_0, 3d_0, \ldots \), which in our case is 1, 4, 16, 64, 256, ... If nothing appears at this level, we can construct the difference table of depth 3 from the difference table of depth 2 using the same method. If you continue the process you will be led to the powers of 2.
Now we can combine these methods with our inversion formula from the May issue (corrected to remove a typo):

\[ t_{n+k} = \sum_{i=0}^{n} \binom{n}{i} d_k . \]

**Example:** 0, 2, 9, 31, 97, 291, ...

Constructing the difference table, we get

<table>
<thead>
<tr>
<th>( t_n )</th>
<th>( 1d_n )</th>
<th>( 2d_n )</th>
<th>( 3d_n )</th>
<th>( 4d_n )</th>
<th>( 5d_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>19</td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>22</td>
<td>44</td>
<td>84</td>
<td>55</td>
<td>29</td>
</tr>
<tr>
<td>31</td>
<td>66</td>
<td>128</td>
<td>36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>97</td>
<td>194</td>
<td>291</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This isn't much help. Let's construct the difference table of depth 2. Starting with the sequence \( \{s_n\} = \{0, 2, 5, 10, 19, 36, \ldots\} \), we get

<table>
<thead>
<tr>
<th>( s_n )</th>
<th>( 1d'_n )</th>
<th>( 2d'_n )</th>
<th>( 3d'_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>10</td>
<td>19</td>
<td>4</td>
<td>36</td>
</tr>
</tbody>
</table>

We see that \( i d_n = 2^n \) for \( i \geq 2 \). Using our inversion formula with \( k = 0 \), we get

\[
 s_n = \sum_{i=0}^{n} \binom{n}{i} d_0 = 0 + 2n + \sum_{i=2}^{n} \binom{n}{i} \\
= \sum_{i=0}^{n} \binom{n}{i} + n - 1 = 2^n + n - 1 .
\]

Hence, applying the inversion formula to the original sequence, we obtain

\[
 t_n = \sum_{i=0}^{n} \binom{n}{i} (2^i + i - 1) .
\]

You should now have a variety of tools for attacking sequences. Until next time, happy problem solving!