Mayhem Solutions

M83. Proposed by the Mayhem Staff.

Five balls numbered 1 to 5 are put into a box. A ball is drawn at random, its number recorded, and the ball returned to the box. This process is repeated until five numbers have been recorded. If the sum of the recorded numbers is 15, what is the probability that the number 3 was drawn each time?

Solution by Geneviève Lalonde, Massey, ON.

The table below shows the number of ways of drawing five balls with a sum of 15. The total number of ways is 381. Therefore, the probability that every draw was a 3 is \( \frac{1}{381} \).

<table>
<thead>
<tr>
<th>Balls</th>
<th>Number of Ways</th>
<th>Balls</th>
<th>Number of Ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 5, 3, 1, 1</td>
<td>( \frac{51}{2721} ) = 30</td>
<td>5, 3, 3, 2, 2</td>
<td>( \frac{51}{2721} ) = 30</td>
</tr>
<tr>
<td>5, 5, 2, 2, 1</td>
<td>( \frac{51}{2721} ) = 30</td>
<td>4, 4, 4, 2, 1</td>
<td>( \frac{51}{2721} ) = 20</td>
</tr>
<tr>
<td>5, 4, 4, 1, 1</td>
<td>( \frac{51}{2721} ) = 30</td>
<td>4, 4, 3, 3, 1</td>
<td>( \frac{51}{2721} ) = 30</td>
</tr>
<tr>
<td>5, 4, 3, 2, 1</td>
<td>5! = 120</td>
<td>4, 4, 3, 2, 2</td>
<td>( \frac{51}{31} ) = 30</td>
</tr>
<tr>
<td>5, 4, 2, 2, 2</td>
<td>( \frac{51}{31} ) = 20</td>
<td>4, 3, 3, 3, 2</td>
<td>( \frac{51}{31} ) = 20</td>
</tr>
<tr>
<td>5, 3, 3, 3, 1</td>
<td>( \frac{51}{31} ) = 20</td>
<td>3, 3, 3, 3, 3</td>
<td>( \frac{51}{31} ) = 1</td>
</tr>
</tbody>
</table>

One incorrect solution was received.

M84. Proposed by the Mayhem Staff.

Consider the two functions \( f(x) = x^2 - 2ax + 1 \) and \( g(x) = 2b(a-x) \), where \( a, b, x \in \mathbb{R} \). We will consider each pair of constants \( a \) and \( b \) as a point \((a, b)\) in the \( ab\)-plane. Let \( A \) be the set of points \((a, b)\) for which the graphs of \( y = f(x) \) and \( y = g(x) \) do not intersect. Find the area of \( A \).

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

The graphs of \( y = f(x) \) and \( y = g(x) \) do not intersect when the equation \( x^2 - 2ax + 1 = 2b(a-x) \) has no solution. This happens if and only if \( b^2 + a^2 < 1 \). Hence, \( A \) is the open disk of radius 1 centred at the origin. Thus, the area of \( A \) is \( \pi \).

Also solved by Robert Bilinski, Outremont, QC.

M85. Proposed by the Mayhem Staff.

Find a triangle whose integer sides are in arithmetic progression with a common difference of 2 and which has an area of 336.
Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let \( a, a+2, \) and \( a+4 \) be the sides of the triangle. By Heron's Formula, we have

\[
336 = \sqrt{\frac{1}{16}(a + 6)(a + 2)(a - 2)(3a + 6)}.
\]

Hence, \((a + 6)(a + 2)(a - 2)(3a + 6) = 16 \cdot 336^2 = 1806336\), which has the unique positive integer solution \( a = 26 \). Then the triangle sides are 26, 28, and 30.

Also solved by Robert Bilinski, Outremont, QC.

**M86.** Proposé par l'équipe de Mayhem.

Soit deux nombres entiers positifs \( a \) et \( b \). Parmi les nombres

\[
a, 2a, 3a, \ldots, (b-1)a, ba
\]

combien y a-t-il qui soient divisibles par \( b \)?

Solution par Robert Bilinski, Outremont, QC.

Si \((a, b) = 1\), alors il y a seulement \( ba \) qui est divisible par \( b \). Il y en a donc 1.

Si \((a, b) = b\), alors tous les nombres sont divisibles par \( b \). Il y en a donc \( b \).

Si \((a, b) = k\), alors \( a = ks \) et \( b = kt \) avec \((s, t) = 1\). La question devient : "Parmi les nombres \( s, 2s, \ldots, (kt-1)s, kts \), combien sont divisibles par \( t \) ?" Dans la suite de nombre on trouvera \( ts, 2ts, \ldots, (k-1)ts, kts \), donc \( k \) nombres sont divisibles par \( t \). Autrement dit, il y aura toujours \((a, b)\) nombres divisibles par \( b \) dans la liste.

En outre résolu par Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine.

**M87.** Proposed by the Mayhem Staff.

Seven people, \( A, B, C, D, E, F, G \), are on one side of a river. To get across the river they have a rowboat, but it can only fit two people at a time. The times that would be required for the people to row across individually are 1, 2, 3, 5, 10, 15, and 20 minutes, respectively. However, when two people are in the boat, the time it takes them to row across is the same as the time necessary for the slower of the two to row across individually. Assuming that no one can cross without the boat, what is the minimum time for all seven people to get across the river?

Solution by Geneviève Lalonde, Massey, ON.

The minimum time is 46 minutes and can be accomplished as shown in the table below:
<table>
<thead>
<tr>
<th>Cross</th>
<th>Time</th>
<th>Return</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A, B$</td>
<td>2</td>
<td>$A$</td>
<td>1</td>
</tr>
<tr>
<td>$F, G$</td>
<td>20</td>
<td>$B$</td>
<td>2</td>
</tr>
<tr>
<td>$A, B$</td>
<td>2</td>
<td>$A$</td>
<td>1</td>
</tr>
<tr>
<td>$D, E$</td>
<td>10</td>
<td>$B$</td>
<td>2</td>
</tr>
<tr>
<td>$A, C$</td>
<td>3</td>
<td>$A$</td>
<td>1</td>
</tr>
<tr>
<td>$A, B$</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One incorrect solution was received.

**M88. Proposed by the Mayhem Staff.**

A set $S$ consists of six numbers. When we take all possible subsets of $S$ containing 5 elements, the sums of the elements of these subsets are 87, 92, 98, 99, 104, and 110, respectively. Determine the six numbers in $S$.

**Solution by Allen O’Hara, grade 10 student, Oakridge Secondary School, London, ON.**

Let $S = \{a, b, c, d, e, f\}$. From the information provided, we can create the following set of equations:

\[
\begin{align*}
    a + b + c + d + e & = 87, \\
    a + b + c + d + f & = 92, \\
    a + b + c + e + f & = 98, \\
    a + b + d + e + f & = 99, \\
    a + c + d + e + f & = 104, \\
    b + c + d + e + f & = 110.
\end{align*}
\]

By subtracting each equation from the last, we can relate each variable to $a$:

\[
\begin{align*}
    f & = a + 23, \\
    e & = a + 18, \\
    d & = a + 12, \\
    c & = a + 11, \\
    b & = a + 6.
\end{align*}
\]

Now we can rewrite one of the equations (1)–(6) to determine $a$, and then obtain the rest of the numbers. For example, we can rewrite equation (6) as

\[
(a + 6) + (a + 11) + (a + 12) + (a + 18) + (a + 23) = 110,
\]

which can be simplified to:

\[
5a + 70 = 110,
\]

from which we get $a = 8$. Then $b = 14$, $c = 19$, $d = 20$, $e = 26$, and $f = 31$. Thus, $S = \{8, 14, 19, 20, 26, 31\}$.

Also solved by Robert Bilinski, Outremont, QC; Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

**M89. Proposé par l’Équipe de Mayhem.**

Trouver tous les entiers positifs $x$ pour lesquels $x(x + 60)$ est un carré parfait.
I. Solution par Robert Bilinski, Outremont, QC.

Posons \( x = n - 30 \), alors chercher \( x > 0 \) tel que \( x(x + 60) \) soit un carré parfait revient à chercher \( n > 30 \) tel que \( (n - 30)(n + 30) = n^2 - 900 \) soit un carré parfait. Nommons \( k^2 \) ce carré parfait que l'on recherche. Ainsi, \( n^2 - 900 = k^2 \) ou bien \( n^2 - k^2 = (n - k)(n + k) = 900 = 2^2 \cdot 3^2 \cdot 5^2 \).

Nous devons maintenant séparer 900 en deux facteurs \( a \) et \( b \) tels que \( n - k = a \) et \( n + k = b \). On aura ainsi que \( n = \frac{a + b}{2} \) et \( k = \frac{b - a}{2} \). Puisque nous voulons que \( n \) et \( k \) soient des entiers, il faut que \( a \) et \( b \) soient de même parité. Puisque un des deux facteurs de 900 doit être pair, les deux le seront. Le problème revient donc à trouver deux facteurs \( r \) et \( s \) de 225 qui donneront \( n = r + s > 30 \) et \( k = r - s > 0 \). Or les paires différentes de facteurs de 225 sont \( (1, 225), (3, 75), (5, 45), (9, 25) \) et \( (15, 15) \). Elles donnent \( n \in \{226, 78, 50, 34\} \) ou bien \( x \in \{196, 48, 20, 4\} \).

II. Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let \( k \) be a positive integer. The equation \( x^2 + 60x - k^2 = 0 \) has positive integer solutions if and only if \( \sqrt{30^2 + k^2} \) is an integer. Then we must find all Pythagorean triples of the form \( (30, k, \sqrt{30^2 + k^2}) \). This is equivalent to finding all primitive Pythagorean triples \( (a, b, c) \) where \( a \mid 30 \) or \( b \mid 30 \). These triples are \( (3, 4, 5), (5, 12, 13), (8, 15, 17), \) and \( (15, 112, 113) \). Hence, \( x = 20, x = 48, x = 4, \) and \( x = 196, \) respectively.

Also solved by Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC.

M90. Proposed by the Mayhem Staff.

Determine the largest positive integer \( n \) for which \( 2002^n \) is a factor of 2002!. What happens if 2002 is replaced by 2003 or 2004?

Solution by Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC.

Let \( F = 2002! \). Since \( 2002 = 2 \cdot 7 \cdot 11 \cdot 13 \), in 2002! there will be \( [2002/13] = 154 \) multiples of 13 and \( [2002/13^2] = 11 \) multiples of \( 13^2 \) and \( [2002/13^3] = 0 \) multiples of \( 13^3 \). There will be more multiples of 11, 7, and 2. Therefore, the largest integer \( n \) such that \( 2002^n \) divides \( 2002! \) is \( n = 154 + 11 = 165 \). Thus, \( n = 165 \) for 2002.

Since 2003 is a prime number, in 2003! there will be only one multiple of 2003. Thus, only \( 2003^1 \) will divide \( 2003! \). Thus, \( n = 1 \) for 2003.

Since \( 2004 = 2^2 \cdot 3 \cdot 167 \), in 2004! there are \( [2004/167] = 12 \) multiples of 167, and since \( 167^2 > 2004 \), there are 0 multiples of \( 167^2 \). Therefore, the largest integer \( n \) such that \( 2004^n \) divides \( 2004! \) is \( n = 12 \).

Also solved by Robert Bilinski, Outremont, QC; and Allen O'Hara, grade 10 student, Oakridge Secondary School, London, ON.
M91. Proposed by Robert Morewood, Burnaby South Secondary School, Burnaby, BC.

Let $k$ be a four-digit integer. Determine all possible values of $k$ for which $k^{2003}$ ends in the four digits 2003. What happens if 2003 is replaced by 2002 or 2004?

Solution by Geneviève Lalonde, Massey, ON.

Looking at the problem modulo 10, we see that $k^{2003} \equiv 3 \pmod{10}$. Thus, $k \equiv 7 \pmod{10}$.

We next notice that $7^2 \equiv 49 \pmod{100}$, $7^3 \equiv 43 \pmod{100}$, and $7^4 \equiv 1 \pmod{100}$. Let $k \equiv 10a + 7 \pmod{100}$, where $a$ is a single non-negative digit. Then, using the Binomial Theorem, we get

$$k^{2003} \equiv (10a + 7)^{2003} \equiv 7^{2003} + 2003 \cdot 10a \cdot 7^{2002} \equiv 43 + 3 \cdot 10a \cdot 49 \equiv 43 + 70a \pmod{100}.$$

Then, since $k^{2003}$ ends in the digits 2003, we have

$$43 + 70a \equiv 3 \pmod{100},$$
$$70a \equiv 60 \pmod{100},$$
$$7a \equiv 6 \pmod{10},$$
$$a \equiv 8 \pmod{10}.$$

Hence, $k \equiv 87 \pmod{100}$. With a little help from Maple, we find that $k \equiv 587 \pmod{1000}$ and $k \equiv 587 \pmod{10000}$. Thus, the only “four-digit” number is $k = 0587$.

When 2003 is replaced with 2002 or 2004, there is no number that satisfies the congruence modulo 10. Thus, there are no solutions.

M92. Proposed by the Mayhem Staff.

A $3 \times 3$ magic square consists of nine distinct values, such that each of the rows, columns, and diagonals have a constant sum. Below is an example of a $3 \times 3$ magic square.

Suppose that a $3 \times 3$ magic square has a constant sum of $T$. Let the middle entry of this square be $E$. Prove that $T = 3E$.

```
<table>
<thead>
<tr>
<th>2</th>
<th>9</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>
```
Solution by Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC.

Let the magic square be

\[
\begin{array}{ccc}
 a_1 & a_2 & a_3 \\
a_4 & a_5 & a_6 \\
a_7 & a_8 & a_9
\end{array}
\]

Then \( a_5 = E \). Since the sum of the elements horizontally, vertically, or diagonally is equal to \( T \), then

\[
3T = (a_1 + a_5 + a_9) + (a_2 + a_5 + a_8) + (a_3 + a_5 + a_7)
\]

\[
= (a_1 + a_2 + a_3) + 3a_5 + (a_9 + a_8 + a_7)
\]

\[
= T + 3E + T = 3E + 2T,
\]

and hence, \( T = 3E \).

Also solved by Robert Bilinski, Outremont, QC; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

M93. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

In triangle \( ABC \), suppose that \( \tan A, \tan B, \tan C \) are in harmonic progression. Show that \( a^2, b^2, c^2 \) form an arithmetic progression.

[Note: \( x, y, z \) are in harmonic progression if \( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \) form an arithmetic progression.]

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let \( ABC \) be an acute-angled triangle. Let \( AH_a, BH_b, CH_c \) be the altitudes corresponding to sides \( a, b, \) and \( c \), respectively.

Since \( \tan A, \tan B, \tan C \) are in harmonic progression, there exists \( d \in \mathbb{R} \) such that:

\[
\frac{1}{\tan A} = \frac{1}{\tan B} - d \quad \text{(1)}
\]

\[
\frac{1}{\tan C} = \frac{1}{\tan B} + d \quad \text{(2)}
\]

In addition, we have the following relations:

\[
\frac{1}{\tan A} = \frac{AH_b}{BH_b} = \frac{AH_c}{CH_c} \quad \text{(3)}
\]

\[
\frac{1}{\tan B} = \frac{BH_c}{CH_c} = \frac{BH_a}{AH_a} \quad \text{(4)}
\]

\[
\frac{1}{\tan C} = \frac{CH_a}{AH_a} = \frac{CH_b}{BH_b} \quad \text{(5)}
\]
From (3) and (4), (4) and (5), (3) and (5), respectively, we get:
\[
\frac{1}{\tan A} + \frac{1}{\tan B} = \frac{AH_c + BH_c}{CH_c} = \frac{c}{CH_c}, \tag{6}
\]
\[
\frac{1}{\tan B} + \frac{1}{\tan C} = \frac{BH_a + CH_a}{AH_a} = \frac{a}{AH_a}, \tag{7}
\]
\[
\frac{1}{\tan A} + \frac{1}{\tan C} = \frac{AH_b + CH_b}{BH_b} = \frac{b}{BH_b}. \tag{8}
\]

From (1) and (6), (2) and (7), and (1), (2) and (8), respectively, we obtain:
\[
\frac{2}{\tan B} = \frac{c}{CH_c} + d, \tag{9}
\]
\[
\frac{2}{\tan B} = \frac{a}{AH_a} - d, \tag{10}
\]
\[
\frac{2}{\tan B} = \frac{b}{BH_b}. \tag{11}
\]

Let \( k = \frac{2}{\tan B} \). From equations (9), (10) and (11), it follows that:
\[
c^2 = (k - d)^2 \cdot (CH_c)^2, \tag{12}
\]
\[
a^2 = (k + d)^2 \cdot (AH_a)^2, \tag{13}
\]
\[
b^2 = k^2 \cdot (BH_b)^2. \tag{14}
\]

Let \( \Delta \) be the area of \( \triangle ABC \). Since
\[
\Delta = \frac{c \cdot CH_c}{2} = \frac{a \cdot AH_a}{2} = \frac{b \cdot BH_b}{2},
\]
we have:
\[
(CH_c)^2 = \frac{4 \Delta^2}{c^2}, \tag{15}
\]
\[
(AH_a)^2 = \frac{4 \Delta^2}{a^2}, \tag{16}
\]
\[
(BH_b)^2 = \frac{4 \Delta^2}{b^2}. \tag{17}
\]

Combining (12) with (15), (13) with (16), and (14) with (17), we obtain:
\[
c^2 = 2 \Delta(k - d) = 2k \Delta - 2d \Delta,
\]
\[
b^2 = 2 \Delta k,
\]
\[
a^2 = 2 \Delta(k + d) = 2k \Delta + 2d \Delta.
\]

These equations show that \( c^2, b^2, \) and \( a^2 \) are in arithmetic progression with common difference \( 2d \Delta \).