THE OLYMPIAD CORNER
No. 240

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Our first set of problems for this issue comes from Argentina. My thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them for our use, and to Alberto Nettel and Luz Palacios for translating them into English.

XVII ARGENTINIAN MATHEMATICAL OLYMPIAD
National Competition
Level Three

1. The natural numbers are written in succession, forming a sequence of digits. Determine how many numerical characters are in the natural number that contributes to this sequence the digit at position $10^{2000}$.

   Note: The natural number that contributes to the sequence the digit at position 10 has 2 numerical characters because it is 10; the natural number that contributes to the sequence the digit at position $10^2$ has 2 numerical characters because it is 55.

2. Given a triangle $ABC$ with side $AB$ greater than $BC$, let $M$ be the midpoint of $AC$, and let $L$ be the point at which the bisector of $\angle B$ cuts side $AC$. A straight line is drawn through $M$ parallel to $AB$, cutting the bisector $BL$ at $D$, and another straight line is drawn through $L$ parallel to $BC$, cutting the median $BM$ at $E$. Show that $ED$ is perpendicular to $BL$.

3. A board has 32 rows and 10 columns. Paul writes in each square of his board either 1 or $-1$. Matthew, with Paul's board in sight, chooses one or more columns from his own board and, in these columns, negates all of Paul's numbers (putting 1 where Paul has $-1$, and $-1$ where Paul has 1). In the rest of the columns he puts the same numbers as Paul's.

   Matthew wins if he can get each row of his board to be different from each row of Paul's board. Otherwise (that is, if any row of Matthew's board is the same as any row of Paul's board), Paul wins.

   If they both play to perfection, determine which of them has victory guaranteed.

4. Determine how many pairs of natural numbers $(a, b)$ there are such that 4620 is a multiple of $a$, 4620 is a multiple of $b$, and $b$ is a multiple of $a$. 
5. A computer program generates a sequence of numbers by the following rule. The first number is written by Camilo. From that point on, the program does integer division of the last generated number by 18, obtaining an integer quotient and remainder. The sum of this integer quotient and this remainder becomes the next generated number. For example, if Camilo's number is 5291, the computer does $5291 = 293 \times 18 + 17$, and generates the number $310 = 293 + 17$. The next generated number will be 21, given that $310 = 17 \times 18 + 4$ and $17 + 4 = 21$, etc.

For every initial number that Camilo chooses, there comes a point at which the computer generates the same number over and over. Determine which number this will be, if Camilo's initial number is $2^{110}$.

6. An equilateral triangle with area equal to 9 is made out of paper. It is folded in two along a straight line passing through the centre of the triangle and not passing through any of the vertices of the triangle. The result is a four-sided polygon formed by the overlap of the two pieces (created by the fold) and three triangles with no overlap. Determine the minimum possible area of the four-sided polygon created by the overlap.

As the second problem set for this number, we give the 12th Form of the XXI Albanian Mathematical Olympiad for High Schools, Third Round. Thanks again go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them.

XXI ALBANIAN MATHEMATICAL OLYMPIAD
FOR HIGH SCHOOLS
Third Round
March 20, 2000

1. (a) Prove the inequality

\[
\frac{(1 + x_1)(1 + x_2) \cdots (1 + x_n)}{1 + x_1 x_2 \cdots x_n} \leq 2^{n-1}, \quad \forall x_1, x_2, \ldots, x_n \in [1, +\infty).
\]

(b) When does the equality hold?

2. Consider the sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for all $n > 1$. Find

(a) $\lim_{n \to \infty} x_n$;

(b) $\lim_{n \to \infty} 4^n(2 - x_n)$.

3. Prove that, if $0 < a < b < \frac{\pi}{2}$, then

(a) $\frac{a}{b} < \frac{\sin a}{\sin b}$;

(b) $\frac{\sin a}{\sin b} < \frac{\pi}{2} \frac{a}{b}$.
4. (a) Prove that for any convex $n$-gon, where $n > 4$, the arithmetic mean of the lengths of its sides is no greater than the arithmetic mean of the lengths of its diagonals.

(b) Does equality hold for any case?

5. Let $a$, $b$, $c$ be the sides of a triangle, and let $\alpha$, $\beta$, $\gamma$ be the angles opposite the sides $a$, $b$, $c$, respectively.

(a) Prove that $\gamma = 2\alpha$ if and only if $c^2 = a(a + b)$.

(b) Find all triangles such that $a$, $b$, $c$ are natural numbers, $b$ is a prime, and $\gamma = 2\alpha$.

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**MATHEMATICS COMPETITIONS IN FINLAND, 2000–2001**

High School Mathematics Contest, Senior Division,

1st Round

October 10, 2000 — Time: 100 minutes

1. Four unit discs are packed together as shown in the figures. The discs are enclosed by a curve of minimal length. In which one of the configurations is the enclosing curve shorter?

2. Determine real $x$ and $y$ such that $5x^2 - 5y^2 - 24xy + 11y + 3x = 0$.

3. Determine all positive integers $m$ and $n$ such that

$$m^2 - n^2 = 270.$$

4. A number of cross-shaped pieces, as shown, are placed on an $8 \times 8$ chessboard in such a way that the squares of the pieces and the squares of the chessboard are aligned and the pieces do not overlap each other. We say that the board has been filled if no more pieces can be placed on the board satisfying the conditions above. Determine the smallest possible number of pieces with which the board can be filled.
MATHEMATICS COMPETITIONS IN FINLAND, 2000–2001
High School Mathematics Contest, Senior Division, Final Round
February 2, 2001 — Time: 180 minutes

1. Let $ABC$ be a right triangle with hypotenuse $AB$ and altitude $CF$, where $F$ lies on $AB$. The circle through $F$ centred at $B$ and another circle of the same radius centred at $A$ intersect on the side $CB$. Determine $FB : BC$.

2. Two non-intersecting curves have equations $y = ax^2 + bx + c$ and $y = dx^2 + cx + f$, where $ad < 0$. Prove that there exists a straight line having no points in common with the two curves.

3. The positive integers $a$, $b$, and $c$ satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42}.$$

4. In the weekly State Lottery, a sequence of seven numbers is picked at random. Each number may be any of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Determine the probability that the sequence is composed of only five different numbers.

5. Determine $n \in \mathbb{N}$ such that $n^2 + 2$ divides $2 + 2001n$.

Now we turn to the solutions we have received to problems of the St. Petersburg Contests 1965–1984 given in May, 2002 [2002: 201–203].

1. There are $n$ glasses each big enough to hold all the water. Initially, all glasses contain the same amount of water. It is allowed to pour from any glass to any other glass as much water as in the second glass. For which values of $n$ is it possible to collect all water into one glass?

Solution by Pierre Bornsztein, Maisons-Laffitte, France, adapted by the editor.

The integers $n$ for which it is possible to put all the water into one glass are those such that $n = 2^k$ for some non-negative integer $k$.

Let $G_1, G_2, \ldots, G_n$ denote the glasses. We will denote by $(i, j)$ the operation which consists of pouring water from $G_i$ to $G_j$.

If $n = 1$, then the glass $G_1$ contains all the water. If $n = 2$, then the operation $(1, 2)$ puts all the water into $G_2$. If $n = 2^k$ where $k > 1$, then the sequence $(1, 2), (3, 4), \ldots, (2^k - 1, 2^k)$ gives a configuration with $2^{k-1}$ glasses containing equal amounts of water. Repeating this process, we reach a configuration where all the water is in a single glass, which proves that $n = 2^k$ is a possible value for $n$. 
Now consider any \( n \geq 2 \) for which there is a finite sequence of operations by which all the water can be put into one glass, say \( G_n \). Consider any such sequence, and let \( N \) be the number of operations in the sequence. Let \( x \) be the initial amount of water in each glass. Thus, the total amount of water is \( nx \). Without any loss of generality, we may assume that the last operation in the sequence is \((n-1,n)\). Just before this operation, both \( G_n \) and \( G_{n-1} \) contained an amount of water equal to \( nx/2 \), and every other glass was empty.

We proceed by descending induction. Let \( i \) be an integer such that \( 0 < i \leq N \). Suppose that after \( i \) operations, for each \( j \in \{1, 2, \ldots, n\} \), the amount of water contained in \( G_j \) is of the form \( p_j nx/2^a_j \), where \( a_j \) and \( p_j \) are non-negative integers.

Let \((\ell, m)\) be the \( i^{\text{th}} \) operation. For each glass \( G_j \) other than \( G_\ell \) and \( G_m \), the amount of water in \( G_j \) is the same before the \( i^{\text{th}} \) operation as after it. By our induction hypothesis, the amount of water in \( G_m \) after the \( i^{\text{th}} \) operation is \( p_m nx/2^{a_m} \); therefore, the amount before the \( i^{\text{th}} \) operation was \( p_m nx/2^{a_m+1} \), which is of the desired form. The amount of water in \( G_\ell \) after the \( i^{\text{th}} \) operation is \( p_\ell nx/2^{a_\ell} \); therefore, the amount before this operation was \( \left( \frac{p_\ell}{2^{a_\ell}} + \frac{p_m}{2^{a_m+1}} \right) nx \), which can be expressed in the desired form.

Thus, immediately after the \((i-1)^{\text{th}}\) operation in the sequence, for each \( j \in \{1, 2, \ldots, n\} \), the amount of water contained in \( G_j \) is of the form \( p_j nx/2^{a_j} \), where \( a_j \) and \( p_j \) are non-negative integers. This ends the induction step.

It follows that the initial amount of water in \( G_n \) is \( x = pnx/2^a \), where \( a \) and \( p \) are non-negative integers. Thus, \( n = 2^a/p \), from which we deduce that \( p \) is a power of 2 and so is \( n \).

2. The point \( C \) is on the segment \( AB \). A straight line through \( C \) intersects the circle with diameter \( AB \) at \( E \) and \( F \), the circle with diameter \( AC \) again at \( M \), and the circle with diameter \( BC \) again at \( N \). Prove that \( MF = EN \).

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; Toshio Seimiya, Kawasaki, Japan; and Bob Serkey, Leonia, NJ, USA. We give the write-up of Miguel Amengual Covas.

In the figure, the line \( AM \) is extended to meet the circle with diameter \( AB \) at \( M' \). Since \( AC \) and \( AB \) are diameters, \( \angle AMC \) and \( \angle AM'B \) are right angles. The two chords \( EF \) and \( BM' \), both perpendicular to \( AM' \), are parallel to each other. Hence, \( EFM'B \) is an isosceles trapezoid.

Since \( BC \) is a diameter, \( \angle BNC = 90^\circ \). The right triangles \( MFM' \) and \( NEB \) are congruent, and hence, \( MF = EN \).
4. The sides of a heptagon \( A_1A_2A_3A_4A_5A_6A_7 \) have equal length. From a point \( O \) inside, perpendiculare are dropped to the sides \( A_1A_2, A_2A_3, \ldots, A_7A_1 \), meeting them, and not their extensions, at \( H_1, H_2, \ldots, H_7 \), respectively. Prove that
\[
A_1H_1 + A_2H_2 + \cdots + A_7H_7 = H_1A_2 + H_2A_3 + \cdots + H_7A_1.
\]

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.

We let \( a \) denote the length of the sides of the heptagon, and we put \( A_8 = A_1 \). For \( i = 1, 2, \ldots, 7 \), since \( OH_i \perp A_iA_{i+1} \), we have
\[
OA_i^2 - OA_{i+1}^2 = A_iH_i^2 - H_iA_{i+1}^2 = (A_iH_i + H_iA_{i+1})(A_iH_i - H_iA_{i+1}) = a(A_iH_i - H_iA_{i+1}).
\]

Thus,
\[
\sum_{i=1}^{7}(OA_i^2 - OA_{i+1}^2) = a \sum_{i=1}^{7}(A_iH_i - H_iA_{i+1}).
\]

Since the sum on the left side is equal to zero, the sum on the right side is also zero. Consequently,
\[
\sum_{i=1}^{7}A_iH_i = \sum_{i=1}^{7}H_iA_{i+1},
\]
which is the desired result.

5. There are \( 2N \) people at a party. Each knows at least \( N \) others. Prove that one can always choose four people and place them at a round table so that each person knows both neighbours.
Solution by Pierre Bornsztein. Maisons-Laffitte, France.

We assume that the relation "A knows B" is symmetric. We also assume that $N \geq 2$, since there is no set of four distinct persons if $N = 0$ or $N = 1$.

First consider the case $N = 2$. Let $A$, $B$, $C$, $D$ be the people. If they all know one another, then each person at the table will know his neighbours, no matter what seating arrangement is used. Otherwise, we may suppose that $A$ does not know $C$. Then $A$ and $C$ must each know $B$ and $D$, and the seating arrangement $A-B-C-D$ is a solution.

Now suppose $N \geq 3$. Consider the simple graph whose vertices are the people, two of which are joined by an edge if and only if they know each other. The problem is to prove that this graph contains a quadrilateral (that is, a 4-circuit). According to [1] (exercise 361, p. 69), every simple graph with $2N$ vertices and at least $\frac{N}{2}(1 + \sqrt{8N - 3})$ edges contains a quadrilateral.

Let $A$ be any one of the vertices, and let $d(A)$ be its degree in the graph. From the given information, we have $d(A) \geq N$. Since any edge $AB$ is counted in both $d(A)$ and $d(B)$, we deduce that the total number of edges is at least $N^2$. It is easy to verify that $N^2 \geq \frac{N}{2}(1 + \sqrt{8N - 3})$ for $N \geq 3$. The conclusion follows.

Reference


6. Prove that any non-negative even integer can be uniquely represented as $(x+y)^2 + 3x + y$ where $x$ and $y$ are non-negative integers.

Solved by Pierre Bornsztein. Maisons-Laffitte, France; and Christopher J. Bradley. Bristol. UK. We give Bornsztein's solution, adapted by the editor.

Let $n$ be a non-negative even integer, and let $x$ and $y$ be non-negative integers such that

$$(x+y)^2 + 3x + y = n$$

It is easy to see that $n = 0$ if and only if $x = y = 0$. From now on, we assume that $n \geq 2$.

Let $a$ be the positive integer such that $a^2 \leq n < (a+1)^2$. Note that $a^2 + a - 2$ and $a^2 + a$ are two consecutive even integers. Consequently, the following two cases exhaust all possibilities:

Case 1. $a^2 + a \leq n < (a+1)^2$.

If $x + y \geq a + 1$, then

$$n = (x+y)^2 + 3x + y \geq (a+1)^2 > n,$$

a contradiction. If $x + y \leq a - 1$, then

$$n = (x+y)^2 + 3x + y \leq (a-1)^2 + 3(a-1)$$
$$= a^2 + a - 2 < a^2 + a \leq n,$$
a contradiction. Therefore, \( x + y = a \). Then
\[
   n = (x + y)^2 + 3x + y = a^2 + a + 2x.
\]
Hence, \( x = \frac{1}{2}(n - a^2 - a) \) and \( y = a - x = \frac{1}{2}(a^2 + 3a - n) \).

Now let \( x \) and \( y \) be defined by these expressions. Since \( n \) is even and
\( a^2 \equiv a \pmod{2} \), both \( x \) and \( y \) are integers. Since \( a \geq 1 \), we have
\[
   a^2 + 3a \geq a^2 + 2a + 1 = (a + 1)^2 > n,
\]
from which we deduce that \( y > 0 \). Since \( n \geq a^2 + a \) (for this case), we also have \( x > 0 \). Thus, the pair \((x, y)\) gives a unique representation of \( n \) in the desired form.

**Case 2.** \( a^2 \leq n \leq a^2 + a - 2 \).

If \( x + y \leq a - 2 \), then
\[
   n = (x + y)^2 + 3x + y \leq (a - 2)^2 + 3(a - 2) = a^2 - a - 2 < a^2 \leq n,
\]
a contradiction. If \( x + y \geq a \), then
\[
   n = (x + y)^2 + 3x + y \geq a^2 + a + 2x \geq a^2 + a > n,
\]
a contradiction. Therefore, \( x + y = a - 1 \). Then
\[
   n = (x + y)^2 + 3x + y = (a - 1)^2 + a - 1 + 2x = a^2 - a + 2x.
\]
Hence, we have \( x = \frac{1}{2}(n - a^2 + a) \) and \( y = a - 1 - x = \frac{1}{2}(a^2 + a - n - 2) \).

Now let \( x \) and \( y \) be defined by these expressions. As in Case 1, both \( x \) and \( y \) are integers. Since \( a^2 \leq n \) and \( a > 0 \), we have \( x > 0 \). Since \( n \leq a^2 + a - 2 \) (for this case), we also have \( y > 0 \). Thus, the pair \((x, y)\) gives a unique representation of \( n \) in the desired form.

7. In triangle \( ABC \), the sides satisfy \( AB + AC = 2BC \). Prove that the bisector of \( \angle A \) is perpendicular to the line segment joining the incentre and circumcentre of \( ABC \).

*Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya’s argument.*

We assume that \( AB \neq AC \). Let \( I \) and \( O \) be the incentre and circumcentre of \( \triangle ABC \), respectively, and let \( D \) be the intersection of \( AI \) with \( BC \). Since \( BI \) and \( CI \) bisect \( \angle ABD \) and \( \angle ACD \), respectively, we have
\[
\frac{AI}{ID} = \frac{AB}{BD} = \frac{AC}{CD} = \frac{AB + AC}{BD + CD} = \frac{2}{BC} = 2.
\]
Hence, \( AB = 2BD \) and \( AI = 2ID \). Consequently,

\[ AD = 3ID. \quad (1) \]

Let \( M \) be the second intersection of \( AD \) with the circumcircle of \( \triangle ABC \). Then

\[ \angle MBD = \angle MBC = \angle MAC = \angle MAB. \]

Since \( \angle BMD = \angle BMA \), we get \( \triangle MBD \sim \triangle MAB \). It follows that

\[ \frac{DM}{BM} = \frac{BM}{AM} = \frac{BD}{AB} = \frac{1}{2}. \]

Thus,

\[ \frac{DM}{AM} = \frac{DM}{BM} \cdot \frac{BM}{AM} = \left( \frac{1}{2} \right)^2 = \frac{1}{4}. \]

Hence, \( AM = 4DM \), from which we obtain

\[ AD = 3DM. \quad (2) \]

From (1) and (2) we get \( ID = DM \). Thus, \( AI = 2ID = 2D + DM = IM \). Therefore, \( OI \perp AM \). This implies that \( AI \perp OI \).

[Ed. See the solution to Problem 2870 later in this issue, especially the editorial comments following the solution.]

9. Four pedestrians were moving at uniform velocities along four straight roads in general positions. Two of them met each other as well as the other two. Prove that the other two also met.

Solved by Pierre Bornzein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bradley’s write-up.

Denote the pedestrians by \( A, B, C, D \). Without loss of generality, we may assume that the first pair to meet was \( A \) and \( D \), and that they met at time 0. Next suppose that \( A \) and \( C \) met at \( t_1 \) and that \( A \) and \( B \) met at \( t_2 \), where \( t_2 > t_1 > 0 \). Suppose further that \( B \) and \( D \) met at \( t_3 \) and that \( B \) and \( C \) met at \( t_4 \). We want to show that \( C \) and \( D \) also met.

The roads along which \( C \) and \( D \) were travelling meet at a point \( X \). Suppose that \( D \) reached \( X \) at time \( t_5 \) and \( C \) reached \( X \) at time \( t_6 \). Distances
are as shown in the figure, assuming that \( t_4 \) is between \( t_2 \) and \( t_3 \). We have also denoted by \( u, v, w, \) and \( x \) the respective speeds of \( A, B, C, \) and \( D \).

Using Menelaus’ Theorem on \( \triangle YRS \) with transversal \( PQX \), we have

\[
\frac{RX}{XS} \cdot \frac{SQ}{QY} \cdot \frac{QP}{PR} = -1 = \frac{t_5}{t_3 - t_5} \cdot \frac{t_4 - t_3}{t_2 - t_4} \cdot \frac{t_1 - t_2}{-t_1}.
\]  

(1)

Using Menelaus’ Theorem on \( \triangle PRX \) with transversal \( YQS \), we have

\[
\frac{PY}{YR} \cdot \frac{RS}{SX} \cdot \frac{XQ}{QP} = -1 = \frac{t_2 - t_1}{-t_2} \cdot \frac{t_3}{t_5 - t_3} \cdot \frac{t_4 - t_6}{t_1 - t_4}.
\]  

(2)

Equation (1) gives \( t_1 t_2 t_3 + t_1 t_3 t_5 + t_2 t_4 t_5 = t_1 t_2 t_5 + t_1 t_3 t_4 + t_2 t_4 t_5 \), and equation (2) gives \( t_1 t_2 t_3 + t_1 t_3 t_6 + t_2 t_4 t_6 = t_1 t_2 t_5 + t_1 t_3 t_4 + t_2 t_3 t_6 \).

Subtracting yields \( (t_1 t_3 - t_2 t_3)(t_5 - t_6) = 0 \). But \( t_1 \neq t_2 \); whence, \( t_6 = t_5 \), implying that \( C \) and \( D \) meet.

[Ed. Here is another approach. We regard the time \( t \) as a third dimension in the problem, along with the two spatial dimensions \( x \) and \( y \). As the pedestrians move in the \( xy \)-plane, they follow paths in \( xyt \)-space. Since each pedestrian moves at a uniform velocity along a straight line in the \( xy \)-plane, the corresponding paths of the pedestrians in \( xyt \)-space are straight lines. Two pedestrians meet at some time if and only if their paths in \( xyt \)-space intersect.

Let the paths of the pedestrians in \( xyt \)-space be the lines \( L_1, L_2, L_3, \) and \( L_4 \). According to the given information, each pair of these lines intersects, with one possible exception—say \( L_3 \) and \( L_4 \). The three lines \( L_1, L_2, \) and \( L_3 \) must be coplanar, since each pair among them has an intersection. Similarly, \( L_1, L_2, \) and \( L_4 \) are coplanar. Therefore, all four lines lie in the same plane. Now we note that \( L_3 \) and \( L_4 \) cannot be parallel, because this would imply that pedestrians 3 and 4 were walking on parallel roads (at the same speed), which is not allowed from the problem statement. (One way to see this is to consider the projections of \( L_3 \) and \( L_4 \) onto the three coordinate planes in \( xyt \)-space.) Therefore, \( L_3 \) and \( L_4 \) intersect. Thus, pedestrians 3 and 4 meet.]

10. At King Arthur’s Court, \( 2n \) knights gathered at the Round Table. Each has at most \( n - 1 \) enemies among the others. Prove that Merlin the wizard can devise a seating arrangement such that no knight will be next to any of his enemies.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We assume that the relation “\( A \) is not an enemy of \( B \)” is symmetric. Consider the simple graph whose vertices are the knights, two of which are joined by an edge if and only if they are not enemies of each other. The problem is then to prove that this graph contains a Hamiltonian circuit.

Let \( K_1, K_2, \ldots, K_{2n} \) be the knights, and, for \( i = 1, 2, \ldots, 2n \), let \( d_i \) be the degree of \( K_i \). From the statement of the problem, we have \( d_i \geq n \) for each \( i \). Thus, \( \min\{d_i\} \geq n \).

But, according to [1] (exercise 21-a, p. 67), every simple graph with \( k \) vertices satisfying \( \min\{d_i\} \geq \frac{k}{2} \) (this is known as Dirac’s condition) has a Hamiltonian circuit. The conclusion follows.

Reference

11. Construct a set of circles with non-zero radii such that exactly one of them passes through each point of three-dimensional space.

Comment by Pierre Bornsztein. Maisons-Laffitte, France.

This problem, with its solution, appears as Exercise 12G in P. Halmos, Problems for Mathematicians Young and Old, MAA, 1991.

12. Does there exist a positive integer \( n \) such that
\[
27^n + 84^n + 110^n + 133^n = 144^n ?
\]

Solved by Pierre Bornsztein. Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; D.J. Smeenk, Zaltbommel, the Netherlands. We give Bornsztein’s solution.

Let
\[
f(x) = \left( \frac{27}{144} \right)^x + \left( \frac{84}{144} \right)^x + \left( \frac{110}{144} \right)^x + \left( \frac{133}{144} \right)^x
\]
defined on \((0, \infty)\). It is easy to see that the function \( f \) is decreasing. Then the equation \( f(x) = 1 \) has at most one solution in positive real numbers. Since \( f(5) = 1 \), it follows that \( n = 5 \) is the only solution of the problem.

14. Prove that
\[
\sum_{i,j=1}^{\infty} \frac{a_i a_j}{i+j} \leq \pi \sum_{k=1}^{\infty} a_k^2 .
\]

Solution by Pierre Bornsztein. Maisons-Laffitte, France.

This is a special case \((p = 2)\) of Theorem 315 in [1], which states: Let \( p > 1 \) and \( p' = \frac{p}{p - 1} \). If \( \sum_{n=1}^{\infty} a_n p \leq A \) and \( \sum_{n=1}^{\infty} b_n p' \leq B \), then
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)} A^{\frac{1}{p}} B^{\frac{1}{p'}} .
\]

Moreover, the constant \( \frac{\pi}{\sin(\pi/p)} \) is the best possible.

Reference

16. Decompose \( 235^2 + 972^2 \) into two factors.

Solved by Miguel Amengual Covas. Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Bristol, UK; and Edward T.H. Wang. Wilfrid Laurier University, Waterloo, ON. We give the write-up of Amengual Covas.

Since
\[
235^2 + 972^2 = 55225 + 944784 = 1000009 = 1000^2 + 3^2 ,
\]
we apply the identity

\[(ab + cd)^2 + (ad - bc)^2 = (a^2 + c^2)(b^2 + d^2),\]

with \(a = 17, b = 58, c = 2, d = 7\), to obtain

\[1000^2 + 3^2 = (17^2 + 2^2)(58^2 + 7^2) = 293 \cdot 3413;\]

whence,

\[235^2 + 972^2 = 293 \cdot 3413.\]

17. Students in a school go for ice cream in groups of at least two. No two students will go together more than once. After \(k \geq 1\) groups have gone, every two students have gone together exactly once. Prove that the number of students in the school is at most \(k\).

**Solution by Pierre Bornszein, Maisons-Laffitte, France.**

First note that the published statement of this problem was obviously wrong, since it asked for proof that "the number of students in the school is at least \(k\)". We prove the corrected version as stated above.

The given conditions have the following consequences:

(a) Any two groups have at most one student in common. (Otherwise, two students would be together in two different groups, contrary to the hypothesis.)

(b) No single group contains all the students. (Otherwise, we could not have another group of at least two students, as required, which would contradict (a).)

(c) Each student belongs to at least two groups. (If some student were in only one group, then either that group would contain all the students, contradicting (b), or else the student would not be together with every other student exactly once, contradicting the hypothesis.)

Let \(S\) be the set of all the students, and let \(n = |S|\). If \(x \in S\), we denote by \(d(x)\) the number of groups containing \(x\).

Let \(a\) be a student. By (c) above, there are two distinct groups \(G\) and \(G'\) such that \(a \in G\) and \(a \in G'\). Since \(|G| \geq 2\) and \(|G'| \geq 2\), there exist students \(b \in G\) and \(c \in G'\) such that \(b \neq a\) and \(c \neq a\). Moreover, by (a), we have \(b \notin G'\) and \(c \notin G\). There is a group \(G''\) which contains \(b\) and \(c\) (since \(b\) and \(c\) must be together once). Using (a), we have \(a \notin G''\). It follows that \(d(a) < k\).

We have

\[
\sum_{\substack{x \in G \cap G' \setminus \{a\}}} 1 = \sum_{x} \left( \sum_{c \in G \cap G' \setminus \{a\}} 1 \right) = \sum_{x} d(x).
\]
On the other hand,
\[
\sum_{x \in G} 1 = \sum_{x \in G} \left( \sum_{a \in G} 1 \right) = \sum_{G} |G|.
\]

Therefore,
\[
\sum_{x} d(x) = \sum_{G} |G|.
\] (1)

Let \( x \) be a student, and let \( G \) be a group not containing \( x \). For each student \( a \in G \), there exists a group \( G_{a} \) which contains \( a \) and \( x \). Note that \( G \neq G_{a} \), since \( x \notin G \). Moreover, if \( b \in G \) and \( b \neq a \), the groups \( G_{a} \) and \( G_{b} \) are distinct (since \( G \) is the unique group containing both \( a \) and \( b \)). It follows that
\[
d(x) \geq |G|.
\] (2)

Now suppose, for a contradiction, that \( n > k \). Then, from (2), for each pair \((x, G)\) such that \( x \notin G \), we have
\[
0 < k - d(x) \leq k - |G| < n - |G|.
\]

Then
\[
\frac{|G|}{n - |G|} < \frac{d(x)}{k - d(x)}.
\] (3)

Summing all the inequalities (3) obtained for all the pairs \((x, G)\) such that \( x \notin G \), we get
\[
\sum_{G} \frac{|G|}{n - |G|} < \sum_{G} \frac{d(x)}{k - d(x)}.
\] (4)

Let \( G \) be a group. The number of students not belonging to \( G \) is \( n - |G| \). Therefore,
\[
\sum_{G} \frac{|G|}{n - |G|} = \sum_{G} \left( \sum_{a \in G} \frac{|G|}{n - |G|} \right)
\]
\[
= \sum_{G} (n - |G|) \frac{|G|}{n - |G|} = \sum_{G} |G|.
\]

Also,
\[
\sum_{G} \frac{d(x)}{k - d(x)} = \sum_{x} \left( \sum_{a \in G} \frac{d(x)}{k - d(x)} \right)
\]
\[
= \sum_{x} (k - d(x)) \frac{d(x)}{k - d(x)} = \sum_{x} d(x).
\]
It follows that (4) may be rewritten as
\[ \sum_G |G| < \sum_x d(x), \]
which contradicts (1). Then \( n \leq k \), as claimed.

**Remark.** The inequality is the best possible, in the following sense. Let \( S = \{x_1, x_2, \ldots, x_n\} \). If \( G_1 = \{x_2, \ldots, x_n\} \), and \( G_i = \{x_1, x_i\} \) for \( i = 2, \ldots, n \), then the conditions of the problem are satisfied and \( k = n \).

**18.** We choose \( 2^{p-1} \) subsets from a set with \( p \) elements such that any three have a common element. Prove that they all have a common element.

*Solution by Pierre Bornsztein, Maisons-Laffitte, France.*

**Lemma.** Let \( n \geq 1 \) be an integer, and let \( S \) be a set with \( n \) elements. Let \( A_1, A_2, \ldots, A_k \) be distinct subsets of \( S \) such that, for all \( i \) and \( j \), the intersection \( A_i \cap A_j \) is not empty. Then \( k \leq 2^{n-1} \).

**Proof of lemma.** For \( i = 1, \ldots, k \), let \( B_i = S \setminus A_i \). Then \( B_1, B_2, \ldots, B_k \) are distinct subsets of \( S \). If \( B_i = A_j \) for some \( i \) and \( j \), then we find that \( A_i \cap A_j = A_i \cap (S \setminus A_i) = \emptyset \), which contradicts the hypothesis. It follows that \( A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k \) are distinct subsets of \( S \). Thus, \( 2k \leq 2^n \); that is, \( k \leq 2^{n-1} \).

We now prove the desired result by induction for \( p \geq 3 \). (If \( p = 1 \) or \( p = 2 \), then the condition that any three subsets have a common element is vacuously true, since we can choose only \( 2^{p-1} < 3 \) subsets. The result is false for these cases.)

If \( p = 3 \), let \( S = \{x_1, x_2, x_3\} \). We choose four sets \( A_1, A_2, A_3, A_4 \) among \( \emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\} \), such that the intersection of any three of them is non-empty. Clearly, none of the four can be \( \emptyset \). Also, the sets \( \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\} \) cannot all be chosen, because their intersection is empty. It follows that one of the 1-element sets must be chosen, say \( \{x_1\} \). Then each of the chosen sets must contain \( x_1 \). Thus, the conclusion holds for \( p = 3 \).

Let \( p \geq 4 \) be a fixed integer. Suppose that the result holds for a set with \( p-1 \) elements. Let \( S = \{x_1, x_2, \ldots, x_p\} \), and let \( A_1, A_2, \ldots, A_{2^{p-1}} \) be distinct subsets of \( S \) such that

\[ A_i \cap A_j \cap A_k \neq \emptyset, \quad \text{for all} \ i < j < k. \]

Let \( n_p \) be the number of sets \( A_i \) that contain \( x_p \). If \( n_p < 2^{p-2} \), then the number of sets \( A_i \) not containing \( x_p \) is \( 2^{p-1} - n_p > 2^{p-1} - 2^{p-2} = 2^{p-2} \). These sets form a collection of more than \( 2^{p-2} \) distinct subsets of \( \{x_1, x_2, \ldots, x_{p-1}\} \) such that \( A_i \cap A_j \neq \emptyset \) for all \( i \) and \( j \), contradicting the lemma. Thus \( n_p \geq 2^{p-2} \).

With no loss of generality, we may now suppose that \( x_p \) belongs to \( A_1, A_2, \ldots, A_{n_p} \).
Case 1. \( n_p > 2^{p-2} \).

Let \( A'_i = A_i \setminus \{ x_p \} \) for \( i = 1, 2, \ldots, n_p \). Then \( A'_1, A'_2, \ldots, A'_{n_p} \) form a collection of more than \( 2^{p-2} \) distinct subsets of \( \{ x_1, x_2, \ldots, x_{p-1} \} \). Using the lemma, we deduce that there exist \( r \) and \( s \) such that \( A'_r \cap A'_s = \emptyset \). It follows that \( A_r \cap A_s = \{ x_p \} \). For each \( k \in \{ r, s \} \), since \( A_r \cap A_s \cap A_k \neq \emptyset \), we have \( A_r \cap A_s \cap A_k = \{ x_p \} \), from which we deduce that \( x_p \) is a common element of all the \( A_i \)'s.

Case 2. \( n_p = 2^{p-2} \).

Let \( k \) be such that \( x_p \notin A_k \) and define \( B_k = A_k \cup \{ x_p \} \). Then \( B_k \neq A_k \).

Suppose that \( B_k \neq A_j \) for all \( j \). Then the sets \( A_1, A_2, \ldots, A_{2^{p-1}} \) form a collection of \( 2^{p-1} + 1 \) distinct subsets of \( S \), and the intersection of any pair of distinct sets in this collection is non-empty, contradicting the lemma.

Therefore, for each \( k \) such that \( x_p \notin A_k \), there exists \( j \) such that \( A_k \cup \{ x_p \} = A_j \). Since the number of \( A_i \)'s which do not contain \( x_p \) is the same as the number of \( A_i \)'s which contain \( x_p \), we deduce that the collection of the \( A_i \)'s is

\[
A_1, A_2, \ldots, A_{n_p}, A_1 \cup \{ x_p \}, A_2 \cup \{ x_p \}, \ldots, A_{n_p} \cup \{ x_p \},
\]

where \( x_p \notin A_i \) for \( i = 1, 2, \ldots, n_p \).

Now \( A_1, A_2, \ldots, A_{n_p} \) are \( 2^{p-2} \) distinct subsets of \( \{ x_1, \ldots, x_{p-1} \} \) such that, for all \( i < j < k \), we have \( A_i \cap A_j \cap A_k \neq \emptyset \). By the induction hypothesis, they share a common element, say \( a \). Clearly, \( a \in A_i \cup \{ x_p \} \) for each \( i \), and therefore, \( a \) is a common element of all the \( A_i \)'s. This ends the induction step and the proof.

19. Let \( a, b \) and \( c \) be real numbers with sum 0. Prove that

\[
\frac{a^7 + b^7 + c^7}{7} = \left( \frac{a^5 + b^5 + c^5}{5} \right) \left( \frac{a^2 + b^2 + c^2}{2} \right).
\]

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; Pierre Bornsztein, Maisons-Laffitte, France; and Vedula N. Murty, Dover, PA, USA. We give Bradley's solution.

Let \( a, b, c \) be the roots of \( x^3 + qx - r = 0 \), and let \( S_k = a^k + b^k + c^k \). Then \( S_1 = a + b + c = 0 \) and

\[
S_2 = a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = -2q
\]

Now we have

\[
\begin{align*}
S_3 + qS_1 - 3r &= 0 \quad \implies \quad S_3 = 3r \\
S_4 + qS_2 - rS_1 &= 0 \quad \implies \quad S_4 = 2q^2 \\
S_5 + qS_3 - rS_2 &= 0 \quad \implies \quad S_5 = -3qr - 2qr = -5qr \\
S_7 + qS_5 - rS_4 &= 0 \quad \implies \quad S_7 = 5q^2r + 2q^2r = 7q^2r.
\end{align*}
\]

Then \( \frac{S_7}{7} = q^2r = \frac{S_5}{5} \cdot \frac{S_2}{2} \).
21. Segments $AC$ and $BD$ intersect at point $E$. Points $K$ and $M$, on segments $AB$ and $CD$, respectively, are such that the segment $KM$ passes through $E$. Prove that $KM \leq \max\{AC, BD\}$.

_Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution._

The following lemma is well known, and its proof will not be given.

**Lemma.** If $P$ is a point on the side $BC$ of triangle $ABC$, then $AP \leq \max\{AB, AC\}$.

**Case 1.** $AB \parallel CD$. (See the diagram on the left below.)

Let $X$, $Y$ be points on $CD$ such that $KX \parallel AC$ and $KY \parallel BD$. Since $AKXC$ and $BKYD$ are both parallelograms, we have

$$KX = AC \quad \text{and} \quad KY = BD.$$  

Since $M$ is a point on the segment $XY$, we have, by the lemma,

$$KM \leq \max\{KX, KY\}.$$  

Therefore,

$$KM \leq \max\{AC, BD\}.$$  

![Diagram](attachment:image.png)

**Case 2.** $AB \parallel CD$. (See the diagram on the right above.)

In this case, when we consider quadrilateral $ABCD$, we have either $\angle A + \angle D > 180^\circ$ or $\angle B + \angle C > 180^\circ$. We may assume without loss of generality that $\angle A + \angle D > 180^\circ$. The line through $D$ parallel to $AB$ meets $EC$ and $EM$ at $P$ and $Q$, respectively. Then $P$ and $Q$ are points on the segments $EC$ and $EM$, respectively. By Menelaus' Theorem for $\triangle DPC$, we have

$$\frac{PQ}{QD} \cdot \frac{DM}{MC} \cdot \frac{CE}{EP} = 1.$$
Therefore, 
\[
\frac{PQ}{QD} \cdot \frac{DM}{MC} = \frac{EP}{CE} < 1;
\]
that is, \( \frac{CM}{MD} > \frac{PQ}{QD} \). Since \( DP \parallel AB \), we get \( \frac{PQ}{QD} = \frac{AK}{KB} \). Thus,
\[
\frac{CM}{MD} > \frac{AK}{KB}.
\]
(1)

Let \( X \) be a point such that \( CX \parallel AK \) and \( KX \parallel AC \). Let \( Y \) be a point such that \( DY \parallel BK \) and \( KY \parallel BD \). Let \( S \) be the intersection of \( XY \) with \( CD \).

Then \( CX = AK, KX = AC, YD = KB, \) and \( KY = BD \). Since \( CX \parallel AB \parallel YD \), we have
\[
\frac{CS}{SD} = \frac{CX}{YD} = \frac{AK}{KB}.
\]
Consequently, using (1), we have \( \frac{CS}{SD} < \frac{CM}{MD} \). Hence, \( M \) is a point on the segment \( DS \).

Let \( T \) be the intersection of \( KM \) with \( SY \). Then \( M \) is a point on the segment \( KT \). Thus, \( KM \leq KT \). Since \( T \) is a point on the segment \( XY \), we have, by the lemma,
\[ KT \leq \max\{KX, KY\} = \max\{AC, BD\}. \]
Therefore, \( KM \leq \max\{AC, BD\} \).

22. Prove that
\[
\sum_{k=0}^{n} \binom{n}{k} (a + k)^{k-1} (b + n - k)^{n-k-1} = (a + b + n)^n \left( \frac{1}{a} + \frac{1}{b} \right). \]

Solution by Michel Bataille. Rouen, France.

Let \( S \) denote the left side of the above equation, and let \( N = n - 1 \). Using \( \binom{n}{k} = \binom{N}{k} + \binom{N}{k-1} \), we write \( S = S_1 + S_2 \), where
\[
S_1 = \sum_{k=0}^{N} \binom{N}{k} (a + k)^{k-1} (b + N + 1 - k)^{N-k}
\]
and
\[
S_2 = \sum_{k=1}^{N+1} \binom{N}{k-1} (a + k)^{k-1} (b + N + 1 - k)^{N-k}. \]
Working with $S_1$ first, we note that, for each $k = 0, 1, \ldots, N$,

\[
\binom{N}{k}(a + k)^{k-1}(b + N + 1 - k)^{N-k}
\]

\[
= \binom{N}{k}(a + k)^{k-1}(a + b + N + 1 - (a + k))^{N-k}
\]

\[
= \binom{N}{k}(a + k)^{k-1} \sum_{j=0}^{N-k} \binom{N-k}{j}(a + b + N + 1)^j(- (a + k))^{N-k-j}
\]

\[
= \sum_{j=0}^{N-k} \binom{N}{k} \binom{N-k}{j}(-1)^{N-k-j}(a + b + N + 1)^j(a + k)^{N-1-j}
\]

\[
= \sum_{j=0}^{N-k} \binom{N}{j} \binom{N-j}{k}(-1)^{N-k-j}(a + b + N + 1)^j(a + k)^{N-1-j}
\]

Summing over $k$, and interchanging the order of summation, we get

\[
S_1 = \sum_{j=0}^{N} \binom{N}{j}(a + b + N + 1)^j \sum_{k=0}^{N-j} \binom{N-j}{k}(-1)^{N-k-j}(a + k)^{N-1-j}.
\]

Now we make use of the following general result: for every polynomial $P$ with degree $< m$,

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} P(k) = 0.
\]

(The sum on the left side is $\Delta^m P(0)$, where $\Delta$ is the difference operator defined by $\Delta P(x) = P(x+1) - P(x)$.) It follows that the inner sum in the above expression for $S_1$ is 0 except when $j = N$. Thus,

\[
S_1 = \binom{N}{N}(a + b + N + 1)^N \binom{0}{0}(-1)^0(a + 0)^{-1} = \frac{1}{a}(a + b + n)^{n-1}.
\]

For $S_2$, we have

\[
S_2 = \sum_{k=1}^{N+1} \binom{N}{N-k+1}(a + k)^{k-1}(b + N + 1 - k)^{N-k}
\]

\[
= \sum_{j=0}^{N} \binom{N}{j}(a + N + 1 - j)^{N-j}(b + j)^{j-1}
\]

Applying the same argument as for $S_1$, we get $S_2 = \frac{1}{b}(a + b + n)^{n-1}$. The desired result follows.
Comment by Pierre Bornsztein. Maisons-Laffitte, France.

This problem is proposed as exercise 44-b (with solution) in L. Lovász, Combinatorial Problems and Exercises, North-Holland, 1979.

23. The plane is divided into regions by \( n \) lines in general positions. Prove that at least \( n - 2 \) of the regions are triangles.

Solution by Pierre Bornsztein. Maisons-Laffitte, France.

First note that, if the line \( \ell \) meets the interior of the non-degenerate triangle \( ABC \), then \( \ell \) must cross two sides of the triangle, say \( [AB] \) at \( M \) and \( [AC] \) at \( N \). It follows that \( AMN \) is a non-degenerate triangle. Thus, if a line meets the interior of a triangle, it divides the triangle into two regions, at least one of which is a triangle.

Let \( n \geq 3 \) be an integer. Let the plane be divided into regions by \( n \) lines \( \ell_1, \ell_2, \ldots, \ell_n \) in general position. Without loss of generality, we may suppose that, for \( i = 1, 2, \ldots, n-1 \), the line \( \ell_i \) meets \( \ell_n \) at \( M_i \), such that the points \( M_1, M_2, \ldots, M_{n-1} \) are pairwise distinct (since the lines are in general position) and in this order on \( \ell_n \).

For \( i \leq n-2 \), the lines \( \ell_i, \ell_{i+1}, \text{and} \ell_n \) form a triangle \( T_i \), where \( M_i \) and \( M_{i+1} \) are two vertices of \( T_i \). Moreover, from the order of the \( M_i \)'s on the line \( \ell_n \), any two of the triangles \( T_1, T_2, \ldots, T_{n-2} \) have no interior point in common. Thus, we have exactly \( n-2 \) triangles.

Let \( k \leq n-2 \) be fixed. If none of the lines \( \ell_i \) meets the interior of \( T_k \), then \( T_k \) is one of the regions. Otherwise, a line \( \ell_i \) meets the interior of \( T_k \). Let \( i \) be the least integer such that \( \ell_i \) meets the interior of \( T_k \). Then, from the initial remark above, the line \( \ell_i \) divides \( T_k \) into two regions, at least one of which is a triangle, say \( T_{k_1} \). Note that none of the lines \( \ell_i \) meets the interior of \( T_{k_1} \) for \( i \leq i_1 \).

If, for \( i > i_1 \), none of the lines \( \ell_i \) meets the interior of \( T_{k_1} \), then \( T_{k_1} \) is one of the regions. In the other case, substituting \( T_{k_1} \) for \( T_k \), we follow the same reasoning as above. Since, at each step, the number of lines remaining to be considered is decreasing, this process will eventually stop, giving us a region which is a triangle.

With this process, each of the \( T_i \)'s leads to a triangular region. Since any two of the regions \( T_i \) have no interior point in common, the triangular regions are distinct, and we are done.

That completes the Corner for this issue. Send me your nice solutions and generalizations. Over the next several issues, we will be making an effort to shorten the time between the appearance of a problem set and publication of the solutions.