SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for misspelling the name of JAMES HOLETON in the list of solvers of 2826, and for omitting the name of ŞEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina from the list of solvers of 2843, and the name of LI ZHOU, Polk Community College, Winter Haven, FL, USA from the list of solvers of 2845.


In a convex quadrilateral $ABCD$, we have $\angle ABC = \angle BCD = 120^\circ$. Suppose that $AB^2 + BC^2 + CD^2 = AD^2$.

Prove that $ABCD$ has an inscribed circle.

Solution by Michel Bataille, Rouen, France; Walther Janous, Ursulinen-gymnasium, Innsbruck, Austria; Mitko Kunchev, Baba Tonka School of Mathematics, Rousse, Bulgaria; David Loeffler, student, Trinity College, Cambridge, UK; Andrei Simion, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. Smeenk, Zalkbomel, the Netherlands; and the proposer.

Let $a = AB$, $b = BC$, $c = CD$, $d = DA$, and let $E$ be the point of intersection of the lines $AB$ and $CD$. Since $\angle ABC = \angle BCD = 120^\circ$, we have $\angle EBC = \angle ECB = 60^\circ$, and therefore, $BE = CE = BC = b$.

Applying the Law of Cosines to $\triangle AED$, we obtain

$$d^2 = (a + b)^2 + (b + c)^2 - (a + b)(b + c)$$
$$= a^2 + b^2 + c^2 + ab + bc - ac.$$

Using the given condition $d^2 = a^2 + b^2 + c^2$, we get $ab + bc - ac = 0$. But then $d^2 = a^2 + b^2 + c^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac = (a + c - b)^2$, so that $d = a + c - b$ or $d = b - a - c$. The latter is impossible, because $d^2 > b^2$. Consequently, $a + c = d + b$, which is a well-known necessary and sufficient condition for a convex quadrilateral to have an inscribed circle.

Also solved by MANUEL BENITO, OSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUL, student, Western Michigan University, Kalamazoo, MI, USA; JOHN G. HEUVER, Grande Prairie, AB; NEVEN JURČ, Zagreb, Croatia; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; VACLAV KONEČNÝ, Big Rapids, MI, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania. There were also two incorrect solutions submitted.

Find all integral solutions of

\[ x^2 - 4xy + 6y^2 - 2x - 20y = 29. \]

Composite of essentially the same solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández Logroño, Spain; Con Amore Problem Group, The Danish University of Education, Copenhagen, Denmark; Douglass L. Grant, University College of Cape Breton, Sydney, NS; Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina; Walther Janous, Ursulinen-gymnasium, Innsbruck, Austria; D. Kipp Johnson, Beaverton, OR, USA; David Loeffler, student, Trinity College, Cambridge, UK; Digby Smith, Mount Royal College, Calgary, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

The given equation can be written as

\[(x - 2y - 1)^2 + 2(y - 6)^2 = 102.\]

Thus, \(x - 2y - 1\) is even. Setting \(x - 2y - 1 = 2u\) and \(y - 6 = v\), we then have \(2u^2 + v^2 = 51\) for some integers \(u\) and \(v\). Clearly, \(|u| \leq 5\). By setting \(u = 0, \pm 2, \pm 3, \pm 4, \pm 5\), we find that \(v\) is an integer only when \(u = \pm 1\) or \(\pm 5\). These values yield eight pairs. \((u, v) = (\pm 1, \pm 7)\) or \((\pm 5, \pm 1)\). Simple substitutions then give eight solution pairs \((x, y)\) for the given equation:

\[(29, 13), (25, 13), (1, -1), (-3, -1), (25, 7), (5, 7), (21, 5), (1, 5).\]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); DIONNE T. BAILEY, ELISE M. CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; PAOLO CUSTODI, Faro, Faro, Italy; OVIDIU FURDU, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; NEVEN JURIC, Zagreb, Croatia; VACLAV KONEČNÝ, Big Rapids, MI, USA; GOTTFRIED PERZ, Pestalozzigrundschule, Graz, Austria; BOB SERKEY, Leonia, NJ, USA; MIKE SPIVEY, Samford University, Birmingham, AL, USA; MIHAI STOJENESCU, Bischwiller, France; PANOS E. TSAOSSOGLIOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; ROGER ZARNOWSKI, Angelo State University, TX, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. There were four incorrect or incomplete solutions.

As pointed out by Con Amore Problem Group, Curtis, and Konečný, the given equation represents an ellipse \(E\), on which there can be only a finite number of lattice points. The solutions to the problem simply identify all eight lattice points on \(E\).

Let \( m, n, \) and \( N \) be non-negative integers such that \( m + n \geq 2N + 1 \). Let \( K = m + n - N - 1 \). Prove that

\[
\sum_{j=0}^{\infty} (-1)^j \frac{N + 1}{N + 1 + j} \binom{N}{j} \left( \binom{K - j}{m} + \binom{K - j}{n} \right) = \left( \begin{array}{c} m + n \\ m \end{array} \right) \left( \begin{array}{c} m + n \\ n \end{array} \right) \left( \frac{2N + 1}{N} \right).
\]

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

In \([1]\), Walther Janous (the proposer of the current problem) and the editors of the “Problems and Solutions” column of the American Mathematical Monthly used the so-called “snake-oil” method to establish the identity

\[
\sum_{j=0}^{N} (-1)^j \frac{N}{N + 1 + j} \binom{N}{j} \left( \binom{K - j}{m} + \binom{K - j}{n} \right) = \binom{m + n}{n} \sum_{j=0}^{N} \frac{(-1)^j}{N + 1 + j} \binom{N}{j}.
\]

In the editorial comment that followed their composite solution, it was also proved, by the so-called WZ method, that

\[
\sum_{j=0}^{N} \frac{(-1)^j}{N + 1 + j} \binom{N}{j} = \frac{1}{N + 1} \left( \frac{2N + 1}{N} \right)^{-1}.
\]

Putting these two identities together solves the problem.

Reference


Also solved by C.P. HENDERSON, Garden Hill, Ontario.


In \( \triangle ABC \), we have \( AB < AC \). The internal bisector of \( \angle BAC \) meets \( BC \) at \( D \). Let \( P \) be an interior point of the line segment \( AD \), and let \( E \) and \( F \) be the intersections of \( BP \) and \( CP \) with \( AC \) and \( AB \), respectively.

Prove that \( \frac{PE}{PF} < \frac{AC}{AB} \).
1. Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Lemma. Let the internal bisector of $\angle BAC$ meet $BC$ at $D$. For any interior point $P$ of the segment $AD$ let $E$ and $F$ be the intersections of $BP$ and $CP$ with $AC$ and $AB$, respectively. Then $AB < AC$ implies $BE < CF$; that is, the shorter cevian goes to the longer side.

Proof. Denote by $a$, $b$, $c$ the lengths of the sides $BC$, $CA$, $AB$, as usual. Let $BF/FA = t > 0$. By Ceva's Theorem,

$$\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CD}{DB} = 1.$$ 

Thus, $FA = c/(1 + t)$ and $AE = bc/(c + bt)$. By the Law of Cosines,

$$BE^2 = AB^2 + AE^2 - 2AB \cdot AE \cos A = c^2 + \left(\frac{bc}{c + bt}\right)^2 - \frac{c(b^2 + c^2 - a^2)}{c + bt},$$

and

$$CF^2 = AC^2 + FA^2 - 2AC \cdot FA \cos A = b^2 + \left(\frac{c}{1 + t}\right)^2 - \frac{b^2 + c^2 - a^2}{1 + t}.$$ 

Therefore,

$$(1 + t)^2(c + bt)^2(CF^2 - BE^2)$$

$$= (b^2 - c^2)(1 + t)^2(c + bt)^2 + c^2[(c + bt)^2 - b^2(1 + t)^2]$$

$$- (b^2 + c^2 - a^2)(1 + t)(c + bt)(b - c)t$$

$$> (b - c)[((b + c)(1 + t)^2(c + bt)^2 - c^2(c + b + 2bt)$$

$$- (b^2 + c^2)(1 + t)(c + bt)t]$$

$$= (b - c)[((1 + t)(c + bt)^3 - c^3 - 2bc^2t)$$

$$+ ((1 + t)(c + bt)bc(1 + t^2) - bc^2)] > 0,$$

which proves the lemma.

Turning to the problem, suppose that the lines through $D$ which are parallel to $BE$ and $CF$ intersect $AC$ and $AB$ at $G$ and $H$, respectively. Then

$$\frac{PE}{DG} = \frac{PA}{DA} = \frac{PF}{DH}.$$ 

Also,

$$\frac{DG}{BE} = \frac{CD}{CB}, \quad \frac{DH}{CF} = \frac{DB}{CB}, \quad \text{and} \quad \frac{CD}{DB} = \frac{AC}{AB}$$

(where the last equality holds because $AD$ bisects $\angle BAC$). Hence,

$$\frac{PE}{PF} = \frac{DG}{DH} = \frac{CD}{DB} \cdot \frac{BE}{CF} = \frac{AC}{AB} \cdot \frac{BE}{CF}.$$ 

Since $BE < CF$ from the lemma, we conclude that $PE/PF < AC/AB$, as desired.
II. Editor's comments, and a 1944 solution by L.M. Kelly.

The lemma was also used in solutions submitted by Janous, Seimiya, and (implicitly) Paragiou. It generalizes the Steiner-Lehmus theorem, a result that deals with the case where \( P \) is the incenter: The shorter angle bisector goes to the longer side. According to Coxeter and Greitzer (Geometry Revisited, Mathematical Association of America (1967), page 14), that theorem always excites interest; papers on it have appeared with some regularity since 1842. One would therefore expect our lemma to have been discovered long ago. A cursory computer search for references to the Steiner-Lehmus theorem in the Mathematical Association of America journals turned up problem E613 of the American Mathematical Monthly (51:3 (March 1944), p. 162, and 51:10 (December 1944), pp. 590–591). Both the problem and the solution came from L.M. Kelly. To solve his problem he used the following:

Lemma (Kelly). If the internal cevians \( BE \) and \( CF \) of triangle \( ABC \) are such that \( \angle CBE > \angle BCF \) and \( \angle ABE > \angle ACF \), then \( BE < CF \).

Kelly's Proof. Select a point \( Q \) on the segment \( AE \) so that \( \angle QBE = \angle ACF \). Let \( CF \) meet \( BE \) at \( P \) and \( BQ \) at \( R \). Since triangle \( QBC \) has a greater angle at \( B \) than at \( C \), \( QC > QB \). Since triangles \( QBE \) and \( QCR \) are similar, it follows that \( BE < CR < CF \), as desired. \( \blacksquare \)

Zhou's Lemma is the special case of Kelly's where \( AD \) is the bisector of \( \angle BAC \). To see this, reflect \( C \) in \( AD \) to a point \( C' \) on the extension of segment \( AB \) beyond \( B \), and note that

\[
\angle CBE = \angle DBP > \angle DC'P = \angle DCP = \angle BCF,
\]

and

\[
\angle ABE = \angle ABP > \angle AC'P = \angle ACP = \angle ACF.
\]

In problem E613, Kelly showed that when \( AB < AC \), the medians \( BE \) and \( CF \) satisfy the conditions of his lemma; from that he easily deduced that a triangle cannot have two equal symmedians without being isosceles.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulineninnasium, Innsbruck, Austria; THEO KLITOS PARAGIOU, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete solution (which stated Zhou's lemma without an accompanying proof or reference).

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In \( \triangle ABC \), we have \( AC = 2AB \). The tangents at \( A \) and \( C \) to the circumcircle of \( \triangle ABC \) meet at \( P \).

Prove that the line \( BP \) bisects the arc \( BAC \) (of the circumcircle).
Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Suppose that \( BP \) intersects the arc \( BAC \) at the point \( D \). Since \( \angle CBD = \angle PCD \), the triangles \( PBC \) and \( PCD \) are similar. Likewise, \( \angle ABD = \angle DAP \), implying that the triangles \( PAB \) and \( PDA \) are similar. Hence,

\[
\frac{PC}{PD} = \frac{BC}{CD} \quad \text{and} \quad \frac{AB}{DA} = \frac{PA}{PD}
\]

Since \( PA = PC \), it follows that

\[
\frac{AB}{DA} = \frac{BC}{CD};
\]

that is, \( AB \cdot CD = DA \cdot BC \). Using Ptolemy's Theorem, we have

\[
AC \cdot BD = AB \cdot CD + DA \cdot BC = 2AB \cdot CD.
\]

Since \( AC = 2AB \), we obtain \( BD = CD \).

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, UrsulinenGymnasium, Innsbruck, Austria; THEOKLITOS PARAGICOU, Limassol, Cyprus, Greece; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUEFENG ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Bucharest, Romania; and the proposer.


Suppose that \( M \) and \( N \) are the mid-points of the sides \( AB \) and \( CD \) of quadrilateral \( ABCD \), respectively.

Prove that \( AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2 \).
I. Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

By applying the Law of Cosines to $\triangle AND$, we get

$$AN^2 = AD^2 + ND^2 - 2(AD)(ND) \cos \angle ADN$$

$$= AD^2 + \frac{1}{4} CD^2 - (AD)(CD) \cos \angle ADC .$$

By applying the Law of Cosines to $\triangle ACD$, we get

$$AC^2 = AD^2 + CD^2 - 2(AD)(CD) \cos \angle ADC .$$

We combine these results to obtain

$$AN^2 = \frac{1}{2}(AD^2 + AC^2) - \frac{1}{4} CD^2 .$$

Similarly,

$$DM^2 = \frac{1}{2}(AD^2 + BD^2) - \frac{1}{4} AB^2 ,$$

$$BN^2 = \frac{1}{2}(BC^2 + BD^2) - \frac{1}{4} CD^2 ,$$

$$CM^2 = \frac{1}{2}(AC^2 + BC^2) - \frac{1}{4} AB^2 .$$

Therefore, $AN^2 + DM^2 - BN^2 - CM^2 = AD^2 - BC^2$, which gives the desired result.

II. Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Take any four points (vectors) $A$, $B$, $C$, and $D$ in $n$-dimensional space. (The plane is a special case.) Define $M = \frac{1}{2}(A + B)$ and $N = \frac{1}{2}(C + D)$ (the mid-points of the lines joining $A$ to $B$, and $C$ to $D$, respectively). Then

$$AN^2 + DM^2 + BC^2 - BN^2 - CM^2 - AD^2 = |A - N|^2 + |D - M|^2 + |B - C|^2$$

$$= -2A \cdot N - 2D \cdot M - 2B \cdot C + 2B \cdot N + 2C \cdot M + 2A \cdot D$$

$$= -A \cdot (C + D) - D \cdot (A + B) - 2B \cdot C$$

$$+ B \cdot (C + D) + C \cdot (A + B) + 2A \cdot D = 0 ,$$

and thus, $AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2$.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valadolid, Spain; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALther Janous, Ursulinen-gymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; VEDULA N. MURTY, Dover, PA, USA; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; MARCELO RUFINO DE OLIVEIRA, Belém, Brazil; BOB SERKEY, Leonia, NJ, USA; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESUS VILLAR RUBIO, Santander,
Most solvers used vectors in some manner more or less like Solution II above. A shortcut is possible in Solution I: equation (1) and the similar equations that follow are instances of a standard theorem about the median of a triangle. Several solvers used this theorem.

2855. [2003 : 316] Proposed by Andreas P. Hatzipolakis and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Given two points \( B \) and \( C \), find the locus of the point \( A \) such that the centre of the nine-point circle of \( \triangle ABC \) lies on the interior bisector of \( \angle CAB \).

1. Solution by Toshio Seimiya, Kawasaki, Japan.

Denote the centre of the nine-point circle of \( \triangle ABC \) by \( J \), and let \( O \) and \( H \) denote the circumcentre and the orthocentre of \( \triangle ABC \), respectively. Then \( J \) is the mid-point of \( OH \).

If \( AB = AC \) (see the first figure), then \( O \) and \( H \) lie on the interior bisector of \( \angle CAB \), and then so does \( J \).

Now suppose that \( AB \neq AC \). If \( \angle CAB \geq 90^\circ \), we can easily verify that \( J \) does not lie on the interior bisector of \( \angle CAB \). Therefore, we must have \( \angle CAB < 90^\circ \) (see the second figure). Since \( AH \) and \( AO \) are isogonal conjugates with respect to \( \angle BAC \), it follows that \( AJ \) is the interior bisector of \( \angle HAO \). Since \( J \) is the mid-point of \( HO \), we have \( AH = AO \).

Let \( M \) be the mid-point of \( BC \). Then \( OM \perp BC \). It is well known that \( AH = 2OM \). Hence, \( OC = AO = AH = 2OM \). Thus, \( \angle MOC = 60^\circ \). Therefore, \( \angle BAC = \angle MOC = 60^\circ \).

Let \( \triangle XBC \) and \( \triangle YBC \) be equilateral triangles (see the third figure). Denote arc \( BXC \) of the circumcircle of \( \triangle XBC \) by \( \Gamma_1 \) and arc \( BYC \) of the circumcircle of \( \triangle YBC \) by \( \Gamma_2 \). Since \( \angle BAC = \angle BXC = \angle BYC = 60^\circ \), the point \( A \) lies either on \( \Gamma_1 \) or \( \Gamma_2 \).

We conclude that the locus is the figure consisting of the perpendicular bisector of \( BC \) and \( \Gamma_1 \cup \Gamma_2 \) (excluding \( B \), \( C \), and the mid-point \( M \) of \( BC \)).
II. Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

It is known that the centre $X(5)$ of the nine-point circle has trilinear coordinates $\cos(B - C)$, $\cos(C - A)$, $\cos(A - B)$. (See Clark Kimberling’s Encyclopedia of Triangle Centers: http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.) Since the trilinears are proportional to the directed distance to the sides $BC$, $CA$, $AB$, and the distance from $X(5)$ to the sides $AB$ and $AC$ are, by assumption, equal (angle bisector property), we are left with the condition:

$$\cos(C - A) = \cos(A - B).$$

This is equivalent to $C - A = \pm(A - B)$. (There are no other possibilities, since $A, B, C < \pi$.) If $C - A = A - B$, we get $3A = A + B + C = \pi$, and hence, $A = \pi/3$. If $C - A = -(A - B)$, then $B = C$.

Thus, the desired loci of $A$ are

(i) an arc with $\angle CAB = 60^\circ$ constructed over segment $BC$ on one side;

(ii) the same arc constructed over $BC$ on the other side; and

(iii) the perpendicular bisector of segment $BC$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinen-Gymnasium, Innsbruck, Austria; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zalkbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers. Two incomplete solutions were received, where the solvers found the arcs, but not the straight line. Most solvers used trilinears.


Let $\alpha_k = \frac{q^k - 1}{q - 1}$, where $q$ is a real number, $q \neq 1$. For integers $n \geq 0$ and $k \geq 1$, define $C_{n,k}$ as follows: $C_{n,1} = 1$, $C_{0,k} = 0$ for $k \geq 2$, and $C_{n,k} = \sum_{j=0}^{n-1} \frac{\alpha_k^{j+1}}{\alpha_k^j} C_{j,k-1}$ for $n \geq 1$ and $k \geq 2$.

Show that

$$C_{n,k} = -(q - 1)^{k-1} \sum_{i=1}^{k} \left(\frac{q^i - 1}{q^k - 1}\right)^n \frac{q^i}{q^{i-1}} \left\{q, q^{i-1}\right\}_q,$$

where $\left\{a, q\right\}_0 = 1$ and $\left\{a, q\right\}_i = (1 - a)(1 - aq) \cdots (1 - aq^{i-1})$ for $i \geq 1$. 

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Note first that $C_{n,k}$ can be defined by the alternative recurrence

$$C_{n,k} = \frac{a_{k-1}^{n-1}}{a_k^n} C_{n-1,k-1} + C_{n-1,k}$$  \hspace{1cm} (1)

for $n \geq 1$ and $k \geq 2$. For clarity, we let

$$B_{n,k} = -(q - 1)^{k-1} \sum_{i=1}^k \left( \frac{q^i - 1}{q^k - 1} \right)^n \frac{q^i \{q^{k-i}, q\}^i}{\{q, q\}^i_{k-i} \{q, q\}^i_k}$$

and show $B_{n,k}$ satisfies both (1) and the boundary conditions.

Clearly, for $n \geq 0$,

$$B_{n,1} = -\frac{q(1-q^{-1})}{1-q} = 1.$$

Assume next $k \geq 2$. Substituting $z = q$ and $z = q^2$ into the well-known $q$-Binomial Theorem (see [1])

$$\prod_{i=1}^k \left( 1 - \frac{z}{q^i} \right) = \sum_{i=0}^k \frac{\{q^{-k}, q\}^i}{\{q, q\}^i} z^i,$$

we get

$$\sum_{i=0}^k \frac{q^i \{q^{-k}, q\}^i}{\{q, q\}^i} = 0 \quad \text{and} \quad \sum_{i=0}^k \frac{q^{2i} \{q^{-k}, q\}^i}{\{q, q\}^i} = 0.$$

Subtracting the two equations, we obtain

$$0 = \sum_{i=1}^k \frac{q^i (1-q^i) \{q^{-k}, q\}^i}{\{q, q\}^i} = \sum_{i=1}^k \frac{q^i \{q^{-k}, q\}^i}{\{q, q\}^i_{k-i-1}} = -\frac{\{q, q\}^k_{k-1}}{(q-1)^{k-1}} B_{0,k}.$$

Thus, $B_{0,k} = 0$. 
Finally, for \( n \geq 1 \) and \( k \geq 2 \), we have

\[
\frac{a_{n-1}}{a_n} B_{n-1,k-1} + B_{n-1,k}
\]

\[
\begin{array}{c}
= - \frac{(q - 1)^{k-1}}{(q^k - 1)^n} \sum_{i=1}^{k-1} (q^{i-1} q^i - q^{-k} q^i) \\
\quad \cdot \left( \frac{(q^k - 1)^n q^i (1 - q^k)}{q, q, q} \right) \\
= B_{n,k},
\end{array}
\]

completing the proof.

Reference.


http://mathworld.wolfram.com/q-BinomialTheorem.html

Also solved by NATALIO H. GUERENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer.

Guersenzvaig notes that we must require that \( q \neq 0 \) and \( q \neq -1 \). He further notes that the result holds for an arbitrary field \( E \), where \( q \) is any non-zero element which is not a root of unity in \( E \).


Let \( O \) be an interior point of \( \triangle ABC \), and let \( D, E, F \), be the intersections of \( AO, BO, CO \) with \( BC, CA, AB \), respectively.

Suppose that \( P \) and \( Q \) are points on the line segments \( BE \) and \( CF \), respectively, such that \( \frac{BP}{PE} = \frac{CQ}{QF} = \frac{DO}{OA} \).

Prove that \( PF \parallel QE \).
I. Solution by D.J. Smeenk. Zaltbommel, the Netherlands.

Let the parallel to $BC$ through $O$ intersect $AC$ at $R$ and $AB$ at $S$. The equal ratios

$$\frac{BP}{PE} = \frac{DO}{OA} = \frac{BS}{SA}$$

imply that $PS \parallel CA$. In the same way, $QR \parallel BA$ (using $CQ/QF$). Because corresponding sides are parallel, we have $\triangle OPS \sim \triangle OER$ and $\triangle OFS \sim \triangle OQR$. Moreover, the side $OS$ is common to triangles $OPS$ and $OFS$, while the side $OR$ is common to triangles $OER$ and $OQR$. Consequently, the quadrangles $OFSP$ and $OQRE$ are similar and, moreover, have corresponding sides parallel. It follows that their corresponding parts, $\triangle PSF$ and $\triangle ERQ$, are similar. These two triangles have two pairs of corresponding sides parallel, namely $PS \parallel ER$ and $SF \parallel RQ$; hence, their third sides $PF$ and $EQ$ must also be parallel, as desired.

Editor's comment. As an alternative to using similar quadrangles, one can use the dilative rotation that takes $OR$ to $OS$ (defined to be the product of the half-turn about $O$ and the dilatation whose ratio is $OS : OR$). Since this transformation takes each line to a parallel line, it takes $RQ$ to $SF$ and $RE$ to $SP$. Thus, $QE$ must be parallel to its image $FP$.

II. Solution by Peter Y. Woo. Biola University, La Mirada, CA, USA, shortened by the editor.

We will use the given equal ratios in the form

$$\frac{EP}{EB} = \frac{FQ}{FC} = \frac{AO}{AD}.$$ 

Let

$$\frac{CD}{DB} = \frac{\lambda}{1 - \lambda} \quad \text{(so that } CD/CB = \lambda \text{ and } BD/BC = 1 - \lambda).$$

By Menelaus’ Theorem applied to triangle $COD$ and line $AFB$,

$$\frac{FO}{FC} = \frac{AO}{AD} \cdot \frac{BD}{BC} = \frac{FQ}{FC} \cdot \left(1 - \lambda\right);$$

thus, $FO/FQ = 1 - \lambda$, or

$$\frac{FO}{OQ} = \frac{1 - \lambda}{\lambda}.$$ 

Similarly, Menelaus’ Theorem applied to triangle $BDO$ and line $AEC$ yields

$$\frac{EO}{EB} = \frac{AO}{AD} \cdot \frac{CD}{CB} = \frac{EP}{EB} \cdot \lambda;$$

thus, $EO/EP = \lambda$, or

$$\frac{EO}{OP} = \frac{\lambda}{1 - \lambda}.$$ 

It follows that $\frac{FO}{OQ} = \frac{PO}{OE}$ and, therefore, that $FP$ is parallel to $QE$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.
Suppose that \( P \) is an interior point of \( \triangle ABC \), and that \( D, E, F \) are the intersections of \( AP, BP, CP \) with \( BC, CA, AB \), respectively. Suppose that

\[
\frac{AE + AF}{BC} = \frac{BF + BD}{CA} = \frac{CD + CE}{AB}.
\]

Characterize the point \( P \).

A combination of similar solutions by Francisco Bellot Rosado, I.B. Emilio Ferrari, and María Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain; Shailesh Shirali, Rishi Valley School, India; and Andrei Simion, student, Cooper Union for Advancement of Science and Art, New York NY, USA.

We show that the given condition is satisfied if and only if \( P \) is the Nagel point of triangle \( ABC \). As usual, we denote the side lengths of the triangle by \( a, b, c \), and its semiperimeter \( (a + b + c)/2 \) by \( s \).

For equality of the three ratios

\[
\frac{AE + AF}{BC}, \quad \frac{BF + BD}{CA}, \quad \frac{CD + CE}{AB},
\]

each of them must equal

\[
\frac{AE + AF + BF + BD + CD + CE}{BC + CA + AB} = \frac{BC + CA + AB}{BC + CA + AB} = 1.
\]

Consequently,

\[
AE + AF = a, \quad BF + BD = b, \quad CD + CE = c. \tag{1}
\]

Moreover, by definition

\[
BD + DC = a, \quad CE + EA = b, \quad AF + FB = c. \tag{2}
\]

We shall assume (without loss of generality) that \( a \leq b \leq c \), and let \( BD = x \). Then the other relevant segments satisfy

\[
DC = a - x, \quad CE = c - a + x, \quad EA = a + b - c - x,
\]

\[
AF = c - b + x, \quad FB = b - x.
\]

Since the three cevians are concurrent at \( P \), Ceva's Theorem implies

\[
x \cdot (c - a + x) \cdot (c - b + x) = (a - x) \cdot (a + b - c - x) \cdot (b - x). \tag{3}
\]

Since all factors in equation (3) are positive when \( 0 < x < a \), the product on the left increases with \( x \) while the product on the right decreases with \( x \); therefore, the equation must have exactly one real root. To find that root, we note that the equations in (1) and (2) would be satisfied by \( BF = CE \), \( AF = CD \), and \( AE = BD \). Each of these equalities is equivalent to
2x = a + b - c, or x = s - c. One easily checks that, indeed, x = s - c is a solution (and therefore, the solution) of (3). Our segments are therefore

\[ BF = CE = s - a, \quad AF = CD = s - b, \quad AE = BD = s - c. \]

These equalities tell us that D, E, F are the points of tangency of the excircles with the sides of triangle, which makes P the Nagel point of triangle ABC.

Also solved by ŠEKFET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK (partial solution); CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinenymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; D.J. SHEMEK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Two solvers submitted solutions that were correct in every way except for the terminology. For their benefit, and for the benefit of readers who have not yet memorized Clark Kimberling's list of triangle centres [Kimberling, C., Encyclopedia of Triangle Centers (X(8) = Nagel Point); http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X8], the Nagel point is defined to be the intersection point of the lines joining each vertex to the point where the opposite side touches the opposite excircle. Equivalently, the Nagel point is the isogonal conjugate of the Gergonne point. Moreover, each of the points D, E, F bisects the distance around the perimeter from the opposite vertex. More details can be found on the Mathworld web page, http://mathworld.wolfram.com/, or in books dealing with the geometry of the triangle.


Prove that

\[ \sum_{\text{cyclic}} \frac{ab}{c(c+a)} \geq \sum_{\text{cyclic}} \frac{a}{c+a}, \]

where a, b, c represent the three sides of a triangle.

Solution by Pierre Bornstein, Maisons-Laffitte, France; and Nick Skombris and Babis Stergiu, Chalkida, Greece.

Let \( x = a/b, y = b/c \) and \( z = c/a \). Then \( xyz = 1 \) and the inequality becomes

\[ \frac{x - 1}{y + 1} + \frac{y - 1}{z + 1} + \frac{z - 1}{x + 1} \geq 0, \]

which is equivalent to

\[ x^2 + y^2 + z^2 - x - y - z + xy^2 + yz^2 + zx^2 - 3 \geq 0. \]

(1)

Applying the AM–QM and AM–GM inequalities, we obtain

\[ x^2 + y^2 + z^2 \geq \frac{1}{3}(x + y + z)^2 \geq \sqrt[3]{xyz}(x + y + z) = x + y + z \]

(2)

and

\[ xy^2 + yz^2 + zx^2 \geq 3\sqrt[3]{x^3y^3z^3} = 3. \]

(3)

Adding (2) and (3) gives (1). Equality holds when \( x = y = z \). Therefore, the desired inequality is true, with equality when \( a = b = c \).
Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); VASILE CIRTOAJE, University of Ploesti, Romania; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Most solvers noticed that the condition that \( a, b, \) and \( c \) are the sides of a triangle is unnecessarily restrictive; the inequality is true for any three positive real numbers \( a, b, \) and \( c \), as seen from the presented solution. Specht and Stergiu indicated that this problem was one of the problems of the 1999 Moldavian Mathematical Olympiad. Specht gave the following web address as a reference: http://www.olomedim.com/olympiad/99.html


In \( \triangle ABC \) and \( \triangle A'B'C' \), the lengths of the sides satisfy \( a \geq b \geq c \) and \( a' \geq b' \geq c' \). Let \( h_a \) and \( h_{a'} \) denote the lengths of the altitudes to the opposite sides from \( A \) and \( A' \), respectively. Prove that

(a) \( bb' + cc' \geq ah_{a'} + a'h_a \);

(b) \( bc' + b'c \geq ah_{a'} + a'h_a \).

1. Solution by Michel Bataille, Rouen, France.

Note that \( A \) is the largest angle. Thus, the foot of the altitude from \( A \) certainly falls between \( B \) and \( C \). It follows that \( \sin B = \frac{h_a}{c} \) and \( \sin C = \frac{h_a}{b} \).

Similarly, \( \sin B' = \frac{h_{a'}}{c'} \) and \( \sin C' = \frac{h_{a'}}{b'} \). We have

\[
ah_{a'} + a'h_a = ab' \sin C' + a'c \sin B = b' \sin C'(b \cos C + c \cos B) + c \sin B(b' \cos C' + c' \cos B') = bb' \sin C' \cos C + b'c \sin C' \cos B + b' \sin B \cos C' + cc' \sin B \cos B'.
\]

(1)

(a) From (1),

\[
ah_{a'} + a'h_a = bb' \sin C' \cos C + cc' \sin B' \cos B + bb' \sin C \cos C' + cc' \sin B \cos B' = bb' \sin (C + C') + cc' \sin (B + B') \leq bb' + cc'.
\]

(b) From (1),

\[
ah_{a'} + a'h_a = bc' \sin B' \cos C + b'c \sin C' \cos B + b'c \sin B \cos C' + bc' \sin C \cos B' = bc' \sin (B' + C) + b'c \sin (B + C') \leq bc' + b'c.
\]
II. Solution by Li Zhou. Polk Community College, Winter Haven, FL, USA.

(a) Scale \( \triangle ABC \) by \( a' \) to get \( \triangle PQS \) with \( PQ = a'c, QS = a'a, \) and \( SP = a'b. \) Scale \( \triangle A'B'C' \) by \( a \) to get \( \triangle RSQ \) with \( RS = ac', QR = ab', \) and \( R \) and \( P \) on opposite sides of \( SQ. \) Applying Ptolemy's Inequality to \( PQRS, \) we get

\[
aa'bb' + aa'cc' = SP \cdot QR + PQ \cdot RS \geq SQ \cdot PR \geq aa'(ah_{a'} + a'h_{a}).
\]

Equality holds if and only if \( PQRS \) is cyclic and \( PR \perp SQ. \)

(b) We obtain this part similarly by modifying the above proof so that \( RS = ab' \) and \( QC = ac'. \)

Also solved by WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; and the proposer.

Klamkin observes that \( b'(b-c) \geq c'(b-c), \) which is equivalent to \( bb' + cc' \geq bc' + b'c. \)

Thus, it is sufficient to prove part (b).


The circle \( \Gamma(P, r) \) intersects the side \( AB \) of \( \triangle ABC \) at \( A_3 \) and \( B_3, \) the side \( BC \) at \( B_1 \) and \( C_1, \) and the side \( CA \) at \( C_2 \) and \( A_2. \)

Given that \( |A_3B_3| : |B_1C_1| : |C_2A_2| = |AB| : |BC| : |CA|, \) determine the locus of \( P. \)

Partial solution by the proposer.

Denote the lengths of \( B_1C_1, C_2A_2, \) and \( A_3B_3 \) by \( 2\lambda a, 2\lambda b, \) and \( 2\lambda c, \) respectively, and let \( x, y, \) and \( z \) be the distances from \( P \) to the sides of the triangle (that is, to the mid-points of these three segments). Since \( P \) is the centre of a circle through the end-points of these segments,

\[
x^2 + \lambda^2 a^2 = y^2 + \lambda^2 b^2 = z^2 + \lambda^2 c^2.
\]

Eliminating \( \lambda^2, \) we find

\[
x^2(b^2 - c^2) + y^2(c^2 - a^2) + z^2(a^2 - b^2) = 0,
\]

which we recognize as the equation of a conic in trilinear coordinates. Indeed, it is the conic through the circumcentre (where \( \lambda = 1/2, \)) and the four tritangent centres (that is, the incentre and the excentres, where \( \lambda = 0). \)

Moreover, since the conic passes through the tritangent centres, it must be a rectangular hyperbola (because each degenerate conic in the pencil determined by \( I, IA, IB, IC \) consists of a perpendicular pair of lines).

The only submitted solution came from the proposer. That is a pity—this seems to be a nice problem. Smeenk's argument establishes only that if \( P \) is a point of the locus, then it lies on the conic. To determine the locus, of course, one must also say which points of the conic are appropriate. It is clear that to be able to solve for \( \lambda \) we must have the magnitudes of the
distances $x, y, z$ in an order opposite to the corresponding side lengths $a, b, c;$ more precisely, if we label the triangle so that $a \geq b \geq c,$ then for $\lambda$ to exist we must have $x \leq y \leq z.$ For an equilateral triangle this implies that the locus consists of four points—the four tritangent centres; the fifth point $O$ will coincide with $I.$ For an isosceles triangle, say with $b = c,$ the hyperbola degenerates into the two bisectors of angle $A.$ In this case the locus consists of the segment between the two tritangent centres on one bisector, and of the complement of that segment on the other (depending on whether the circumcentre lies between $I$ and $I_A$ or not). For a scalene triangle it is not obvious to this editor how $\lambda$ is related to the locus, nor is it clear how much of the hyperbola belongs to the locus. Is it not the case that one branch of the hyperbola contains one excentre while the other branch contains the other excentres along with the incentre and the circumcentre?


The sequence $\{x_n\}$ is defined by $(1 + \frac{1}{n})^{n+x_n} = e.$

(a) Prove that $\{x_n\}$ is convergent, and determine its limit.

(b) Determine the asymptotic expansion of the sequence.

Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA.

We solve the given equation for $x_n$ as follows:

$$(n + x_n) \log \left(1 + \frac{1}{n}\right) = 1,$$

$$x_n \log \left(1 + \frac{1}{n}\right) = 1 - n \log \left(1 + \frac{1}{n}\right),$$

$$x_n = \frac{1 - n \log \left(1 + \frac{1}{n}\right)}{\log \left(1 + \frac{1}{n}\right)}.$$

Using a change of variables and three applications of L'Hôpital's Rule, we obtain

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1 - n \log \left(1 + \frac{1}{n}\right)}{\log \left(1 + \frac{1}{n}\right)} = \lim_{t \to 0^+} \frac{1 - \frac{1}{t} \log (1 + t)}{\log (1 + t)}$$

$$= \lim_{t \to 0^+} - \frac{1}{t(1 + t)} + \frac{1}{t^2} \log (1 + t)$$

$$= \lim_{t \to 0^+} - \frac{t + (1 + t) \log(1 + t)}{t^2}$$

$$= \lim_{t \to 0^+} \frac{\log(1 + t)}{2t} = \lim_{t \to 0^+} \frac{1}{2(1 + t)} = \frac{1}{2}.$$
In the discussion following equation (7.59) on p. 352 of [1], it is pointed out that
\[
\frac{z}{\log(1 + z)} = - \sum_{k \geq 0} (-z)^k \sigma_k(k - 1),
\]
where \( \sigma_k(x) \) is the \( k^{th} \) Stirling polynomial ([1], p. 271). Since
\[
x_n = \frac{1}{\log \left( 1 + \frac{1}{n} \right)} - n,
\]
we have
\[
x_n = -n \sum_{k \geq 0} \left( \frac{1}{n} \right)^k \sigma_k(k - 1) - n = \sum_{k \geq 1} \frac{(-1)^{k-1}}{nk-1} \sigma_k(k - 1).
\]
Again from [1], p. 271, we have the expressions for \( \sigma_k(x) \) shown on the left below. Corresponding values of \( \sigma_k(k - 1) \) are shown on the right.

\[
\begin{align*}
\sigma_0(x) &= \frac{1}{x}, & \sigma_0(-1) &= -1, \\
\sigma_1(x) &= \frac{1}{2}, & \sigma_1(0) &= \frac{1}{2}, \\
\sigma_2(x) &= \frac{3x - 1}{24}, & \sigma_2(1) &= \frac{1}{12}, \\
\sigma_3(x) &= \frac{x^2 - x}{48}, & \sigma_3(2) &= \frac{1}{24}, \\
\sigma_4(x) &= \frac{15x^3 - 30x^2 + 5x + 2}{5760}, & \sigma_4(3) &= \frac{19}{720}, \\
\end{align*}
\]

Hence,
\[
x_n = \frac{1}{2} - \frac{1}{12n} + \frac{1}{24n^2} - \frac{19}{720n^3} + \cdots.
\]

Reference.

Several solvers used the Maclaurin series expansion for \( \ln(1 + x) \) to obtain, for example,

\[
x_n = \frac{1 - n \left( \frac{1}{2} - \frac{1}{2n^2} + \frac{1}{3n^2} - \frac{1}{4n^2} + \cdots \right)}{1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^2} + \cdots} = \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots,
\]

from which the limit of \( \frac{1}{x} \) is clear. Moreover, the asymptotic expansion can be discovered as follows:

\[
x_n = 1 - n \left( \frac{1}{2} - \frac{1}{2n^2} + \frac{3n}{4n^2} + \cdots \right)
\begin{align*}
&= \frac{1}{2} + \frac{1}{12n} + \frac{1}{12n^2} - \frac{1}{4n^2} + \cdots \\
&= \frac{1}{2} - \frac{1}{12n} + \frac{1}{24n^2} - \frac{17}{360n^4} + \cdots
\end{align*}
\]

Benito, Ciauri, Fernandez, and Miguez, provided a solution to the problem using the Rey Pastor numbers and polynomials.

\[2863. \text{[2003 : 319]} \text{ Proposed by Mihály Bencze, Brasov, Romania.}\]

Suppose that \( a, b, c \) are complex numbers such that \(|a| = |b| = |c|\).

Prove that

\[
\left| \frac{ab}{a^2 - b^2} \right| + \left| \frac{bc}{b^2 - c^2} \right| + \left| \frac{ca}{c^2 - a^2} \right| \geq \sqrt{3}.
\]

Solution by CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA, modified slightly by the editor.

[Ed: Clearly, we must assume that \( a^2, b^2, \) and \( c^2 \) are all distinct and hence, \( abc \neq 0 \).]

Set \( a = re^{i\alpha}, b = re^{i\beta}, \) and \( c = re^{i\gamma} \). Then

\[
\left| \frac{ab}{a^2 - b^2} \right| = \frac{1}{|a|} \left| \frac{b}{a} - \frac{b}{a} \right| = \frac{1}{|a|} \left| e^{i(\alpha - \beta)} - e^{i(\alpha - \beta)} \right| = \frac{1}{2} \csc(\alpha - \beta) |.
\]

By symmetry, the given inequality is then equivalent to

\[
|\csc(\alpha - \beta)| + |\csc(\beta - \gamma)| + |\csc(\gamma - \alpha)| \geq 2\sqrt{3}.
\]

By the AM–GM Inequality, we have

\[
|\csc(\alpha - \beta)| + |\csc(\beta - \gamma)| + |\csc(\gamma - \alpha)| \geq 3|\csc(\alpha - \beta)\csc(\beta - \gamma)\csc(\gamma - \alpha)|^{1/3}
\]

\[
= 3|\csc(\alpha - \beta)\csc(\beta - \gamma)\csc(\pi - (\alpha - \gamma))|^{1/3}
\]

\[
= 3|\csc A\csc B\csc C|^{1/3},
\]

(2)
where \( A = \alpha - \beta, \ B = \beta - \gamma, \) and \( C = \pi - (A + B). \)
Consider the function \( f(t) = \ln |\sin t|. \) Since \( f'''(t) = -\csc^2 t < 0, \)
we see that \( f \) is convex. Hence, by Jensen's Inequality, we have
\[
f(A) + f(B) + f(C) \leq 3f \left( \frac{A + B + C}{3} \right) = 3 \ln \left( \frac{\pi}{3} \right) \]
\[
\ln |\sin A \sin B \sin C| \leq \left( \frac{\sqrt{3}}{2} \right)^3 \]
\[
|\sin A \sin B \sin C| \leq \left( \frac{\sqrt{3}}{2} \right)^3 \]
\[
|\csc A \csc B \csc C|^{1/3} \geq \frac{2}{\sqrt{3}}. \quad (3) \]
Substituting (3) into (2), we see that (1) follows immediately.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; OSCAR CIAURRI, Universidad de La Rioja, Logroño, Spain and JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHE JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; ANDREI SIMION, student, Cooper Union for the Advancement of Science and Art, New York, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Batallé also showed that equality holds if and only if the triangle with \( a, \ b, \ c \) as vertices is either equilateral or isosceles with an angle of \( 2\pi/3. \) Janous established the stronger inequality that for non-zero complex numbers \( a, \ b, \) and \( c, \) we have
\[
\left| \frac{a^2 - b^2}{ab} \right| + \left| \frac{b^2 - c^2}{bc} \right| + \left| \frac{c^2 - a^2}{ca} \right| \leq 9. \]
It is easy to show, via the Power–Mean Inequality and the AM–HM Inequality, that this result would imply the inequality in the proposed problem.

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**Crux Mathematicorum with Mathematical Mayhem**

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