M130. Proposé par l'Équipe de Mayhem.

On numérote des billets 1, 2, 3, 4, ..., N. La moitié d'entre eux comporte le chiffre 1.
(a) Sachant que N est un nombre de trois chiffres, déterminer toutes les valeurs possibles de N.
(b) Trouver certaines valeurs plus grandes que N pourraient avoir, si c'était un nombre de quatre chiffres, de cinq chiffres, etc.

M131. Proposé par l'Équipe de Mayhem.

Huit rangées de nombres forment un triangle avec les propriétés suivantes :
1. La base comporte les huit entiers 1, 2, ..., 8, écrits dans n'importe quel ordre.
2. En dessus de la base, chaque nombre est la somme des deux nombres de la ligne inférieure immédiatement voisins, pour autant que cette somme soit plus petite que 10; sinon on soustrait 9 de la somme.

Est-il possible de former un triangle de nombres avec les deux propriétés ci-dessus en utilisant chaque nombre de 1 à 9 exactement quatre fois?

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Pólya's Paragon

Au Contraire, Mon Ami

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Sometimes in mathematics, it is easier to do what you don't want to do than what you do want to do! In the September 2003 issue, we looked at proofs by contradiction. The main idea behind these proofs is that, instead of showing that what you want to prove is true, you show that the opposite must be untrue because, if it were true, it would produce a contradiction.
For example, let’s look at the second problem that was left for you in September:

**Prove that there are infinitely many primes.**

**Proof:** We start by assuming the opposite, namely, that there are finitely many primes. Label these primes \( p_1, p_2, \ldots, p_n \). Let’s create the number

\[
N = p_1 \times p_2 \times \cdots \times p_n + 1.
\]

We know that any integer greater than 1 is either prime or composite. Therefore, we have only two possibilities for \( N \):

**Case 1:** \( N \) is prime.

Clearly \( N \) is not one of the primes \( p_1, p_2, \ldots, p_n \), since, by its construction, \( N \) must be larger than all of these primes. Our original assumption is that we had all primes in our list. Thus, we have a contradiction.

**Case 2:** \( N \) is composite.

Then there must be some prime \( p \) that divides \( N \). Clearly, \( p \) is not one of the primes \( p_1, p_2, \ldots, p_n \), since, by the construction of \( N \), each of the primes \( p_1, p_2, \ldots, p_n \) leave a remainder of 1 when dividing into \( N \); thus, none of them divides \( N \). Therefore, \( p \) is a “new” prime (that is, one that was not in our list), and again we have a contradiction.

Our proof shows that, no matter what, we end up with a contradiction. That is, if we have a list of “all” primes, we can always construct a number that will give us a prime that isn’t on the list. So, we have proved that we can never list them all; that is, there are not finitely many primes.

You can try this yourself. If you assume that 2 is the only prime, you can create

\[
2 + 1 = 3,
\]

which is prime. Then, adjoining 3 to the list of primes, you get

\[
2 \times 3 + 1 = 7,
\]

which is prime. Repeating this process yields

\[
2 \times 3 \times 7 + 1 = 43,
\]

which is prime, and then

\[
2 \times 3 \times 7 \times 43 + 1 = 1807 = 13 \times 139,
\]

where 13 and 139 are both prime. If you continue this process, you will always get new primes at each step (although the factoring will become very difficult very quickly!)

A proof by contradiction is called an “indirect proof” because we get at the result indirectly, by considering another problem. Very often counting problems can be attacked indirectly as well, rather than looking at all the possibilities. For example, consider the following problem:
In how many ways can parents line up their 6 children for a picture if Sally and John cannot be put beside each other (since they fight)?

If we were to do this directly, we would have to find all the possible ways in which we can place these two children so that they are not side by side (which turns out to be 10 different cases). Alternatively, we can look at the “opposite” problem:

In how many ways can parents line up their 6 children for a picture if Sally and John must be put beside each other?

In this case, we can “attach” Sally and John and treat them as one “child”. We can do this in 2 ways, since we can attach them as Sally & John or John & Sally. Now we are arranging only 5 “children”, and there are $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ ways to do this. Thus, there are $2 \times 120 = 240$ ways to arrange the children so that Sally and John will be together. Going back to the original problem, we know there are $6! = 720$ ways to line up all 6 children, if we don’t care how we do it. Therefore, there must be $720 - 240 = 480$ ways to line them up with Sally and John apart.

Thus, the main idea behind the indirect method is to calculate the cases you don’t want and remove them from the total. This will save you some work in quite a few problems.

Here are a couple of problems for you to consider.

1. In how many ways can we rearrange the letters in the word MAYHEM so that the A and E are apart and the two M’s are together?

2. Thirteen people each pick a number from 1 to 100 and secretly write it on a piece of paper. What is the probability that at least two people picked the same number? (This is a variation of the classic “birthday problem” that we will visit in an upcoming issue.)