On Expanding $4/n$ into Three Egyptian Fractions

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A notable conjecture of P. Erdős and E. Straus states that the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is solvable in positive integers, for $n \in \{4, 5, \ldots\}$. The fractions on the right side above, reciprocals, are called "Egyptian" fractions. According to R. Guy [1], the problem has been verified by N. Franceschine for $n < 10^8$. In addition, Guy notes that, when the results of L. Bernstein, R. Obláth, K. Yamamoto, and L. Rosati on this problem are summarized, the conjecture is seen to be true for all n, except possibly where

$$n \equiv 1, 121, 169, 289, 361, 529 \pmod{840}.$$

Also in [1], it is indicated that A. Schinzel has relaxed the condition that x, y, z be positive. It is known that $4/n$ can be expressed as the sum of Egyptian fractions where the algebraic sign of at least one of the Egyptian fractions is negative.

The objective of this note is to present a universal three-term Egyptian fraction decomposition for $4/n$, where $n > 3$. Specifically, we introduce a single formula that allows a proper fraction of the form $4/n$ to be expressed as

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} - \frac{1}{z},$$

where x, y, $z \in \mathbb{Z}^+$.

Without loss of generality, we may assume that n is prime. Otherwise, if $n = 2^k$ for $k \geq 2$, then $4/n$ is equivalent to the Egyptian fraction $1/2^{k-2}$, and we may choose $x = 1/2^{k-2}$, and $y = z$, any positive integer. Furthermore, if $n = rp$, where p is an odd prime, then $4/n = (1/r) \cdot (4/p)$, and a three-term Egyptian fraction decomposition for $4/n$ may be obtained by expanding $4/p$ in terms of Egyptian fractions and then multiplying through by $1/r$.

Theorem. Let n be an odd prime. Then

$$\frac{4}{n} = \frac{1}{\left(\frac{n-1}{2}\right)} + \frac{1}{\left(\frac{n+1}{2}\right)} - \frac{1}{n\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)}.$$

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Examples.

1. $n = 109$:

$$\begin{aligned}\frac{4}{109} &= \frac{1}{\left(\frac{109-1}{2}\right)} + \frac{1}{\left(\frac{109+1}{2}\right)} - \frac{1}{109\left(\frac{109-1}{2}\right)\left(\frac{109+1}{2}\right)} \\ &= \frac{1}{54} + \frac{1}{55} - \frac{1}{323730}.\end{aligned}$$

2. $n = 1001$:

$$\begin{aligned}\frac{4}{1001} = \frac{1}{7\cdot 11}\left(\frac{4}{13}\right) &= \frac{1}{77}\left[\frac{1}{\left(\frac{13-1}{2}\right)} + \frac{1}{\left(\frac{13+1}{2}\right)} - \frac{1}{13\left(\frac{13-1}{2}\right)\left(\frac{13+1}{2}\right)}\right] \\ &= \frac{1}{462} + \frac{1}{539} - \frac{1}{42042}.\end{aligned}$$

References.

[1] Richard K. Guy, *Unsolved Problems in Number Theory*, 2nd Ed., Springer Verlag, New York, (1994).

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