THE OLYMPIAD CORNER
No. 235

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Here is the start of another volume of *Crux Mathematicorum* and the occasion to thank all those who contributed problems, comments, solutions and generalizations used in the 2003 number of the *Corner*.

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Special thanks also go to Joanne Longworth for her exceptional skills deciphering my handwriting, producing high quality *\LaTeX* materials, and being far more organized, on track, and on schedule than I seem to be.

To rekindle your problem solving for the new year, we first give a set of five Klamkin Quickies. Try them before looking up his "Quickie" solutions later in this number. Thanks to Murray S. Klamkin, University of Alberta, Edmonton, AB for creating them for us.

**FIVE KLAMKIN QUICKIES**
February 2004

1. Determine all polynomials $P(x)$ such that either $(x+1)P(x) = xP(x-1)$ or else $(x+1)P(x) = xP(x+1)$ for all $x$.

2. Show that the following polynomial has no real roots:

$$P(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - \cdots - 2nx + 2n + 1.$$
3. Determine all integral solutions of $x^3 + y^3 = z^6 + 3$.

4. Determine the maximum value of $\tan x + \tan y$, where

$$(1 + \sin x)(1 + \sin y) = \cos x \cos y.$$ 

5. Determine whether or not the inequality $(x^2 + y^2)^4 \geq 4xy(x^3 + y^3)^2$ is valid for all real $x, y$.

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Next we give the problems of the 40th IMO Vietnam Team Selection Test written in May 2001. My thanks go to Pham van Thuan for translating them and submitting them for our use.

**40th INTERNATIONAL MATHEMATICAL OLYMPIAD**

*Vietnam Team Selection Test*

_Hanoi, Vietnam  May 8–9, 2001_

1. Let a sequence of integers $\{a_n\}$, $n \in \mathbb{N}$, be defined by

$$a_0 = 1, \quad \text{and} \quad a_n = a_{n-1} + a_{\lfloor n/3 \rfloor} \quad \text{for every} \quad n \in \mathbb{N}^*,$$

where $\lfloor x \rfloor$ denotes the integer part of the real number $x$. Prove that for each prime number $p$ there exists a natural number $k$ such that $a_k$ is divisible by $p$.

2. Two circles are given, intersecting each other at two points $A$ and $B$. Let $\ell$ be a common tangent of the two circles, and let $P$ and $T$ be the points of tangency. Denote by $S$ the intersection point of the two tangents to the circumcircle of triangle $APT$ at $P$ and $T$. Let $H$ be the reflection of point $B$ across the line $\ell$. Prove that $A, S, H$ are collinear.

3. There are 42 members in a club. Among any 31 of them, there is a pair consisting of a man and a woman who know each other. Prove that there are at least 12 man-woman pairs who know each other.

4. Given positive real numbers $x, y, z$ satisfying the inequality

$$21xy + 2yz + 8zx \leq 12,$$

determine the minimum value of the function

$$f(x, y, z) = \frac{1}{x} + \frac{2}{y} + \frac{3}{z}.$$ 

5. Let $n$ be an integer, $n > 1$. Denote by $A$ the set of points $P(x, y, z)$ such that $x, y, z$ are integers satisfying $1 \leq x, y, z \leq n$. Colour some of the points in $A$ in such a manner that if point $M(x_0, y_0, z_0)$ is coloured, point $N(x_1, y_1, z_1)$ with $x_1 \leq x_0, y_1 \leq y_0, z_1 \leq z_0$ is not coloured.

Determine, with proof, the maximum possible number of points that can be coloured.
6. Denote by \( \{a_n\} \), \( n \in \mathbb{N}^* \), a sequence of positive integers satisfying the following condition

\[
0 < a_{n+1} - a_n \leq 2001 \quad \text{for all} \quad n \in \mathbb{N}^*.
\]

Prove that there are infinitely many pairs of positive integers \((p, q)\) such that \( p < q \) and \( a_p \) is a divisor of \( a_q \).

As a second Olympiad set we give the XVII National Mathematical Contest of Italy written at Cesenatico, May 4, 2001. My thanks go to Chris Small, Canadian Team Leader to the 41st IMO, for collecting them for us.

**XVII NATIONAL MATHEMATICAL CONTEST OF ITALY**

**May 4, 2001**

1. In a hexagon with all angles equal, the lengths of four consecutive edges are 5, 3, 6 and 7 (in this order). Find the lengths of the remaining two edges.

2. In a basketball tournament, each team played twice against each other team. Two points were awarded for a win and no points for a loss. (No game could finish in a draw.) A single team won the tournament with 26 points, and exactly two teams finished last with 20 points. How many teams participated in the tournament?

3. Given the equation \( x^{2001} = y^x \),

   (a) find all solution pairs \((x, y)\) consisting of positive integers with \( x \) prime;

   (b) find all solution pairs \((x, y)\) consisting of positive integers.

   (Recall that \( 2001 = 3 \cdot 23 \cdot 29 \).)

4. Call a positive integer **monotone** if, in its decimal representation, there are at least two digits, all digits are different from zero, and the digits appear in a strictly increasing or strictly decreasing order. (For instance, 127 and 9742 are monotone, whereas 172, 1224, and 7320 are not.)

   (a) Compute the sum of all monotone numbers having five digits.

   (b) Determine the number of final zeroes of the decimal representation of the least common multiple of all monotone numbers (without any restriction on the number of digits).

5. The incircle \( \gamma \) of triangle \( ABC \) touches the side \( AB \) at \( T \). Let \( D \) be the point on \( \gamma \) diametrically opposite to \( T \), and let \( S \) be the intersection of the line through \( C \) and \( D \) with the side \( AB \). Show that \( AT = SB \).
6. A square is filled with \( n^2 \) lamps, arranged in \( n \) rows and \( n \) columns. Some of them are alight, the others are out. To each lamp corresponds a push-button that, when pressed, switches all lamps of its row and its column (including the lamp itself). Determine the states from which it is possible to light all the lamps if

(a) \( n = 10 \); 
(b) \( n = 9 \).

As a third Olympiad set we give the problems of the 52nd Polish Mathematical Olympiad, Final Round, April 2–3, 2001. Thanks again go to Chris Small, Canadian Team Leader to the XLI IMO, for collecting them.

52nd POLISH MATHEMATICAL OLYMPIAD
Final Round
April 2 (Day 1), 2001 – Time: 5 hours

1. Show that the inequality

\[
\sum_{i=1}^{n} x_i \leq \binom{n}{2} + \sum_{i=1}^{n} x_i
\]

holds for every integer \( n \geq 2 \) and all real numbers \( x_1, x_2, \ldots, x_n \geq 0 \).

2. Consider an arbitrary point \( P \) inside the regular tetrahedron with an edge of length 1. Show that the sum of the distances from \( P \) to the vertices of the tetrahedron does not exceed 3.

3. The sequence \( x_1, x_2, x_3, \ldots \) is defined recursively by

\[
x_1 = a, \quad x_2 = b, \quad \text{and} \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 1, 2, 3, \ldots,
\]

where \( a \) and \( b \) are real numbers. A number \( c \) will be called a repeated value if \( x_k = x_l = c \) for at least two distinct indices \( k \) and \( l \). Prove that the initial terms \( a \) and \( b \) can be chosen so that there are more than 2000 repeated values, but it is impossible to choose \( a \) and \( b \) so that there are infinitely many repeated values.

April 3 (Day 2), 2001 – Time: 5 hours

4. The integers \( a \) and \( b \) have the property that, for every non-negative integer \( n \), the number \( 2^n a + b \) is the square of an integer. Show that \( a = 0 \).

5. Let \( ABCD \) be a parallelogram, and let \( K \) and \( L \) be points lying on the sides \( BC \) and \( CD \), respectively, such that \( BK \cdot AD = DL \cdot AB \). The segments \( DK \) and \( BL \) intersect at \( P \). Show that \( \angle DAP = \angle BAC \).

6. Let \( n_1 < n_2 < \cdots < n_{2000} < 10^{100} \) be positive integers. Prove that the set \( \{ n_1, n_2, \ldots, n_{2000} \} \) has two non-empty disjoint subsets \( A \) and \( B \) with equally many elements, equal sums of their elements, and equal sums of the squares of their elements.
SOLUTIONS TO FIVE KLAMKIN QUICKIES

February 2004

1. Setting \( x = -1 \) in the first relation, we get \( P(-2) = 0 \). It then follows that \( P(-n) = 0 \) for all \( n \geq 2 \). Then \( P(x) \) cannot be a polynomial. Setting \( P(x) = xQ(x) \) in the second relation, we get \( Q(x) = Q(x+1) \), so that \( Q(x) \) must be a constant. Therefore, the only solution is \( P(x) = kx \).

2. Clearly, \( P(x) \) has no negative roots. To show there are no non-negative real roots, we note that for all \( x \geq 0 \),

\[
(1 + x)P(x) = x^{2n+1} - x^{2n} + \cdots + x + 2n + 1 \]

\[
= \frac{x(x^{2n+1} + 1)}{x + 1} + 2n + 1 > 0.
\]

3. We consider values modulo 7. Cubic residues are 0, -1, and 1. Thus, the right side is either 3 or 4, while the left side can only be 0, 1, 2, -1, or -2. Hence, there are no solutions.

4. Let \( x \) and \( y \) satisfy the given relation. Letting \( \tan x = \sinh u \) and \( \tan y = \sinh v \), we have \( \sec x = \pm \cosh u \) and \( \sec y = \pm \cosh v \). The given relation becomes \((\sinh u \pm \cosh u)(\sinh v \pm \cosh v) = 1\), which implies that \( e^{u+v} = 1 \) and hence, \( u + v = 0 \). Then \( \tan x + \tan y = \sinh u + \sinh v = 0 \).

5. The inequality is clearly valid for \( x = y = 1 \). Next, we try \( x = 1 + \varepsilon \) and \( y = 1 \) where \( \varepsilon \) is arbitrarily small. Expanding both sides to order \( \varepsilon^2 \), the left side is \( 16 + 64 + 128\varepsilon^2 \), while the right side is \( 16 + 64\varepsilon + 132\varepsilon^2 \). Thus, the inequality is invalid.

Next we turn to readers' contributions for problems of the 47th Czech and Slovak Mathematical Olympiad 1998 given [2001 : 360–361].

1. Find all solutions in the real domain of the equation

\[ x \cdot [x \cdot |x|] = 88, \]

where \( |a| \) is the integer part of a real number \( a \); that is, the integer satisfying \( |a| \leq a < |a| + 1 \). For instance, \( |3.7| = 3 \), \( |-3.7| = -4 \) and \( |6| = 6 \).

Solved by Mohammed Assila, Strasbourg, France; Michel Bataille, Rouen, France; Robert Biński, Outremont, QC; Pierre Bornsztein, Maisons-Laffitte, France; Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; George Evagelopoulos, Athens, Greece; Pavlos Maragoudakis, Pireas, Greece. We give the solution by Maragoudakis.

A straightforward calculation shows that \( x = \frac{22}{7} \) is a solution. We will prove that this is the only solution.
Let $x$ be a solution and let $n = \lfloor x \rfloor$. Then $n \leq x < n + 1$.

**Case 1.** $x > 0$.

Then

$$n^2 \leq x \cdot \lfloor x \rfloor < n^2 + n,$$

$$n^2 \leq \lfloor x \cdot \lfloor x \rfloor \rfloor < n^2 + n,$$

$$n^3 \leq x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor < (n + 1)^2,$$

$$n^3 \leq \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor < (n + 1)^2,$$

$$n^4 \leq x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor < (n + 1)^3,$$

$$n^4 \leq 88 < (n + 1)^3.$$

The inequalities in the last line are true only if $n = 3$. Now the equation becomes $x \cdot \lfloor x \cdot [3x] \rfloor = 88$.

Let $k = [3x]$. Then $k \leq 3x < k + 1$. Thus,

$$\frac{k}{3} \leq x < \frac{k + 1}{3},$$

$$\frac{k^2}{9} \leq x \cdot [3x] < \frac{k(k + 1)}{3},$$

$$\frac{k^2 - 1}{3} < \lfloor x \cdot [3x] \rfloor \leq \frac{k(k + 1)}{3},$$

$$\frac{k^3 - 3k}{9} < x \cdot \lfloor x \cdot [3x] \rfloor \rfloor < \frac{k(k + 1)^2}{9},$$

$$\frac{k^3 - 3k}{9} < 88 < \frac{k(k + 1)^2}{9}.$$

These last inequalities are true only if $k = 9$. Now the equation becomes $x[9x] = 88$.

Let $\ell = [9x]$. Then $\ell \leq 9x < \ell + 1$. Hence,

$$\frac{\ell}{9} \leq x < \frac{\ell + 1}{9},$$

$$\frac{\ell^2}{9} \leq x \cdot [9x] < \frac{\ell^2 + \ell}{9},$$

$$\frac{\ell^2}{9} \leq 88 < \frac{\ell^2 + \ell}{9},$$

$$\ell^2 \leq 792 < \ell^2 + \ell.$$

These inequalities are true only if $\ell = 28$. Finally, from the equation $x[9x] = 88$, we get $x = \frac{88}{28} = \frac{22}{7}$.  

Case 2. \( x < 0. \)

Then (since \( n < 0 \)) we have

\[
\begin{align*}
\quad n^2 & \geq x \cdot |x| > n^2 + n, \\
\quad n^2 & \geq |x \cdot |x|| \geq n^2 + n, \\
\quad n^3 & \leq x \cdot |x \cdot |x|| \leq n(n + 1)^2, \\
\quad n^3 & \leq |x \cdot |x \cdot |x|| \leq n(n + 1)^2, \\
\quad n^4 & \geq x \cdot |x \cdot |x \cdot |x|| \geq n(n + 1)^3, \\
\quad n^4 & \geq 88 \geq n(n + 1)^3.
\end{align*}
\]

No integer \( n < 0 \) satisfies these inequalities.

2. Show that from any fourteen different natural numbers it is possible to choose, for a suitable \( k (1 \leq k \leq 7) \), two disjoint \( k \)-element subsets \( \{a_1, a_2, \ldots, a_k\} \) and \( \{b_1, b_2, \ldots, b_k\} \) in such a way that the sums

\[
A = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \quad \text{and} \quad B = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_k}
\]

differ by less than 0.001; that is, \( |A - B| < 0.001 \).

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornstein, Maisons-Laffitte, France; and George Evangelopoulos, Athens, Greece. We present Bornstein’s write-up.

Let \( E \) be a set of fourteen different natural numbers. The number of subsets of \( E \) having cardinality 7 is \( \binom{14}{7} = 3432 \). Let \( X \) be one of these 7-element subsets, and set \( f(X) = \sum_{a \in X} \frac{1}{a} \). We then have

\[
f(X) \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{363}{140} < 2.6.
\]

Thus, each of the 3432 values of \( f(X) \), when \( X \) varies over all the possible 7-element subsets of \( E \), belongs to one of the 2600 pairwise disjoint intervals of the form \( (\frac{p}{1000}, \frac{p+1}{1000}) \), where \( p = 0, 1, \ldots, 2599 \). Applying the Pigeon-Hole Principle, we deduce that there exist two different 7-element subsets of \( E \), say \( X \) and \( Y \), such that \( f(X) \) and \( f(Y) \) belong to the same interval.

Let \( X' \) and \( Y' \) be the sets formed from \( X \) and \( Y \) by removing any elements common to \( X \) and \( Y \), and let \( A = f(X') \), \( B = f(Y') \). Then \( X' \) and \( Y' \) are two disjoint subsets of \( E \) with the same cardinality, and we have \( |A - B| = |f(X') - f(Y')| = |f(X) - f(Y)| < 0.001 \).
3. A sphere is inscribed in a given tetrahedron $ABCD$. Its four tangent planes, which are parallel to the faces of the tetrahedron, cut four smaller tetrahedra from the tetrahedron. Prove that the sum of lengths of all their 24 edges is equal to twice the sum of the lengths of the edges of the tetrahedron $ABCD$.

![Diagram of a tetrahedron with inscribed sphere]

Solution by George Evagelopoulos, Athens, Greece.

Denote by $r$ the radius of the inscribed sphere and by $U_A, U_B, U_C, U_D$ the four heights of the given tetrahedron, labelled according to the vertices from which they emanate. The smaller tetrahedron $AKLM$ (see figure) is homothetic (with centre $A$) to the whole tetrahedron $ABCD$. Thus, the ratio of the sum of the edge lengths of $AKLM$ to the sum of the edge lengths of $ABCD$ is the same as the ratio of their heights from the common vertex $A$. This ratio is $(U_A - 2r) : U_A$ (since $2r$ is the distance between the parallel planes $KLM$ and $BCD$, both of which are tangent to the inscribed sphere).

A similar argument applies to the other three small tetrahedra.

Our task is then to show that

$$
\frac{U_A - 2r}{U_A} + \frac{U_B - 2r}{U_B} + \frac{U_C - 2r}{U_C} + \frac{U_D - 2r}{U_D} = 2,
$$

or equivalently,

$$
r \left( \frac{1}{U_A} + \frac{1}{U_B} + \frac{1}{U_C} + \frac{1}{U_D} \right) = 1.
$$

We prove this by an argument involving the volume $V$ and surface area $S$ of the tetrahedron $ABCD$. Let $S_X$ denote the area of the face not containing the vertex $X$. Then $S = S_A + S_B + S_C + S_D$. Furthermore,

$$
V = \frac{1}{3}S_A U_A = \frac{1}{3}S_B U_B = \frac{1}{3}S_C U_C = \frac{1}{3}S_D U_D,
$$

and $V = \frac{1}{3}rS$. Combining these formulas, we obtain

$$
r \left( \frac{1}{U_A} + \frac{1}{U_B} + \frac{1}{U_C} + \frac{1}{U_D} \right) = \frac{3V}{S} \left( \frac{S_A}{3V} + \frac{S_B}{3V} + \frac{S_C}{3V} + \frac{S_D}{3V} \right) = 1,
$$

which completes the proof.
4. For each date of the current year (1998) we evaluate the expression
\[ \text{day}^\text{month} - \text{year} \]
and then find the highest power of 3 dividing it. For instance, for April 21 we obtain \(21^4 - 1998 = 192,483 = 3^3 \cdot 7129\), which is a multiple of \(3^3\), but not of \(3^4\). Find all days for which the corresponding power is the greatest.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and George Evagelopoulos, Athens, Greece. We give Bornsztein's write-up.

Let \(f(d, m) = d^m - 1998\), where \(d\) and \(m\) are the day and the month, respectively. Let \(\nu(d, m)\) be the exponent of 3 in the prime decomposition of \(f(d, m)\). Since \(3^3\) is the highest power of 3 which divides 1998, we can have \(\nu(d, m) \geq 4\) only if \(d^m \equiv 0 \mod 3^3\) and \(d^m \not\equiv 0 \mod 3^4\). To have \(\nu(d, m) \geq 4\), we then need \(\alpha m = 3\), where \(\alpha\) is the exponent of 3 in the prime decomposition of \(d\).

**First case.** \(\alpha = 3\) and \(m = 1\).

Since \(d \in \{1, 2, \ldots, 31\}\), we have \(d \equiv 0 \mod 3^3\) if and only if \(d = 27\). Then, the date is January 27, and \(f(27, 1) = 27 - 1998 = -1971 = -3^3 \times 73\); whence, \(\nu(27, 1) = 3\).

**Second case.** \(\alpha = 1\) and \(m = 3\).

Then \(d \in \{3, 6, 12, 15, 21, 24, 30\}\). Direct computation in these seven cases leads to:

\[
\nu(3, 3) = \nu(12, 3) = \nu(21, 3) = \nu(30, 03) = 3, \\
\nu(6, 3) = \nu(15, 3) = \nu(24, 3) = 4.
\]

Thus, the power of 3 is greatest for March 6, March 15, and March 24.

**Conjecture.** This exercise was given to the competitors March 24, 1998 (according to the dates given in [2001 : 360]).

5. In the exterior of a circle \(k\) a point \(A\) is given. Show that the diagonals of all trapezoids which are inscribed into the circle \(k\) and whose extended arms intersect at the point \(A\) intersect at the same point \(U\).

*Solution by George Evagelopoulos, Athens, Greece.*
Let $S$ be the centre of the circle $k$, and let $KLMN$ be an inscribed trapezoid whose extended arms $KN$ and $LM$ intersect at $A$ (see figure). We use the axial symmetry of the trapezoid $KLMN$ with respect to the line $AS$ to conclude that the intersection $U$ of its diagonals must also lie on this line. Let $T$ be one of the endpoints of the chord of the circle $k$ which is perpendicular to $SA$ and passes through $U$. The power of the point $U$ with respect to the circle $k$ is $|KU| \cdot |MU| = |TU|^2$. Considering the triangle $KAM$, in which $AU$ bisects $\angle KAM$, we see that

$$|AU|^2 = |AK| \cdot |AM| - |KU| \cdot |MU| = |AK| \cdot |AN| - |TU|^2.$$ 

Hence,

$$|AK| \cdot |AN| = |AU|^2 + |TU|^2 = |AT|^2.$$ 

Since $|AK| \cdot |AN| = |AT|^2$, the point $T$ is the point of tangency of one of the two tangents to $k$ passing through the point $A$. These two tangents are independent of the choice of the trapezoid $KLMN$. Hence, the same is true for the point $U$.

Let us also remark that, by the Theorem of Euclid, the leg $ST$ of the right triangle $ATS$ satisfies $|ST|^2 = |SU| \cdot |SA|$, showing that $U$ and $A$ are images of each other in the inversion with respect to the circle $k$.

6. Let $a$, $b$, $c$ be positive numbers. Show that the triangle with sides $a$, $b$, $c$ exists if and only if the system of equations

$$\frac{y}{z} + \frac{z}{y} = \frac{a}{x}, \quad \frac{z}{x} + \frac{x}{z} = \frac{b}{y}, \quad \frac{x}{y} + \frac{y}{x} = \frac{c}{z}$$

has a solution in the real domain.

Solved by Michel Bataille, Rouen, France; George Evagelopoulos, Athens, Greece; and Pavlos Maragoudakis, Pireas, Greece. We feature Bataille’s solution.

Suppose first that the given system has a solution $(x, y, z)$, where $x$, $y$, and $z$ are (non-zero) real numbers. Then,

$$b + c - a = y \left( \frac{z}{x} + \frac{x}{z} \right) + z \left( \frac{x}{y} + \frac{y}{x} \right) - x \left( \frac{y}{z} + \frac{z}{y} \right) = \frac{2yz}{x} = \frac{2(y^2 + z^2)}{a} > 0.$$ 

Thus, $b + c > a$. Similarly, $c + a > b$ and $a + b > c$. Therefore, $a$, $b$, $c$ are the sides of some triangle.

Conversely, if $a$, $b$, $c$ are the sides of a triangle, let $s = \frac{a + b + c}{2}$ and

$$x = \sqrt{(s - b)(s - c)}, \quad y = \sqrt{(s - c)(s - a)}, \quad z = \sqrt{(s - a)(s - b)}.$$
Then \((x, y, z)\) is a solution of the system. Indeed,

\[
\frac{y}{z} + \frac{z}{y} = \frac{\sqrt{s-c} + \sqrt{s-b}}{\sqrt{s-c}} + \frac{\sqrt{s-b}}{\sqrt{s-c}} = \frac{s-c + s-b}{x} = \frac{a}{x},
\]
and the other two equations are similarly verified.

Next we turn to the November 2001 number of the Corner and readers' solutions and comments to Selection Questions for the Armenian Team for IMO99 given [2001 : 419-420].

1. Let \(O\) be the centre of the circumcircle of the acute triangle \(ABC\). The lines \(CO, AO,\) and \(BO\) intersect for the second time the circumcircles of the triangles \(AOB, BOC,\) and \(AOC\) at \(C_1, A_1,\) and \(B_1\) respectively.

Prove that

\[
\frac{AA_1}{OA_1} + \frac{BB_1}{OB_1} + \frac{CC_1}{OC_1} \geq 4.5.
\]

*Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's argument.*

![Diagram of triangle ABC with circumcircles and incentre O](image)

Let \(\Gamma\) be the circumcircle of \(\triangle ABC\), and let \(R\) be its radius. Let \(f\) be the inversion in \(\Gamma\). For any point \(P\) distinct from \(O\), let \(P' = f(P)\). Then \(A' = A, B' = B,\) and \(C' = C\). The image of the circumcircle of \(\triangle OBC\) is a line containing \(B'\) and \(C'\) and is therefore the line \(BC\). Since the line through \(A, O,\) and \(A_1\) is mapped onto itself by \(f\), we deduce that \(A_1'\) is the intersection of this line with \(BC\), and we have

\[
OA_1 \cdot OA_1' = R^2. \tag{1}
\]

For any points \(M\) and \(N\) distinct from \(O\), it is well known that

\[
M'N' = \frac{R^2 \cdot MN}{OM \cdot ON}.
\]
Thus, 
\[ AA_1 = \frac{R^2 \cdot AA'_1}{OA \cdot OA'_1}, \]
and hence, using (1),
\[ \frac{AA_1}{OA_1} = \frac{R^2 \cdot AA'_1}{OA \cdot OA'_1} \cdot \frac{OA'_1}{R^2} = \frac{AA'_1}{OA} \] \hspace{1cm} (2)

Let \( I \) and \( J \) be the respective orthogonal projections of \( A \) and \( O \) onto the line \( BC \). Let \( x = [OBC] \) and \( S = [ABC] \). From Thales’ Theorem, we have
\[ \frac{OA'_1}{AA'_1} = \frac{OJ}{AI} = \frac{x}{S}. \]

Since \( O \) is interior to \( \triangle ABC \) (because \( \triangle ABC \) is acute), it follows that \( OA = AA'_1 - OA'_1 \), and therefore
\[ \frac{OA}{AA'_1} = 1 - \frac{OA'_1}{AA'_1} = \frac{S - x}{S}. \]

Then, using (2),
\[ \frac{AA_1}{OA_1} = \frac{S}{S - x}. \]

Similarly,
\[ \frac{BB_1}{OB_1} = \frac{S}{S - y} \quad \text{and} \quad \frac{CC_1}{OC_1} = \frac{S}{S - z}, \]
where \( y = [OAC] \) and \( z = [OAB] \). Then
\[ \frac{AA_1}{OA_1} + \frac{BB_1}{OB_1} + \frac{CC_1}{OC_1} = \frac{S}{S - x} + \frac{S}{S - y} + \frac{S}{S - z}. \] \hspace{1cm} (3)

We have \( x, y, z > 0 \) and \( S = x + y + z \). From the Cauchy-Schwarz Inequality, we have
\[ \left( \frac{S}{S - x} + \frac{S}{S - y} + \frac{S}{S - z} \right) \left( (S - x) + (S - y) + (S - z) \right) \geq \left( \sqrt{S} + \sqrt{S} + \sqrt{S} \right)^2; \]
that is, \( 2S \left( \frac{S}{S - x} + \frac{S}{S - y} + \frac{S}{S - z} \right) \geq 9S \). Thus,
\[ \frac{S}{S - x} + \frac{S}{S - y} + \frac{S}{S - z} \geq \frac{9}{2}. \] \hspace{1cm} (4)

The result follows directly from (3) and (4). Note that equality occurs if and only if \( \triangle ABC \) is equilateral.
2. Let the escribed circle (opposite \( \angle A \)) of the triangle \( ABC \) (\( \angle A, \angle B, \angle C < 120^\circ \)) with centre \( O \) be tangent to the sides of the triangle \( AB, BC \) and \( CA \) at points \( C_1, A_1 \) and \( B_1 \) respectively. Denote the mid-points of the segments \( AO, BO, \) and \( CO \) by \( A_2, B_2, \) and \( C_2, \) respectively.

Prove that lines \( A_1A_2, B_1B_2, \) and \( C_1C_2 \) intersect at the same point.

**Solution by Christopher J. Bradley, Clifton College, Bristol, UK.**

We will use areal coordinates. The following coordinates are known:

\[
A: (1, 0, 0); \quad B: (0, 1, 0); \quad C: (0, 0, 1);
\]

\[
A_1: \frac{1}{2a}(0, a - b + c, a + b - c);
\]

\[
B_1: \frac{1}{2b}(b - c - a, 0, a + b + c);
\]

\[
C_1: \frac{1}{2c}(c - a - b, a + b + c, 0);
\]

\[
O: \frac{1}{b + c - a}(-a, b, c).
\]

The coordinates of the mid-point of a line segment are the averages of the coordinates of the endpoints. Thus, the coordinates of \( A_2, B_2, \) and \( C_2 \) are:

\[
A_2: \frac{1}{2(b + c - a)}(b + c - 2a, b, c);
\]

\[
B_2: \frac{1}{2(b + c - a)}(-a, 2b + c - a, c);
\]

\[
C_2: \frac{1}{2(b + c - a)}(-a, b, 2c + b - a).
\]

The equation of \( A_1A_2 \) is

\[
\text{det} \begin{pmatrix}
  x & y & z \\
 b + c - 2a & b & c \\
 0 & a - b + c & a + b - c
\end{pmatrix} = 0,
\]

which, on expansion, becomes

\[
x(b - c)(a + b + c) - y(b + c - 2a)(a + b - c) + z(b + c - 2a)(a - b + c) = 0.
\]

Similarly, \( B_1B_2 \) has the equation

\[
x(a + b + c)(2b + c - a) + y(a + c)(a + b - c) + z(a - b + c)(2b + c - a) = 0,
\]

and \( C_1C_2 \) has the equation

\[
x(a + b + c)(2c + b - a) + y(a + b - c)(2c + b - a) + z(a + b)(a - b + c) = 0.
\]
The three lines meet at the point with unnormalized areal coordinates

\[ x = \frac{(b + c)(2a - b - c)}{a + b + c}, \]
\[ y = \frac{(2b + c - a)(c - a)}{a + b - c}, \]
\[ z = \frac{(2c + b - a)(b - a)}{a - b + c}. \]

5. Any 9 squares are removed from the 40 white squares of a 9 \times 9 chess-like painted board. Prove that the remaining board is impossible to cover using 24 pieces of the kind as shown in the figure.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

Since the original board has 40 white squares and 41 black squares, the four squares at its corners are black. Label the rows from 1 to 9. Colour the black squares red or yellow according to the parity of their row, as shown in the figure below. After removing the 9 white squares, there are 25 red squares, 16 yellow squares, and \( 40 - 9 = 31 \) white squares.

\[
\begin{array}{cccccccc}
1 & R & W & R & W & R & \cdots \\
2 & W & Y & W & Y & W & \cdots \\
3 & R & W & R & W & R & \cdots \\
4 & W & Y & W & Y & W & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
9 & R & W & R & W & R & \cdots \\
\end{array}
\]

Now, suppose that a covering is possible. It uses three types of pieces:

- Type \( R \)
- Type \( Y \)
- Type \( W \)

Let \( r, y, w \) be the numbers of pieces of types \( R, Y, W \), respectively, used for the covering. Then \( r + y + w = 24 \). Moreover, since there are 25 red squares, we must have \( r + w = 25 \). Thus, \( y = -1 \), a contradiction. The conclusion follows.

6. Solve the equation

\[
\frac{1}{x^2} + \frac{1}{(4 - \sqrt{3}x)^2} = 1.
\]
Solved by Christopher J. Bradley, Clifton College, Bristol, UK; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; Pavlos Maragoudakis, Pireas, Greece; D.J. Smeenk, Zaltbommel, the Netherlands; Panos E. Tsoussoglou, Athens, Greece; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Chen and Wang.

For any solution $x$, there is some number $\theta$ such that $\cos \theta = 1/x$ and $\sin \theta = \frac{1}{4 - \sqrt{3}x}$; hence, there is some $\theta$ such that

$$4 = \sqrt{3} \sec \theta + \csc \theta.$$  (1)

Conversely, if $\theta$ satisfies the above equation, then, letting $x = \sec \theta$, we have $4 - \sqrt{3}x = \csc \theta$, and $x$ is a solution of the given equation.

Equation (1) is equivalent to

$$2 \sin \theta \cos \theta = \frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \cos \theta;$$

that is,

$$\sin(2\theta) = \sin \left( \theta + \frac{\pi}{6} \right).$$

This equation is satisfied either when $2\theta = \theta + \frac{\pi}{6} + 2k\pi$ or when $2\theta = (2k + 1)\pi - (\theta + \frac{\pi}{6})$, where $k \in \mathbb{Z}$. Thus, the solutions of (1) are given by

$$\theta = \frac{\pi}{6} + 2k\pi$$  (2)

or

$$\theta = \frac{5\pi}{18} + 2k\pi$$  (3)

Due to the periodicity of the secant function, all values of $\theta$ given by (2) yield $x = \sec \left( \frac{\theta}{4} \right) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Three more distinct values of $x$ are obtained from the values of $\theta$ given by (3). These are $x = \sec \left( \frac{5\pi}{18} \right)$, $x = \sec \left( \frac{17\pi}{18} \right)$, and $x = \sec \left( \frac{20\pi}{18} \right)$.

7. It is known that all members of the infinite sequence $a - b$, $a^2 - b^2$, $a^3 - b^3$, . . . , are natural numbers. Prove that $a$ and $b$ are integers.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Pavlos Maragoudakis, Pireas, Greece. We feature the solution by Maragoudakis.

Let $n_1 = a - b$ and $n_2 = a^2 - b^2$, where $n_1, n_2 \in \mathbb{N}^*$. Then

$$a = \frac{1}{2} \left( n_1 + \frac{n_2}{n_1} \right) \in \mathbb{Q} \text{ and } b = a - n_1 \in \mathbb{Q}.$$ Let $a = \frac{k}{\ell}$ and $b = \frac{m}{n}$, where $k, \ell, m, n \in \mathbb{Z}$, with $\ell, n > 0$ and gcd$(k, \ell) = \gcd(m, n) = 1$.

Now, $\frac{k}{\ell} - \frac{m}{n} = n_1$; that is,

$$kn - m\ell = n_1 \ell.$$ 

Since $n \mid kn$ and $n \mid n_1 n\ell$, it follows that $n \mid m\ell$. Since gcd$(m, n) = 1$, we must have $n \mid \ell$. Similarly, $\ell \mid n$. Therefore, $\ell = n$. 

Now \( a = \frac{k}{\ell} \) and \( b = \frac{m}{\ell} \). We must show that \( \ell = 1 \). Suppose instead that \( \ell \neq 1 \). Then there exists a prime \( p > 0 \) such that \( p \mid \ell \). Note that \( p \mid k \), since \( \gcd(k, \ell) = 1 \). For each \( j = 1, 2, \ldots \), we have \( a^j - b^j \in \mathbb{N}^* \), by hypothesis, and hence \( \ell^j \mid (a^j - b^j) \). Then \( p^j \mid (a^j - b^j) \), and therefore \( k^j \equiv m^j \pmod{p} \). Thus, for \( j = 2, 3, 4, \ldots \), \[
\frac{k^j - m^j}{k - m} = k^{j-1} + k^{j-2}m + \cdots + km^{j-2} + m^{j-1} \equiv jk^{j-1} \pmod{p}.
\]

Let \( m_0 \in \mathbb{N}^* \) be such that \( p^{m_0} \) is the largest power of \( p \) that divides \( k - m \), and choose any integer \( n_0 \) such that \( n_0 > m_0 \) and \( p \nmid n_0 \). Then, since \( p \nmid k \), it follows that \( p \nmid n_0k^{m_0-1} \), and thus, \[
p \nmid \frac{k^{m_0} - m_0}{k - m}.
\]

Therefore, the largest power of \( p \) which divides \( k^{m_0} - m_0 \) is \( p^{m_0} \). Since \( n_0 > m_0 \), we see that \( p^{n_0} \nmid (k^{m_0} - m_0) \), a contradiction. Thus, \( \ell = 1 \), and \( a \) and \( b \) are integers.

8. Prove that if \( m \) and \( n \) are natural numbers, such that the number \( 2^{mn} - 1 \) is divisible by the number \( (2^m - 1)(2^n - 1) \), then the number \( 2(3^m - 1) \) is divisible by \( (3^m - 1)(3^n - 1) \).

Solved by Pierre Bornszein, Maisons-Laffitte, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornszein’s solution.

Let \( m \) and \( n \) be natural numbers such that \( 2^{mn} - 1 \) is divisible by \( (2^m - 1)(2^n - 1) \).

Claim. The numbers \( m \) and \( n \) are coprime.

Proof. Let \( d = \gcd(m, n) \). Then \( m = dx \) and \( n = dy \), where \( \gcd(x, y) = 1 \). Since \( d \) divides both \( m \) and \( n \), it follows that \( 2^d - 1 \) divides both \( 2^m - 1 \) and \( 2^n - 1 \); that is, \( 2^m \equiv 2^n \equiv 1 \pmod{2^d - 1} \). Since \( 2^n - 1 \) divides \( \frac{2^{mn} - 1}{2^m - 1} \) (by hypothesis), we deduce that \( 2^d - 1 \) divides \( \frac{2^{mn} - 1}{2^m - 1} \). On the other hand, \[
\frac{2^{mn} - 1}{2^m - 1} = \sum_{k=0}^{n-1} 2^{mk} \equiv \sum_{k=0}^{n-1} 1 \equiv n \pmod{2^d - 1}.
\]

Therefore, \( n \equiv 0 \pmod{2^d - 1} \). In a similar way, we conclude that \( m \equiv 0 \pmod{2^d - 1} \). Thus, \( dy \equiv dx \equiv 0 \pmod{2^d - 1} \).

Let \( p, q \) be positive integers such that \[
dy = p(2^d - 1) \quad \text{and} \quad dx = q(2^d - 1).
\]

Then \( qdy = pdx \), which leads to \( qy = px \). From Gauss' Theorem, we deduce that \( x \) divides \( q \); that is, \( q = ax \) for some positive integer \( a \). Then, from (1), we have \( d = a(2^d - 1) \). It follows that \( 2^d - 1 \) divides \( d \). But an easy induction shows that \( 2^d - 1 > d \) for \( d \geq 2 \). It follows that \( d = 1 \), as claimed. \( \blacksquare \)
It is well known ([1], p. 26) that if \( a, b \in \mathbb{N}^* \), and if \( k \geq 1 \) is an integer, then \( \gcd(k^a - 1, k^b - 1) = k^{\gcd(a,b)} - 1 \). From this and the claim, we deduce that \( \gcd(3^m - 1, 3^n - 1) = 3^1 - 1 = 2 \).

At least one of the numbers \( m \) and \( n \) must be odd, by the claim. Say \( m = 2i + 1 \). Then \( 3^m - 1 = 3 \times 9^i - 1 \equiv 2 \) (mod 4). Thus, \( 3^m - 1 = 2r \), where \( r \) is an odd integer and \( \gcd(r, 3^n - 1) = 1 \). Since each of \( 3^n - 1 \) and \( 3^m - 1 \) is a divisor of \( 3^{mn} - 1 \), we deduce that \( 3^{mn} - 1 \) is divisible by \( r(3^n - 1) \). Then \( 2(3^{mn} - 1) \) is divisible by \( 2r(3^n - 1) = (3^m - 1)(3^n - 1) \).

**Reference.**


**9.** Find all natural numbers \( k \) for which the sequence \( x_n = \frac{S(n)}{S(kn)} \), \((n = 1, 2, \ldots)\) will be bounded. Here \( S(a) \) denotes the sum of the digits of the natural number \( a \).

**Solution by Pierre Bornstein, Maisons-Laffitte, France.**

We will prove that the sequence is bounded if and only if \( k = 2^a 5^a \), for some non-negative integers \( \alpha, \beta \).

We will use the following lemma, which is known to regular readers of *CRUX with MAYHEM* (see, for example, [2001: 179–180]): “If \( n \) is a natural number, then \( S(2n) \leq 2S(n) \leq 18S(2n) \).” From this, we deduce that, for all positive integers \( n \),

\[
\frac{1}{2} \leq \frac{S(n)}{S(2n)} \leq 5. \tag{1}
\]

Replacing \( n \) by \( 5n \) we get

\[
\frac{1}{2} \leq \frac{S(5n)}{S(10n)} \leq 5.
\]

Since \( S(10n) = S(n) \), it follows that

\[
\frac{1}{5} \leq \frac{S(n)}{S(5n)} \leq 2. \tag{2}
\]

Let \( B \) be the set of \( k \in \mathbb{Z}^+ \) such that the sequence \( \left\{ \frac{S(n)}{S(kn)} \right\} \) is bounded.

Note that 1 \( \in B \) (obviously), and also 2 \( \in B \) and 5 \( \in B \), from (1) and (2).

Suppose that \( k = 2^a 5^\alpha \), where \( \alpha \) and \( \alpha \) are positive integers and \( \alpha \) is odd. Let \( n \) be a positive integer. Then

\[
\frac{S(n)}{S(kn)} = \frac{S(n)}{S(an)} \frac{S(2an)}{S(2an)} \frac{S(2^2an)}{S(2^2an)} \cdots \frac{S(2^{\alpha-1}an)}{S(2^{\alpha-1}an)}.
\]

and, using (1),

\[
\frac{1}{2^\alpha} \leq \frac{S(an)}{S(2an)} \frac{S(2an)}{S(2^2an)} \cdots \frac{S(2^{\alpha-1}an)}{S(2^{\alpha}an)} \leq 5^\alpha.
\]


Thus, 
\[ \frac{1}{2^{\alpha}} \frac{S(n)}{S(an)} \leq \frac{S(n)}{S(kn)} \leq 5^{\alpha} \frac{S(n)}{S(an)}. \]

Since \( \alpha \) does not depend on \( n \), it follows that the sequence \( \left\{ \frac{S(n)}{S(kn)} \right\} \) is bounded if and only if the sequence \( \left\{ \frac{S(n)}{S(an)} \right\} \) is bounded. That is,
\[ 2^{\alpha}a \in B \quad \text{if and only if} \quad a \in B. \quad (3) \]

Since \( 1 \in B \), it follows that \( 2^{\alpha} \in B \) for any non-negative integer \( \alpha \).

Next, suppose that \( k = 5^{\beta}b \), where \( \beta \) and \( b \) are positive integers, and \( b \not\equiv 0 \) (mod 5). Using (2), we can prove in a similar way that
\[ 5^{\beta}b \in B \quad \text{if and only if} \quad b \in B. \quad (4) \]

Since \( 2^{\alpha} \in B \), it follows that \( 2^{\alpha}5^{\beta} \in B \), where \( \alpha \) and \( \beta \) are any non-negative integers.

Now let \( k \) be an integer with \( k > 1 \) and \( \text{gcd}(k, 10) = 1 \). Then, from the Fermat-Euler Theorem, we have \( 10^{\varphi(k)} \equiv 1 \pmod{k} \), where \( \varphi \) denotes Euler’s totient function. Set \( a = \frac{10^{\varphi(k)} - 1}{k} \). Then, \( a \in \mathbb{N}^* \), and \( a < 10^{\varphi(k)} - 1 \), from which we deduce that \( k + a \) has at most \( \varphi(k) \) decimal digits.

Let \( q \) be any positive integer such that \( 10^{q\varphi(k)} > k - 1 \). We have \( 10^{q\varphi(k)} \equiv 1 \pmod{k} \), and hence, \( 10^{q\varphi(k)} + (k - 1) = k n_q \), for some \( n_q \in \mathbb{N}^* \). Then \( S(kn_q) = 1 + S(k - 1) \). But
\[ n_q = 1 + \frac{10^{q\varphi(k)} - 1}{k} \]
\[ = 1 + \frac{10^{q\varphi(k)} - 1}{k} \left( 1 + 10^{\varphi(k)} + 10^{2\varphi(k)} + \cdots + 10^{(q-1)\varphi(k)} \right) \]
\[ = 1 + a + a10^{\varphi(k)} + a10^{2\varphi(k)} + \cdots + a10^{(q-1)\varphi(k)}. \]

Therefore, \( S(n_q) = S(1 + a) + (q - 1)S(a) \geq (q - 1)S(a) \). It follows that
\[ \frac{S(n_q)}{S(kn_q)} \geq \frac{(q - 1)S(a)}{1 + S(k - 1)}. \]

Then \( \frac{S(n_q)}{S(kn_q)} \rightarrow +\infty \) as \( q \rightarrow +\infty \), and therefore, \( k \not\in B \).

Thus, if \( k \) is an integer with \( k > 1 \) and \( \text{gcd}(k, 10) = 1 \), then \( k \not\in B \).

It follows, using (3) and (4), that \( k \in B \) if and only if \( k = 2^{\alpha}5^{\beta} \).

That completes the Olympiad Corner for this issue. Send me your contests as well as your nice solutions and generalizations.