The Power of Symmetry: An Example Linking Algebra and Geometry

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The formula for the area of a triangle is first introduced in the middle grades. An approach that is commonly used is to decompose the triangle in order to construct a parallelogram (Figure 1a), which is then decomposed to construct a rectangle (Figure 1b). The area of the rectangle can be calculated using the formula $A = lw$, which has been previously developed, and the area of the triangle is then determined to be $A = \frac{1}{2}bh$.

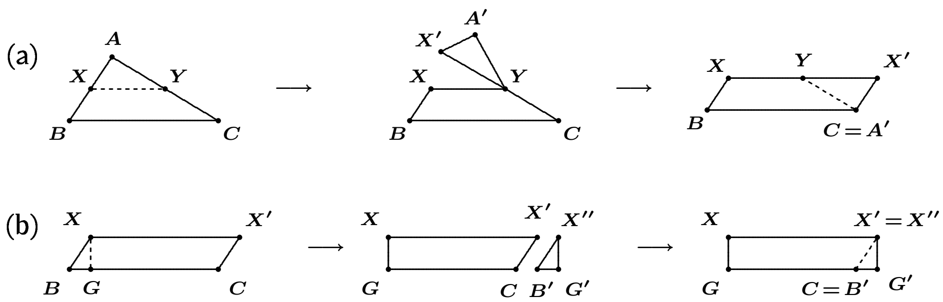

Figure 1

One difficulty students often have in applying the formula $A = \frac{1}{2}bh$ is that they believe the so-called *base* must be oriented horizontally. If the triangle is rotated so that none of the sides is horizontal, some students become confused and unable to apply the formula successfully. The underlying problem often appears to be that the student does not recognize that *any* side can serve as a base. The student fails to appreciate the symmetry.

In this article, exploration of the area of a triangle using only basic algebraic tools leads to Heron's formula and another formula similar to Heron's but not as well known. In the process, it becomes clear that symmetry plays a fundamental role in both the geometry and the algebraic development.

The triangle area formula has three symmetric forms (see Figure 2):

$$A = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c . \qquad (1)$$

From (1), we get $2A = ah_a = bh_b = ch_c$, which, when squared, gives

$$4A^2 = a^2h_a^2 = b^2h_b^2 = c^2h_c^2 . \qquad (2)$$

We still have three distinct symmetric equations that apply to Figure 2. However, if we add the three equations together, we obtain the single combined equation

$$12A^2 \;=\; a^2h_a^2 + b^2h_b^2 + c^2h_c^2\,. \tag{3}$$

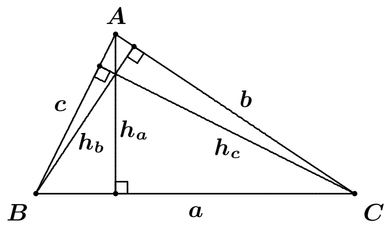

Figure 2

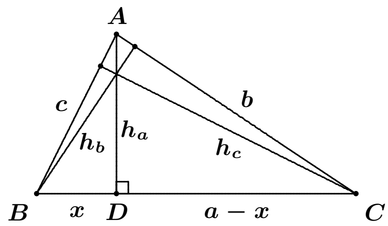

Figure 3

Let the height relative to side BC intersect BC at D (Figure 3), and let $|BD| = x$. Then $|DC| = a - x$. By the Pythagorean Theorem, we have

$$\begin{aligned} h_a^2 + x^2 &= c^2\,, \\ h_a^2 + (a-x)^2 &= b^2\,. \end{aligned} \tag{4}$$

Adding these two equations, expanding, and rearranging the terms yields

$$2h_a^2 \;=\; b^2 + c^2 - a^2 + 2x(a-x)\,. \tag{5}$$

By subtracting the equations in (4) instead of adding them, we obtain, after simplification,

$$x \;=\; \frac{c^2 + a^2 - b^2}{2a}\,. \tag{6}$$

Substituting (6) into (5), we get

$$2h_a^2 \;=\; b^2 + c^2 - a^2 + 2\left(\frac{a^2 - b^2 + c^2}{2a}\right)\left(\frac{a^2 + b^2 - c^2}{2a}\right)\,.$$

This relationship can be simplified as follows:

$$\begin{aligned} 4h_a^2 &= 2b^2 + 2c^2 - 2a^2 + \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{a^2}\,, \\ 4a^2h_a^2 &= 2a^2b^2 + 2a^2c^2 - 2a^4 + \left(a^2 - [b^2 - c^2]\right)\left(a^2 + [b^2 - c^2]\right) \\ &= 2a^2b^2 + 2a^2c^2 - 2a^4 + a^4 - [b^2 - c^2]^2 \\ &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\,. \end{aligned}$$

As there are three symmetric expressions for the area of a triangle in terms of base and height, there are likewise three symmetric expressions for the heights h_a, h_b, h_c in terms of the lengths of the sides:

$$\begin{aligned} 4a^2h_a^2 &= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\,, \\ 4b^2h_b^2 &= 2b^2c^2 + 2a^2b^2 + 2a^2c^2 - b^4 - c^4 - a^4\,, \\ 4c^2h_c^2 &= 2a^2c^2 + 2b^2c^2 + 2a^2b^2 - c^4 - a^4 - b^4\,. \end{aligned} \tag{7}$$

Adding the above equations gives us

$$4(a^2h_a^2+b^2h_b^2+c^2h_c^2) = 3\left[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\right] \,. \quad (8)$$

Using (3) and (8), we get

$$16A^2 \;=\; 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\,. \qquad (9)$$

Equation (9) gives us the area A of a triangle in terms of the lengths of the sides. But what about the expression on the right side of (9)? Can it be simplified using only the basic tools of intermediate algebra?

Let us start by considering what happens when a general trinomial is squared: $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$. The expression on the right side of (9) looks like

$$(a^2+b^2+c^2)^2 \;=\; a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2\,,$$

except that the signs of some terms are negative. In attempting to factor the expression on the right side of (9), it might be worthwhile to try factors of the form $(\pm a \pm b \pm c)$. Due to the symmetry in the problem, it would be reasonable to expect the presence of the three factors $(-a+b+c)$, $(a-b+c)$, and $(a+b-c)$. Since the fourth powers in (9) are negative, the fourth factor would have to be $a+b+c$. Multiplying these four factors together confirms that

$$\begin{aligned} &2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\\ &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c)\,. \end{aligned} \qquad (10)$$

The same result can be obtained in another way. Although the expression on the right side of (9) resists factoring as a perfect square, we can factor it as a difference of squares:

$$\begin{aligned} &2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\\ &= 4a^2b^2+2b^2c^2+2c^2a^2-2a^2b^2-a^4-b^4-c^4\\ &= 4a^2b^2-[c^4-2b^2c^2-2c^2a^2+a^4+2a^2b^2+b^4]\\ &= 4a^2b^2-[c^4-2(a^2+b^2)c^2+(a^2+b^2)^2]\\ &= (2ab)^2-[c^2-(a^2+b^2)]^2\\ &= [2ab-c^2+a^2+b^2][2ab+c^2-a^2-b^2]\\ &= [(a+b)^2-c^2][c^2-(a-b)^2]\\ &= (a+b+c)(a+b-c)(c+a-b)(c-a+b)\,, \end{aligned}$$

which is the same result as (10).

Substituting (10) into (9) yields

$$\begin{aligned} 16A^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c)\,,\\ A^2 &= \left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\,. \end{aligned}$$

We now define the semi-perimeter, s, as $s = \dfrac{a+b+c}{2}$. Then

$$A^2 \;=\; s(s-a)(s-b)(s-c)\,;$$

that is,

$$A \;=\; \sqrt{s(s-a)(s-b)(s-c)}\,.$$

Thus, Heron's formula is established.

Not only does $\triangle ABC$ have three sides, a, b, and c, and three heights, h_a, h_b, and h_c, but it also has three medians, m_a, m_b, and m_c (Figure 4). In the development of Heron's formula, the semi-perimeter, s, was introduced. In a parallel fashion, we define the semi-median sum $s_m = \dfrac{m_a+m_b+m_c}{2}$.

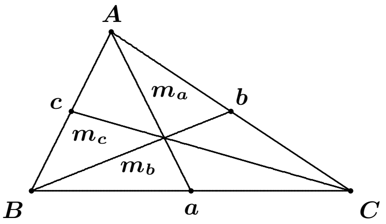

Figure 4

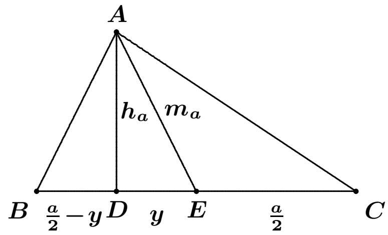

Figure 5

Let the median m_a from A to side BC intersect BC at E, and let $|DE| = y$. Assume that D is closer to B than to C (as in Figure 5). Then $|BD| = (a/2) - y$ and $|EC| = a/2$. In $\triangle ADE$, we have $m_a^2 = h_a^2 + y^2$. Hence, $h_a^2 = m_a^2 - y^2$.

From equation (5), we obtain

$$\begin{aligned} 2h_a^2 &= b^2+c^2-a^2+2|BD|\cdot|DC| \\ &= b^2+c^2-a^2+2\left(\frac{a}{2}-y\right)\left(\frac{a}{2}+y\right) \\ &= b^2+c^2-\frac{a^2}{2}-2y^2\,. \end{aligned} \tag{11}$$

Substituting $h_a^2 = m_a^2 - y^2$ into (11) yields

$$\begin{aligned} 2m_a^2-2y^2 &= b^2+c^2-\frac{a^2}{2}-2y^2\,, \\ 4m_a^2 &= 2(b^2+c^2)-a^2\,. \end{aligned}$$

Again, by symmetry, there are three equations:

$$\begin{aligned} 4m_a^2 &= 2(b^2+c^2)-a^2\,, \\ 4m_b^2 &= 2(c^2+a^2)-b^2\,, \\ 4m_c^2 &= 2(a^2+b^2)-c^2\,. \end{aligned} \tag{12}$$

Adding these three equations produces

$$4(m_a^2 + m_b^2 + m_c^2) \;=\; 3(a^2 + b^2 + c^2)\,. \qquad (13)$$

Adding together the *squares* of the three equations in (12) produces

$$16(m_a^4 + m_b^4 + m_c^4) \;=\; 9(a^4 + b^4 + c^4)\,. \qquad (14)$$

Note that the square of (13) contains all the terms in (14), plus some additional terms. By squaring (13) and subtracting (14) from the result, we get

$$16\left[2(m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2)\right] \;=\; 9[2(a^2b^2 + b^2c^2 + c^2a^2)]\,. \qquad (15)$$

We would like to express the area A strictly in terms of the medians (and the semi-median sum). From (9), we know that

$$A^2 \;=\; \frac{1}{16}\left[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)\right]\,.$$

In this equation, we substitute for $2(a^2b^2 + b^2c^2 + c^2a^2)$ from (15) and for $a^4 + b^4 + c^4$ from (14):

$$\begin{aligned} A^2 &= \frac{1}{16}\cdot\frac{16}{9}\left[2(m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2) - (m_a^4 + m_b^4 + m_c^4)\right] \\ &= \frac{1}{9}\left[2(m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2) - (m_a^4 + m_b^4 + m_c^4)\right]\,. \qquad (16) \end{aligned}$$

The *form* of the expression in the square brackets in (16) should look familiar. It is just the left side of (10) with a, b, and c replaced by m_a, m_b, and m_c. Hence, the same algebraic development as before leads to

$$A \;=\; \frac{4}{3}\sqrt{s_m(s_m - m_a)(s_m - m_b)(s_m - m_c)}\,. \qquad (17)$$

Thus, the area of a triangle can be expressed in terms of the lengths of its medians.

By considering the area of a triangle as *base* $\times$ *height*, we tend to mask the inherent symmetry. On the other hand, by considering the area of a triangle as *length of side* $\times$ *corresponding height*, we open up the richness of the symmetry in the geometric situation. This leads to the remarkable algebraic symmetry in the Sine and Cosine Laws, Heron's formula, and (17).

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