Pólya's Paragon

They All Fall Down

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Often in mathematics we are faced with problems that are expressed in such a way that we are actually asking infinitely many problems at once. Fermat's Last Theorem, for example, asks us to prove that there are no integer solutions x, y, z to the equation $x^n + y^n = z^n$ for any integer $n > 2$. This is asking us to prove that there are no solutions to $x^3 + y^3 = z^3$ AND no solutions to $x^4 + y^4 = z^4$ AND, etc. It would be nice if there were some way we could relate all these different problems and solve them all in one fell swoop. One way of doing this is to use what is called *Mathematical Induction*. While this does not actually help us prove Fermat's Last Theorem, it will allow us to solve some similarly worded problems.

When we need to prove that a particular statement is true for infinitely many successive natural numbers n, we proceed in following way:

1. Prove that the result is true for some particular natural number n—typically an easy case where n is 0 or 1 (the *base case*).
2. Prove that, if the statement is true for some natural number k, then it must also be true for the value $k + 1$ (the *inductive step*).
3. Conclude that the statement is true for all n greater than or equal to the value used in step 1.

To get an intuitive notion as to why we are allowed to make the conclusion in step 3, we examine what the first two steps tell us. First, we show that our statement is true for $n = 1$. Step 2 then tells us that, since it is true for $n = 1$, it is also true for $n = 2$. But then, by the same logic, since it is true for $n = 2$, it will also be true for $n = 3$. Clearly, this will continue for ever, and we will never find a value of n where the statement will not be true.

As an analogy, think of a string of dominoes standing on end next to each other. Step 1 says 'we cause the first domino to fall over'. Step 2 says 'if a domino falls over, the next in line will also fall over'. Knowing only these two facts we know for certain that every domino must fall over. This is the process of Mathematical Induction.

When this is taught as part of the high-school or early university curriculum, many of the problems involve finding the sum of some sequence of n terms. While these problems are adequate, in my experience, students who are only exposed to such problems learn only the specific pattern of how to solve that type of problem rather than really understanding the principle of Mathematical Induction. I will now show two problems that can be solved using Mathematical Induction that are a little different than what you would see in most textbooks.

Problem 1. An ancient puzzle called the *Tower of Hanoi* consists of three pegs on a stand and n punctured discs of different sizes that are placed in decreasing order of size on one of the pegs. The object of the puzzle is to transfer the pile of discs to another peg, by moving one disc at a time, and without placing any disc on top of a smaller disc. Show that it is possible to solve this puzzle in $2^n - 1$ moves.

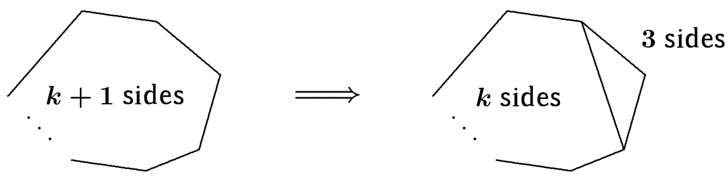
Solution. For a base case, we simply examine the game which begins with only one disc. Clearly, this will only take one move to solve since we can move the disc to the appropriate peg directly. Since $2^1 - 1 = 1$, the statement is true for $n = 1$.

Now we assume that it is possible to solve the puzzle starting with k discs in $2^k - 1$ moves. To solve the puzzle with $k + 1$ discs we first move the first k discs to another peg (taking $2^k - 1$ moves); then we move the last disc to the empty peg (taking 1 move); and finally, we replace the first k discs to rebuild the tower (taking another $2^k - 1$ moves). We have now solved the puzzle and used $(2^k - 1) + (1) + (2^k - 1) = 2 \cdot 2^k - 2 + 1 = 2^{k+1} - 1$ moves. Therefore, we can solve the puzzle with $k + 1$ discs in $2^{k+1} - 1$ moves. By Mathematical Induction, we can conclude that the statement is true for all n.

Problem 2. Prove that the sum of the interior angles of a convex n–gon is exactly $(n - 2) \times 180°$. (I should note that this statement holds for all n–gons, but the proof is much clearer with convex n–gons.)

Solution. Our base case for this problem is $n = 3$, since it is impossible to create a polygon with fewer than 3 sides. For $n = 3$, we have a triangle which has an interior angle sum of $180°$. (Exercise: can you prove this rigorously?) We can also check that $(3 - 2) \times 180° = 180°$. Thus, the statement is true for $n = 3$.

Now assume that every convex k–gon has an interior angle sum of $(k - 2) \times 180°$. We now consider a convex $(k + 1)$–gon. If we draw a line from one vertex to another to form a triangle, we notice that this line divides the $(k + 1)$–gon into a k–gon (which has an interior angle sum of $(k - 2) \times 180°$) and a triangle (which has an interior angle sum of $180°$).



Now we see that the interior angle sum of the $(k+1)$–gon is exactly $(k-2)\times 180° + 180° = ((k+1)-2)\times 180°$, which is what we needed in order to show that the statement is true. Therefore, by Mathematical Induction, we can conclude that every convex n–gon has an interior angle sum of exactly $(n-2)\times 180°$.

Each of the following problems can be proven with or without Mathematical Induction. Try to find as many solutions as possible to each of them.

PROBLEMS:

1. Prove that $2^{3n} - 1$ is divisible by 7 for all natural numbers n.
2. Prove that the sum of the terms in the n^{th} row of Pascal's Triangle is exactly 2^n.
3. Prove that the product of n consecutive integers is divisible by $n!$.