SOLUCIONES

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


Prove that \[ \sum_{1 \leq i \leq j \leq n} \sin^2 \left( \frac{(j-i)\pi}{n} \right) = \frac{n^2}{4}. \]


Suppose \( n \geq 2 \), as the claim does not hold for \( n = 1 \). It is well known that

\[
\sum_{k=1}^{n} \left( \cos \left( \frac{2k\pi}{n} \right) + i \sin \left( \frac{2k\pi}{n} \right) \right) = \sum_{i=1}^{n} e^{2\pi i/n} = 0.
\]

Hence, we have

\[
\sum_{1 \leq i \leq j \leq n} \cos \left( \frac{2(j-i)\pi}{n} \right)
= \sum_{1 \leq i \leq j \leq n} \left( \cos \left( \frac{2j\pi}{n} \right) \cos \left( \frac{2i\pi}{n} \right) + \sin \left( \frac{2j\pi}{n} \right) \sin \left( \frac{2i\pi}{n} \right) \right)
= \left( \sum_{k=1}^{n} \cos \left( \frac{2k\pi}{n} \right) \right)^2 + \left( \sum_{k=1}^{n} \sin \left( \frac{2k\pi}{n} \right) \right)^2
= \left( \sum_{k=1}^{n} \left( \cos \left( \frac{2k\pi}{n} \right) + i \sin \left( \frac{2k\pi}{n} \right) \right) \right)^2 = 0.
\]

Then

\[
\sum_{1 \leq i \leq j \leq n} \sin^2 \left( \frac{(j-i)\pi}{n} \right) = \frac{1}{4} \sum_{1 \leq i \leq j \leq n} \left( 1 - \cos \left( \frac{2(j-i)\pi}{n} \right) \right)
= \frac{n^2}{4} - \frac{1}{4} \sum_{1 \leq i \leq j \leq n} \cos \left( \frac{2(j-i)\pi}{n} \right) = \frac{n^2}{4}.
\]

II. Solution by Peter Y. Woo. Biola University, La Mirada, CA. USA
(modified slightly by the editor).

The claim is clearly false for \( n = 1 \). We assume that \( n \geq 2 \). Let \( S \) denote the sum to be evaluated, and let \( \theta = \frac{\pi}{n} \). Then

\[
S = \sum_{k=1}^{n-1} (n-k) \sin^2(k\theta), \tag{1}
\]
Let $m = n - k$. Then $k\theta = n\theta - m\theta = \pi - m\theta$. Hence,

$$S = \sum_{m=1}^{n-1} m \sin^2(\pi - m\theta) = \sum_{m=1}^{n-1} m \sin^2(m\theta). \tag{2}$$

Adding (1) and (2), and noting that $\sin(n\theta) = 0$, we get

$$2S = \sum_{k=1}^{n-1} n \sin^2(k\theta) = n \sum_{k=1}^{n} \sin^2(k\theta).$$

Since $n \geq 2$ implies $\sin \theta \neq 0$, we have

$$S = \frac{n}{4} \sum_{k=1}^{n} (1 - \cos(2k\theta)) = \frac{n^2}{4} - \frac{n}{8 \sin \theta} \sum_{k=1}^{n} [\sin(2k+1)\theta - \sin(2k-1)\theta]$$

$$= \frac{n^2}{4} - \frac{n}{8 \sin \theta}[\sin(2n+1)\theta - \sin \theta]$$

$$= \frac{n^2}{4} - \frac{n}{8 \sin \theta} [\sin(2\pi + \theta) - \sin \theta] = \frac{n^2}{4}.$$  

Also solved by JEAN-CLAUDE ANDRIEUX, Beaune, France; MICHEL BATALLÉ, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, Concordia University, Montreal, QC; JOSE LUIS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinenkurs, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; NEVEN JURIC, Zagreb, Croatia; DAVID Loeffler, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; STAN WAGON, Macalester College, St. Paul, MN, USA; LIZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Most of the submitted solutions used complex exponential functions.

Wagon commented that "there are well-known algorithms for determining sums such as these (Gosper's algorithm is the most prominent), and they are built into Mathematica, Maple, etc." He stated that the answer $n^2/4$ came out immediately when he typed the given summation into Mathematica. He further commented that "the generation of such symbolic sums is completely algorithmic, analogous to getting 100 digits of $\sqrt{2}$".

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Suppose that $f : [0,1] \to (0,\infty)$ is a continuous function. Prove that if there exists $\alpha > 0$ such that, for $n \in \mathbb{N}$,

$$\int_0^1 x^\alpha (f(x))^n \, dx \geq \frac{1}{(n+1)\alpha + 1} \geq \int_0^1 (f(x))^{n+1} \, dx,$$

then $\alpha$ is unique.
Solution by Li Zhou. Polk Community College, Winter Haven, FL, USA. Assume that $0 < \alpha < \beta$ and that both $\alpha$ and $\beta$ satisfy the given inequalities for every $n \in \mathbb{N}$. Choose $N \in \mathbb{N}$ sufficiently large so that $\alpha < \left( \frac{N + 1}{N + 2} \right) \beta$. Then

$$\frac{1}{(N + 1)\beta + 1} \leq \int_0^1 (f(x))^{N+1} \, dx \leq \int_0^1 x^\alpha (f(x))^{N+1} \, dx \leq \frac{1}{(N + 2)\alpha + 1},$$

a contradiction.

Also solved by MICHEL BATAILLE, Rouen, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer.


Let $A_k \in M_n(\mathbb{R})$ ($k = 1, 2, \ldots, m \geq 2$) for which

$$\sum_{1 \leq i < j \leq m} (A_iA_j + A_jA_i) = 0_n.$$

Prove that

$$\det \left( \sum_{k=1}^m (I_n + A_k)^2 - (m - 2)I_n \right) \geq 0.$$

Solution by Michel Bataille, Rouen, France.

Let $B$ be the matrix $I_n + A_1 + A_2 + \cdots + A_m$. Using the given condition, we get

$$\left( \sum_{k=1}^m A_k \right)^2 = \sum_{k=1}^m A_k^2,$$

and hence,

$$B^2 = \left( I_n + \sum_{k=1}^m A_k \right)^2 = I_n + 2 \sum_{k=1}^m A_k + \sum_{k=1}^m A_k^2.$$

Let

$$C = \sum_{k=1}^m (I_n + A_k)^2 - (m - 2)I_n.$$

Then

$$C = 2I_n + 2 \sum_{k=1}^m A_k + \sum_{k=1}^m A_k^2 = I_n + B^2.$$
Since the entries of $B$ are real numbers, we obtain
\[
\det(C) = \det(I_n + B^2) = \det((I_n + iB)(I_n - iB)) \\
= \det((I_n + iB)(I_n + iB)) = \det(I_n + iB) \det(I_n + iB) \\
= \det(I_n + iB) \det(I_n + iB) = |\det(I_n + iB)|^2 \geq 0,
\]
as desired.

Also solved by CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark, OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALEO H. GUERENZVAIG, Universidad CAECE, Buenos Aires, Argentina; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2793. [2002·534] Proposed by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.

Show that the area of the image of the portion of the unit disc which lies in the first quadrant under the mapping $\zeta = \cosh^{-1}(z)$ is a well-known constant.

Solution by Michel Bataille, Rouen, France.

The required area is Catalan’s constant, $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$.

Let $z = x + iy$ and $\zeta = \xi + i\eta$, where $x$, $y$, $\xi$ and $\eta$ denote real numbers. Let $\Delta$ be the portion of the unit disk lying in the first quadrant. The required area is
\[
A = \iint_{\cosh^{-1}(\Delta)} d\xi d\eta = \iint_{\Delta} \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| dx dy = \iint_{\Delta} \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right|^{-1} dx dy,
\]
where the change of variables is defined by
\[
z = x + iy = \cosh \zeta = \cosh \xi \cos \eta + i \sinh \xi \sin \eta.
\]
We readily obtain that
\[
\frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{vmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{vmatrix} = \sinh^2 \xi \cos^2 \eta + \cosh^2 \xi \sin^2 \eta \\
= \sinh^2 \xi + \sin^2 \eta = |\sinh^2 \xi| = |z^2 - 1|.
\]

Using polar coordinates, we let $z = \rho \cos \theta + i \rho \sin \theta$, for $\theta \in (0, \pi/2)$. Then
\[
|z^2 - 1|^2 = (x^2 - y^2 - 1)^2 + 4x^2 y^2 \\
= (\rho^2 \cos(2\theta) - 1)^2 + \rho^4 \sin^2(2\theta) \\
= \sin^2(2\theta) \left( 1 + \left( \frac{\rho^2 - \cos(2\theta)}{\sin(2\theta)} \right)^2 \right).
\]
It follows that

\[
A = \int_0^{\pi/2} \int_0^1 \frac{\rho}{\sin(2\theta)} \sqrt{1 + \left(\frac{\rho^2 - \cos(2\theta)}{\sin(2\theta)}\right)^2} \, d\rho \, d\theta.
\]

\[
= \frac{1}{2} \int_0^{\pi/2} \int_0^{\tan\theta} \frac{du}{\sqrt{1 + u^2}} \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\pi/2} \left( \ln \left( \tan \theta + \sqrt{1 + \tan^2 \theta} \right) - \ln \left( -\cot(2\theta) + \sqrt{1 + \cot^2(2\theta)} \right) \right) \, d\theta
\]

\[
= \frac{1}{2} \left( \int_0^{\pi/2} \ln(1 + \sin \theta) \, d\theta - \int_0^{\pi/2} \ln(\sin \theta) \, d\theta \right).
\]

It is well known that \( \int_0^{\pi/2} \ln(\sin \theta) \, d\theta = -\frac{\pi}{2} \ln 2 \). Furthermore, we have

\[
\int_0^{\pi/2} \ln(1 + \sin \theta) \, d\theta = \int_0^{\pi/2} \ln(1 + \cos t) \, dt
\]

\[
= \int_0^{\pi/2} \ln \left( 2 \cos^2 \left( \frac{t}{2} \right) \right) \, dt = \frac{\pi}{2} \ln 2 + 2I,
\]

where \( I = \int_0^{\pi/2} \ln \left( \cos \left( \frac{t}{2} \right) \right) \, dt \). In order to compute \( I \), we introduce the complementary integral \( J = \int_0^{\pi/2} \ln \left( \sin \left( \frac{t}{2} \right) \right) \, dt \). Note that

\[
J + I = \int_0^{\pi/2} \ln \left( \frac{1}{2} \sin t \right) \, dt = -\pi \ln 2,
\]

and

\[
J - I = \int_0^{\pi/2} \ln \left( \tan \left( \frac{t}{2} \right) \right) \, dt = -\int_0^{\pi/2} \frac{t}{\sin t} \, dt
\]

\[
= -2 \int_0^1 \frac{\arctan x}{x} \, dx = -2G,
\]

where we have used integration by parts, followed by the substitution \( t = 2 \arctan(x) \), and then the expansion \( \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \), which is uniformly convergent for \( x \in [0, 1] \). It follows that \( 2I = -\pi \ln 2 + 2G \).
Substituting these results into the above expression for $A$, we obtain

$$A = \frac{1}{2} \left( \frac{\pi}{2} \ln 2 - \pi \ln 2 + 2G + \frac{\pi}{2} \ln 2 \right) = G.$$ 

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA, and the proposer. Two incomplete solutions were received, where the solvers did not identify the name of the constant.

Specht and Zhou made reference to WEB sites with this and other "near" constants. Loeffler's authority was MATHEMATICA.

2794. Proposed by Mihály Benczé, Brasov, Romania.
Suppose that $z_k \in \mathbb{C}$ ($k = 1, 2, \ldots, n$) such that

$$|z_1 + z_2 + \cdots + z_n| + |z_2 + z_3 + \cdots + z_n| + \cdots + |z_{n-1} + z_n| + |z_n| = |z_1 + 2z_2 + \cdots + nz_n|.$$

Prove that the $z_k$ are collinear.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.
By the well-known generalization of the Triangle Inequality, we have, for any complex numbers $w_1, \ldots, w_n$,

$$\sum_{k=1}^{n} |w_k| \geq \left| \sum_{k=1}^{n} w_k \right|.$$

Equality holds if and only if the ratio $w_i/w_j$ is real and positive, for any two non-zero terms $w_i$ and $w_j$. Assuming $w_n \neq 0$, equality holds if and only if $w_k = c_k w_n$ for $k = 1, 2, \ldots, n-1$, where each $c_k$ is real and non-negative. [Ed: Zhou called the inequality above Minkowski's Inequality, and gave the reference [1] below.]

Let $w_k = \sum_{j=k}^{n} z_j$ for $1 \leq k \leq n$. Then the given equation becomes

$$\sum_{k=1}^{n} |w_k| = \left| \sum_{k=1}^{n} w_k \right|.$$

Since $w_n \neq 0$, we have $w_k = c_k w_n$ for $k = 1, 2, \ldots, n - 1$, and therefore, $z_k = w_k - w_{k+1} = (c_k - c_{k+1})z_n$. That is, the $z_k$ are collinear.

Reference
A convex polygon with sides \(a_1, a_2, \ldots, a_n\) is inscribed in a circle of radius \(R\). Prove that

\[
\sum_{k=1}^{n} \sqrt{4R^2 - a_k^2} \leq 2nR \sin \left( \frac{(n-2)\pi}{n} \right).
\]

[Editor's comments: The denominator on the right side of the inequality should have been \(2n\) instead of \(n\). All solvers made this correction. There is another error, a missing hypothesis, as explained in the featured solution below. Failure to note the need for this hypothesis did not, in itself, disqualify a solution.]

Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.

The stated result is not true. Just consider a case where all sides have lengths close to zero. The problem should have stated that the centre of the circle is not in the exterior of the polygon.

For \(k = 1, 2, \ldots, n\), let \(\theta_k\) be half the angle at the centre of the circle subtended by \(a_k\). Then \(a_k = 2R \sin \theta_k\) and \(\sqrt{4R^2 - a_k^2} = 2R \cos \theta_k\).

Since \(\cos \theta\) is concave in the interval \((0, \pi/2)\), we have

\[
\sum_{k=1}^{n} \cos \theta_k \leq n \cos \left( \frac{1}{n} \sum_{k=1}^{n} \theta_k \right) = n \cos \left( \frac{\pi}{n} \right) = n \sin \left( \frac{(n-2)\pi}{2n} \right).
\]

Then

\[
\sum_{k=1}^{n} \sqrt{4R^2 - a_k^2} = 2R \sum_{k=1}^{n} \cos \theta_k \leq 2nR \sin \left( \frac{(n-2)\pi}{2n} \right).
\]

Equality holds for regular polygons.

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; OVIDIU FURDU, student, Western Michigan University, Kalamazoo, MI, USA; NATALIU H. GUERENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinenymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.
If \( n \leq 6 \), then \( 2 \sin(\pi/n) \geq 2 \sin(\pi/6) = 1 \) and
\[
2nR \sin \left( \frac{(n-2)\pi}{n} \right) \geq 2nR \cos \left( \frac{\pi}{n} \right) \geq \sum_{k=1}^{n} \sqrt{4R^2 - a_k^2}.
\]
The inequality as given originally is false for a regular \( n \)-gon when \( n > 6 \).

Let \( \{p_n\} \) be the sequence of prime numbers. Prove that, for each \( n \geq 2 \), the set \( I = \{1, 2, \ldots, n\} \) can be partitioned into two sets \( A \) and \( B \), where \( A \cup B = I \), in such a way that,
\[
1 \leq \frac{\prod_{i \in A} p_i}{\prod_{j \in B} p_j} \leq 2.
\]

Comment: Both Michel Bataille, Rouen, France, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria, observe that the proposer has published Problem 2796 with a solution in The Mathematical Gazette, 86 (2002), pp. 264-265.

Li Zhou, Polk Community College, Winter Haven, FL, USA, observes that, from the Prime Number Theorem, we can say that for each \( \epsilon > 0 \), there is an \( N \) such that for any \( n \geq 1 \), the set \( \{N + 1, N + 2, \ldots, N + 2n\} \) can be partitioned into \( A, B \) with \( |A| = |B| = n \) such that
\[
1 < \frac{\prod_{i \in A} p_i}{\prod_{j \in B} p_j} < 1 + \epsilon.
\]

Also solved by PIERRE BORNSZTEIN, Pontoise, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and DANIEL MARCOTTE, student, Bishop’s University, Lennoxville, QC.

In \( \triangle ABC \), suppose that \( AD \) is an altitude. Suppose that perpendiculars from \( D \) meet the sides \( AB \) and \( AC \) at \( E \) and \( F \), respectively. Suppose that \( G \) and \( H \) are points of \( AB \) and \( AC \), respectively, such that \( DG \parallel AC \) and \( DH \parallel AB \). Prove that

(a) \( EF \) and \( GH \) intersect at \( A^* \) on \( BC \).

Defining \( B^* \) and \( C^* \) similarly, prove that

(b) \( A^*, B^* \) and \( C^* \) are collinear.
Solution by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

(a) Since triangle $ADB$ has a right angle at $D$ and $DE \perp AB$, we have \( \frac{AE}{EB} = \frac{AD^2}{BD^2} \). Similarly, $DF \perp AC$ in $\triangle ADC$, and hence, \( \frac{CF}{FA} = \frac{CD^2}{AD^2} \). If we define $A^*$ to be the point where $BC$ meets $EF$, then Menelaus' Theorem applied to triangle $ABC$ and the line $EFA^*$ yields

\[
1 = \frac{BA^*}{CA^*} \cdot \frac{CF}{FA} \cdot \frac{AE}{EB} = \frac{BA^*}{CA^*} \cdot \frac{CD^2}{AD^2} \cdot \frac{AD^2}{BD^2}.
\]

Then

\[
\frac{BA^*}{CA^*} = \frac{BD^2}{CD^2}.
\] (1)

We prove that $A^*$, $H$, and $G$ are collinear by the converse of Menelaus' Theorem applied to triangle $ABC$. Since $DG \parallel AC$, we have \( \frac{AG}{GB} = \frac{CD}{DB} \); since $DH \parallel AB$, we have \( \frac{CH}{HA} = \frac{CD}{DB} \). Thus, the product

\[
\frac{BA^*}{CA^*} \cdot \frac{CH}{HA} \cdot \frac{AG}{GB} = \frac{BA^*}{CA^*} \cdot \frac{CD}{DB} \cdot \frac{CD}{DB} = \frac{BA^*}{CA^*} \cdot \frac{CD^2}{BD^2} = 1,
\]

according to (1). This implies that $A^*$, $H$, and $G$ are collinear, as desired.

(b) In triangle $ABC$, let $P$ be the foot of the altitude from $B$, and let $R$ be the foot of the altitude from $C$. By arguments used to obtain (1), we see that

\[
\frac{AB^*}{CB^*} = \frac{AP^2}{CP^2} \quad \text{and} \quad \frac{AC^*}{BC^*} = \frac{AR^2}{BR^2}.
\] (2)

Ceva's Theorem applied to $D, P$, and $R$ gives us \( \frac{BD}{DC} \cdot \frac{CP}{PA} \cdot \frac{AR}{RB} = 1 \). Hence, by (1) and (2),

\[
\frac{BA^*}{CA^*} \cdot \frac{CB^*}{B^*A} \cdot \frac{AC^*}{C^*B} = \left( \frac{BD}{DC} \cdot \frac{CP}{PA} \cdot \frac{AR}{RB} \right)^2 = 1.
\]

By the converse to Menelaus' Theorem, we conclude that $A^*$, $B^*$, and $C^*$ are collinear.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SATURNINO CAMPO RUIZ, Salamanca, Spain; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Folk Community College, Winter Haven, FL, USA; and the proposer.
Editor’s generalization:

This problem is a special case of a theorem in projective geometry, in which the altitudes of \( \triangle ABC \) in our given problem are replaced by any three concurrent cevians. Of course, the line at infinity could also be replaced by an arbitrary line \( l \), but it is convenient for us to continue to call two lines parallel if they meet on \( l \).

Let \( P \) be any point not on a side of triangle \( ABC \), and let \( D \) be the point where \( AP \) meets \( BC \). Suppose that the parallel to \( CP \) through \( D \) meets \( AB \) at \( E \) and the parallel to \( BP \) through \( D \) meets \( AC \) at \( F \); while \( G \) and \( H \) are points of \( AB \) and \( AC \), respectively, such that \( DG \parallel AC \) and \( DH \parallel AB \). Then \( EF \) and \( GH \) intersect at a point \( A^* \) on \( BC \). Moreover, with \( B^* \) and \( C^* \) defined similarly, \( A^*, B^*, \) and \( C^* \) are collinear.

This result is an immediate consequence of an elementary theorem concerning projective mappings between lines. I will first give this projective argument, and then outline other proofs for readers who prefer life in a Euclidean world.

Let us fix \( D \) on \( BC \) and allow point \( X \) to move along \( AD \). For each position of \( X \) on \( AD \) we define \( X' \) on \( AB \) to be the point where the parallel to \( CX \) through \( D \) meets \( AB \), and we define \( X'' \) on \( AC \) to be the point where the parallel to \( BX \) through \( D \) meets \( AC \). In this way \( X' \) and \( X'' \) are projectively related. Note that when \( X \) is the point at infinity of \( AD \) (that is, the point where \( AD \) meets \( l \)), \( X' = X'' = A \), so that \( A \) is fixed by this projectivity, \( A \) projectivity relating the points of two distinct lines that fixes their common point must be a perspectivity, which means that the lines \( X'X'' \) go through a common point (which is the centre of the perspectivity); see, for example, H.S.M. Coxeter, Projective Geometry, 2nd ed. (Springer, 1967), p. 35, Theorem 4.22. We apply this theorem to our given triangle by noting that when \( X \) is at \( D \), \( X' = B \) and \( X'' = C \); when \( X \) is at \( P \), \( X' = E \) and \( X'' = F \); when \( X \) is at \( A \), \( X' = G \) and \( X'' = H \). Thus, \( BC, EF, \) and \( GH \) pass through the centre of the perspectivity (the point we call \( A^* \)).

Coordinates provide an easy alternative proof. Since in the projective plane we are free to choose four lines for our coordinate axes, let us place \( A \) at the origin, \( B \) and \( C \) at infinity on the \( x \)- and \( y \)-axes, and let our line \( l \) be \( x + y = 1 \). Let \( D \) be the point at infinity of \( y = m \) (where \( m \) is a fixed non-zero number). If our moving point \( X \) has coordinates \( (p, mp) \), then \( X' = \left( \frac{m(1+m)}{m+1}, 0 \right) \), \( X'' = (0, mp(1+m) - m) \), and \( X'X'' \) will have slope \(-m^2\) (making \( A^* \) the point at infinity of \( y = -m^2x \)). For yet another approach, one can leave \( l \) at infinity and modify the featured solution above with a second application of Ceva’s Theorem. The argument involving the line \( GH \) remains unchanged, while the argument for \( EF \) uses Ceva’s Theorem to deduce that \( EF \) is parallel to the line joining the point where \( CP \) meets \( AB \) to the point where \( BP \) meets \( AC \).

For the final claim concerning the collinearity of \( A^*, B^*, \) and \( C^* \), the proof in the featured solution above goes through in the general setting without change (since the points like \( G \) and \( H \) were defined without the use of perpendicularity).

\[ \text{2798★. [2002 : 535]} \] Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Prove or disprove the inequality \( \sum_{j=1}^{n} \frac{1}{1 - \frac{P}{x_j}} \leq \frac{n}{1 - \left( \frac{1}{n} \right)^n} \), where \( \sum_{j=1}^{n} x_j = 1, x_j \geq 0 \ (j = 1, 2, \ldots, n) \), and \( P = \prod_{j=1}^{n} x_j \).
Solution by Peter Y. Woo. Biola University. La Mirada, CA. USA. adapted by the editor.

The proposed result is true for all \( n \geq 3 \). We will prove a slightly more general theorem.

**Theorem.** Let \( n \geq 3 \), \( h > 0 \) and \( k \geq h^{n-1} \). Let

\[
W(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \frac{1}{k - \frac{P}{x_j}},
\]

where \( \sum_{j=1}^{n} x_j = h \), \( x_j \geq 0 \) (\( j = 1, 2, \ldots, n \)), and \( P = \prod_{j=1}^{n} x_j \). Then

\[
W(x_1, x_2, \ldots, x_n) \leq \frac{n}{k - (\frac{h}{n})^{n-1}}.
\]

Equality holds if and only if \( (x_1, x_2, \ldots, x_n) = \left( \frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n} \right) \).

**Proof:** The set of admissible values for \( (x_1, x_2, \ldots, x_n) \) is a compact set in \( \mathbb{R}^n \), and \( W \) is continuous on this set. Therefore, \( W \) attains a maximum value at some point in the set. We will prove that the maximum cannot occur at any point other than \( \left( \frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n} \right) \). Since

\[
W \left( \frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n} \right) = \frac{n}{k - (\frac{h}{n})^{n-1}},
\]

we will then have the desired result.

We proceed by induction on \( n \). The case \( n = 3 \) is essentially CRUX Problem #2786 [2002 : 459; 2003 : 476-477], but we need a stronger version of it. We will leave this until the end of the proof. As our induction hypothesis, we assume that the statement in the theorem is true with \( n \) replaced by \( n-1 \), for some particular \( n \geq 4 \).

Let \( (x_1, x_2, \ldots, x_n) \) be any admissible point other than \( \left( \frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n} \right) \). If \( x_j = 0 \) for more than one value of the index \( j \), then \( P/x_j = 0 \) for all \( j \), and hence \( W(x_1, x_2, \ldots, x_n) = \frac{n}{k} < W \left( \frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n} \right) \). Thus, in this case, \( W \) is not maximal at \( (x_1, x_2, \ldots, x_n) \).

Now suppose that \( x_j = 0 \) for at most one \( j \). By re-labelling the points if necessary, we can assume that \( x_n \neq 0 \) and that \( x_1, x_2, \ldots, x_{n-1} \) are not all equal. Then

\[
W(x_1, x_2, \ldots, x_n) = \frac{1}{k - P'} + \frac{1}{x_n} \sum_{j=1}^{n-1} \frac{1}{k' - \frac{P'}{x_j}},
\]
where $P' = \prod_{j=1}^{n-1} x_j$ and $k' = k/x_n$.

Letting $h' = h - x_n$, we have $\sum_{j=1}^{n-1} x_j = h'$. Consider what happens if we replace $(x_1, x_2, \ldots, x_{n-1})$ by \( \left( \frac{h'}{n-1}, \frac{h'}{n-1}, \ldots, \frac{h'}{n-1} \right) \), keeping $x_n$ fixed. This increases the product $P'$ and, consequently, increases the first term on the right side of (1). Also, by the induction hypothesis, the sum on the right side of (1) is increased. Thus,

\[
W(x_1, x_2, \ldots, x_n) < W\left( \frac{h'}{n-1}, \frac{h'}{n-1}, \ldots, \frac{h'}{n-1}, x_n \right).
\]

We conclude that $W$ is not maximal at the point $(x_1, x_2, \ldots, x_n)$.

It remains for us to prove the case $n = 3$. For convenience, let $x = x_1$, $y = x_2$, and $z = x_3$. Without loss of generality, we assume that $h = 1$. (Otherwise, we could scale the variables $x$, $y$, $z$ by the factor $1/h$, with a corresponding change in the value of $k$.) Let $(x, y, z)$ be any admissible point other than $\left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$. By re-labeling the points if necessary, we can assume that $z \leq \frac{1}{2}$ and $x \neq y$. Now

\[
W(x, y, z) = \frac{1}{k - xy} + \frac{1}{k - yz} + \frac{1}{k - zx} - \frac{3k^2 - 2k(xy + yz + zx) + xyz(x + y + z)}{k^3 - k^2(xy + yz + zx) + kxyz(x + y + z) - x^2y^2z^2} = \frac{3k^2 - 2k(xy + yz + zx) + xyz}{k^3 - k^2(xy + yz + zx) + kxyz - x^2y^2z^2} = \frac{3 + 1}{k} k^2(xy + yz + zx) + xyz - x^2y^2z^2 = \frac{3 + \frac{1}{k}}{k} k^3 - k^2(xy + yz + zx) + kxyz - x^2y^2z^2.
\]

Replacing $(x, y)$ by $\left( \frac{1}{2} (x + y), \frac{1}{2} (x + y) \right)$ causes the product $xy$ to increase (since $x \neq y$), which increases the numerator and decreases the denominator in the second term on the right side above. Therefore, $W(x, y, z) < W \left( \frac{x + y}{2}, \frac{x + y}{2}, z \right)$. We conclude that $W$ is not maximal at the point $(x, y, z)$. Therefore, $W$ must attain its maximum value at $\left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$ and only at this point.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina. There was one incorrect solution.

A generalization of Problems 2798 and 2799 has been proposed by Guerensenzig and independently by Mihaly Benze, Brasov, Romania. See the notes to Problem 2799 below.
2799★. [2002 : 536] Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Prove or disprove the inequality

$$\sum_{\substack{i, j \in \{1, 2, \ldots, n\} \\1 \leq i < j \leq n}} \frac{1}{1 - x_i x_j} \leq \left(\begin{array}{c} n \\ 2 \end{array}\right) \frac{1}{1 - \frac{1}{n^2}}$$

where \(\sum_{j=1}^{n} x_j = 1, x_j \geq 0\).

[Editor’s Remark. The condition \(i < j\) replaced the original \(i \leq j\) in a footnote in [2003 : 47].]

Solution by Natalio H. Guersenzaig, Universidad CAECE, Buenos Aires, Argentina, modified by the editors.

We will prove the given inequality for \(n \geq 2\). To begin, we denote the sum on the left side by \(F(x_1, x_2, \ldots, x_n)\), which defines a continuous function \(F\) whose domain is a compact subset of \(\mathbb{R}^n\), say \(S\). Thus, \(F\) has a maximum value in \(S\), say \(M\). Then \(2M\) is the maximum value over \(S\) of

$$2F(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left( \sum_{\substack{j=1 \\neq i}}^{n} \frac{1}{1 - x_i x_j} \right).$$

Moreover, by considering, for each \(i = 1, \ldots, n\) and each \(x \in [0, 1]\), the compact subset \(S_i(x)\) of \(S\) formed by the points \((y_1, \ldots, y_n) \in S\) such that \(y_i = x\), it is clear that

$$2M \leq \max_{(x_1, \ldots, x_n) \in S} \sum_{i=1}^{n} \left( \max_{(y_1, \ldots, y_n) \in S_i(x)} \sum_{\substack{j=1 \\neq i}}^{n} \frac{1}{1 - x_i y_j} \right).$$

Therefore, since \(F\) is a symmetric function, in order to complete the proof it will be sufficient to prove the following two facts.

(i) For each \(x \in [0, 1]\) and each \((y_1, \ldots, y_{n-1}, x) \in S_n(x)\), the maximum value of the function \(F_{n-1,x}\) defined by

$$F_{n-1,x}(y_1, \ldots, y_{n-1}) = \sum_{j=1}^{n-1} \frac{1}{1 - x y_j}$$

is given by

$$F_{n-1,x} \left( \frac{1 - x}{n - 1}, \ldots, \frac{1 - x}{n - 1} \right) = \frac{n - 1}{1 - x(1 - x) \frac{n - 1}{n - 1}}.$$
(ii) The maximum value of the function $G_n$ defined on $S$ by

$$G_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{n-1}{1 - x_i(1-x_i)}$$

is given by

$$G_n\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = 2 \binom{n}{2} \frac{1}{1 - \frac{1}{n^2}}$$

We will argue inductively beginning with (i). Cases $x = 0$ and $x = 1$ are clearly true. Therefore, we may assume that $x(1-x) \neq 0$. Case $n = 2$ is also clear because $y_1 = 1 - x$. Thus, suppose that $n \geq 3$, and assume (i) holds for $n - 1$.

Let us notice in the first place that $0 \leq x(1-x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$. Hence, for $n \geq 3$, we have

$$\frac{n-2}{1 - \frac{x(1-x)}{n-2}} \leq \frac{n-2}{1 - \frac{1}{4(n-2)}} < \frac{n-1}{1} \leq \frac{n-1}{1 - \frac{x(1-x)}{n-1}}.$$

Suppose $y_j = 0$ for some $j$, $1 \leq j \leq n-1$. Without loss of generality, we may assume $y_{n-1} = 0$ because of the symmetry of $F_{n-1,x}$. Observing that

$$1 + F_{n-2,x}(y_1, \ldots, y_{n-2}) = F_{n-1,x}(y_1, \ldots, y_{n-1}, 0)$$

and using the induction hypothesis, we get

$$F_{n-1,x}(y_1, \ldots, y_{n-1}, 0) \leq \frac{n-2}{1 - \frac{x(1-x)}{n-2}} < F_{n-1,x}\left(\frac{1-x}{n-1}, \ldots, \frac{1-x}{n-1}\right),$$

which proves that all the coordinates of any maximum point of $F_{n-1,x}$ are non-zero.

Consider the Lagrange function $L$ defined for positive $y_1, \ldots, y_{n-1}$, and $\lambda \in \mathbb{R}$ by

$$L(y_1, \ldots, y_{n-1}, \lambda) = \sum_{j=1}^{n-1} \frac{1}{1 - xy_j} - \lambda(1 - x - y_1 - \cdots - y_{n-1}).$$

Looking for the critical points of $L$, we set

$$\frac{\partial L}{\partial y_1} = \cdots = \frac{\partial L}{\partial y_{n-1}} = \frac{\partial L}{\partial \lambda} = 0.$$
to obtain
\[
\frac{x}{(1 - xy_1)^2} = \cdots = \frac{x}{(1 - xy_{n-1})^2} ;
\]
whence, \(y_1 = \cdots = y_{n-1} = \frac{1-x}{n-1}\), which means that (i) holds for \(n\). Thus, the proof of (i) is complete.

Case \(n = 2\) of (ii) is true because \(x_i(1 - x_i) \leq \frac{1}{4}\); whence,
\[
G_2(x_1, x_2) \leq 2\frac{1}{1 - \frac{1}{4}} = 2\left(\frac{2}{1 - \frac{1}{4}}\right) = G_2\left(\frac{1}{2}, \frac{1}{2}\right).
\]

Now suppose \(n \geq 3\), and assume (i) holds for \(n-1\). We first consider the case that \(x_i = 0\) for some \(i\), \(1 \leq i \leq n\). From the symmetry of \(G_n\), we can assume \(x_n = 0\). Since for \(n \geq 3\), we have
\[
(n-1) + 2\binom{n-1}{2} \frac{1}{1 - \frac{1}{(n-1)^2}} < 2\binom{n}{2} \frac{1}{1 - \frac{1}{n^2}},
\]
using the induction hypothesis, we derive
\[
G_n(x_1, \ldots, x_{n-1}, 0) = n - 1 + G_{n-1}(x_1, \ldots, x_{n-1})
\leq n - 1 + 2\binom{n-1}{2} \frac{1}{1 - \frac{1}{(n-1)^2}}
\leq G_n\left(\frac{1}{n}, \ldots, \frac{1}{n}\right),
\]
from which it follows that all the coordinates of any maximum point of \(G_n\) are non-zero.

Consider the Lagrange function \(L\) defined for positive \(x_1, \ldots, x_n\), and \(\lambda \in \mathbb{R}\) by
\[
L(x_1, \ldots, x_n, \lambda) = \sum_{i=1}^{n} \frac{n-1}{1 - x_i(1-x_i)} - \lambda(1-x_1 - \cdots - x_n).
\]
To complete the proof it will suffice to show that \(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\) is the only critical point of \(L\). From the necessary condition
\[
\frac{\partial L}{\partial x_1} = \cdots = \frac{\partial L}{\partial x_n} = \frac{\partial L}{\partial \lambda} = 0,
\]
after canceling \((n-1)^2\) from each term, we obtain
\[
\frac{2x_1 - 1}{(n-1 - x_1 + x_1^2)^2} = \cdots = \frac{2x_n - 1}{(n-1 - x_n + x_n^2)^2}.
\]
Hence, by noting that the function \( f \) defined for \( x \in (0, 1) \) by
\[
f(x) = \frac{2x - 1}{(n - 1 - x + x^2)^2}
\]
is strictly increasing (because \( f'(x) = \frac{2(n - 2 + 3x(1 - x))}{(n - 1 - x - x^2)^3} > 0 \)), we see at once that \( x_1 = \cdots = x_n = \frac{1}{n} \), which means that (ii) holds for \( n \). Thus, the proof of (ii) is complete.

There was one incomplete solution.

The author also adds the following remarks.

(a) Let \( m \) be the minimum value of the given sum. From cases \( x = 0 \) and \( x = 1 \) of (i), it follows that the minimum value of \( F_{n-1,x} \) is equal to \( F_{n-1,0,1} = n - 1 \); whence, \( m = \binom{n}{2} \) because
\[
2m = \min_{(x_1, \ldots, x_n) \in S} \sum_{i=1}^{n} \left( \min_{(y_1, \ldots, y_n) \in S_i(x_i)} \sum_{j=1 \atop j \neq i}^{n} \frac{1}{1 - x_i y_j} \right).
\]

[Ed. We can also see this by observing that the original sum has \( \binom{n}{2} \) terms, each of which is at least one.]

(b) By the same iterative argument, we can prove by induction over \( k \) the following generalization of #2799 (case \( k = 2 \)) and #2798 (case \( k = n - 1 \)).

Let \( n, k \) be arbitrary integers with \( 2 \leq k \leq n \). Then
\[
\binom{n}{k} \leq \sum_{1 \leq x_1 < \cdots < x_k \leq n} \frac{1}{1 - \prod_{j=1}^{k} x_{i_j}} \leq \binom{n}{k} \frac{1}{1 - \frac{1}{n^k}},
\]
where \( \sum_{j=1}^{n} x_j = 1 \), and \( x_i \geq 0 \) for \( i = 1, 2, \ldots, n \).

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**2800.** [2002 : 536] **Proposed by Mihály Benze, Brasov, Romania.**

Suppose that \( z_k \in \mathbb{C}^* \) (\( k = 1, 2, \ldots, n \)) such that \( |z_k| = |M| \) (constant) and
\[
\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} z_k - z_1 = \sum_{k=1}^{n} z_k - z_2 = \cdots = \sum_{k=1}^{n} z_k - z_n
\]
\((k \in \{1, 2, \ldots, n\})

Prove that \( \left( \sum_{k=1}^{n} z_k \right) \left( \sum_{k=1}^{n} \frac{1}{z_k} \right) = \frac{n}{2} \).
1. **Solution by Eckard Specht. Otto-von-Guericke University, Magdeburg, Germany.**

Let $s = \sum_{k=1}^{n} z_k$. Taking the complex conjugate, we have $\bar{s} = \sum_{k=1}^{n} \bar{z}_k$.

Then, for $i = 1, 2, \ldots, n$,

$$s \cdot \bar{s} = \left| \sum_{k=1}^{n} z_k \right|^2 = \left| \sum_{k=1}^{n} z_k - z_i \right|^2 = (s - z_i) \cdot (\bar{s} - \bar{z}_i)$$

$$= (s - z_i) \cdot (\bar{s} - \bar{z}_i) = s \cdot \bar{s} - z_i \cdot \bar{s} - s \cdot \bar{z}_i + |M|^2.$$

Hence, $z_i \cdot \bar{s} + s \cdot \bar{z}_i = |M|^2$. Summing this equation over $i = 1, 2, \ldots, n$ gives $2s \cdot \bar{s} = n |M|^2$. Finally,

$$\left( \sum_{k=1}^{n} z_k \right) \left( \sum_{k=1}^{n} \frac{1}{z_k} \right) = s \cdot \left( \sum_{k=1}^{n} \frac{\bar{z}_k}{z_k} \right) = \frac{s \cdot \bar{s}}{|M|^2} = \frac{n}{2}.$$

II. **Solution by David Loeffler, student, Trinity College, Cambridge, UK.**

Let $s = \sum_{k=1}^{n} z_k$. From the given information, we see that in the complex plane the points $z_k$ all lie on a common circle $\Gamma$ centred at $s$ and passing through the origin. Furthermore, since $|z_k| = |M|$, the points $z_k$ also lie on the circle centred at 0 with radius $|M|$. Hence, each point $z_k$ is at one or the other of the points of intersection of these two circles.

Since the problem is invariant under multiplication of all the points $z_k$ by a complex constant, we can assume without loss of generality that $|M| = 1$ and that $s$ is real and positive. Then $n$ must be even, and $n/2$ of the points $z_k$ lie at each of the points $e^{i\theta}$ and $e^{-i\theta}$, for some constant $\theta$. Therefore, $s = \frac{n}{2}(e^{i\theta} + e^{-i\theta}) = n \cos \theta$. Since the mapping $z \mapsto \frac{1}{z}$ just interchanges the two sets of points, we also have $\sum_{k=1}^{n} \frac{1}{z_k} = n \cos \theta$.

The triangle with vertices 0, $z_1$, and $s$ is isosceles ($s - 0 = |s - z_1|$). From this triangle, we have $1 = |z_1| = 2s \cos \theta$. Hence, $1 = 2n \cos^2 \theta$. Thus,

$$\left( \sum_{k=1}^{n} z_k \right) \left( \sum_{k=1}^{n} \frac{1}{z_k} \right) = (n \cos \theta)^2 = \frac{n}{2} (2n \cos^2 \theta) = \frac{n}{2}.$$

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSTIEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSE LUIS DIAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; NATALIO H. GUERSENZVAIG, Universidad CABA, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinenymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.