THE OLYMPIAD CORNER
No. 234

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

To start off this number of the Corner, we have another five Klamkin Quickies. Thanks go to Murray S. Klamkin, University of Alberta, Edmonton, AB, for creating them for our use. Try them before looking for his Quickie solutions later in this issue.

FIVE KLAMKIN QUICKIES
December 2003

1. The numbers $2^{2003}$ and $5^{2003}$ are written out in base 10 one right after the other to create a single number. How many digits are there in this number? (No log tables or calculators, please!)

2. When an integer-sided cube $C$ is cut into 729 smaller integer-sided cubes, exactly 728 of them are unit cubes. What is the smallest possible volume of $C$?

3. Points $A, B, C, D, E, F$ are the consecutive vertices of an inscribed hexagon such that $AB = BC$, $CD = DE$, and $EF = FA$. Find the ratio of the area of triangle $BDF$ to the area of the hexagon.

4. Express $(x^n + x^{n-1} + \cdots + 1)^2 - x^n$ as a product of non-constant polynomials.

5. Determine the maximum value of

$$\frac{x}{y(1 + z + x)} + \frac{y}{z(1 + x + y)} + \frac{z}{x(1 + y + z)}$$

where $x, y, z \geq 1$. 
As a first Olympiad problem set, we give the Olympiades Académiques de Mathématiques, Session de 2001, Classe de Première. My thanks go to Michel Bataille, Rouen, France for sending them for our use.

**OLYMPIADES ACADÉMIQUES DE MATHEMATIQUES**

**Session de 2001**

**Classe de Première (Durée : 4 heures)**

1. Les faces d'un dé en forme de tétraèdre régulier sont numérotées de 1 à 4. Le dé est posé sur une table, face «1» contre cette table. Une étape consiste à faire basculer le dé autour de l'une quelconque des arêtes de sa base. À l'issue de chaque étape, on note le numéro de la face contre la table. On fait la somme $s$ de tous ces nombres après 2001 étapes, en comptant aussi le «1» initial.

   (i) Donner la valeur maximale et la valeur minimale que l'on peut ainsi obtenir pour $s$.

   (ii) La somme $s$ peut-elle prendre toutes les valeurs entières entre ces deux valeurs?

2. Une lampe entourée d'un abat-jour est suspendue entre deux murs distants de 8 mètres à une rampe. La situation est représentée par le schéma à droite.

   - Les murs ont pour équations $x = 0$, $x = 8$, et la rampe a pour équation $y = \frac{1}{6} (x - 4)^2 + \frac{19}{3}$.
   - L'abat-jour est symbolisé par un triangle rectangle isocèle $UMJ$ de côtés 1 et $\sqrt{2}$.

   (i) Vérifier que les bords de l'abat-jour ne touchent ni la rampe ni les murs lorsque $1 < x < 7$.

   (ii) Calculer l'aire du polygone éclairé $OBMCD$ correspondant à $x = 3$.

   (iii) Trouver la position de la lampe sur la rampe qui donne un éclairage maximal.

Déterminer l'ensemble des points $M$ du terrain d'où l'on voit les trois buts sous des angles $\angle AMB$, $\angle BMC$ et $\angle CMD$ égaux.


Considérons un autre cube $C'$ admettant aussi $(A, B)$ comme couple de sommets opposés. Certaines arêtes de $C$ rencontrent des arêtes de $C'$. Justifiez le fait que, en dehors de $A$ et $B$, on obtient ainsi six points d'intersection entre une arête de $C$ et une arête de $C'$.

Placez l'un d'eux sur le dessin et expliquez comment placer alors les cinq autres. $V$ étant le volume de $C$, quelle est la valeur minimale du volume de la portion d'espace commune aux cubes $C$ et $C'$?

As another set of problems for your pleasure over the holiday season, we give the Selected Problems of the Ukrainian Mathematical Olympiad, March 2001. Thanks go to Chris Small, Canadian Team Leader to the XLI IMO, for collecting them.

**UKRAINIAN MATHEMATICAL OLYMPIAD**

**March 2001**

**Selected Problems**

1. (Grade 9) All 5-digit positive integers with digits in increasing order (from left to right) are given. Is it possible to take away one digit from each number so that we obtain all 4-digit positive integers with digits in increasing order?

2. (Grade 9) Let $I$ be the incentre of triangle $ABC$; let the bisectors of $\angle BAC$ and $\angle ACB$ meet the sides $BC$ and $AB$ at $A_1$ and $C_1$, respectively; and let $M$ be an arbitrary point on the segment $AC$. Lines through $M$ parallel to these angle bisectors meet $AA_1$, $CC_1$, $AB$, and $CB$ at points $H$, $N$, $P$, and $Q$, respectively. Denote $BC = a$, $AC = b$, $AB = c$, and let $d_1$, $d_2$, $d_3$ be the respective distances from $H$, $I$, $N$ to the line $PQ$. Prove that

$$\frac{d_1}{d_2} + \frac{d_2}{d_3} + \frac{d_3}{d_1} \geq \frac{2ab}{a^2 + bc} + \frac{2ca}{c^2 + ab} + \frac{2bc}{b^2 + ca}.$$
3. (Grade 10) Let $a_1, a_2, \ldots, a_n$ be real numbers such that
\[ a_1 + a_2 + \cdots + a_n \geq n^2 \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1. \]
Prove that $n - 1 \leq a_k \leq n + 1$ for all $k$.

4. (Grade 10) There are $n$ mathematicians in each of three countries. Each mathematician corresponds with at least $n + 1$ foreign mathematicians. Prove that there exist three mathematicians who correspond with each other.

5. (Grade 11) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds?
\[ f(xy) = \max \{ f(x), y \} + \min \{ f(y), x \}. \]

6. (Grade 11) Positive integers $a$ and $n$ are such that $n$ divides $a^2 + 1$. Prove that there exists a positive integer $b$ such that $n(n^2 + 1)$ divides $b^2 + 1$.

7. (Grade 11) An acute triangle $ABC$, with $AC \neq BC$, is inscribed in a circle $\omega$. The points $A$, $B$, $C$ divide the circle into disjoint arcs $AB$, $BC$, and $CA$. Let $M$ and $N$ be the mid-points of $BC$ and $AC$, respectively, and let $K$ be an arbitrary point of $AB$. Let $D$ be the point of $MN$ such that $CD \parallel NM$. Let $O, O_1, O_2$ be the incentres of triangles $ABC, CAK, CBK$, respectively. Let $L$ be the intersection point of the line $DO$ and the circle $\omega$, where $L \neq D$. Prove that the points $K$, $O_1$, $O_2$, $L$ are concyclic.

8. (Grade 11) Let $a, b, c$ and $\alpha, \beta, \gamma$ be positive real numbers such that $\alpha + \beta + \gamma = 1$. Prove the inequality
\[ \alpha a + \beta b + \gamma c + 2\sqrt{(\alpha \beta + \beta \gamma + \gamma \alpha)(ab + bc + ca)} \leq a + b + c. \]

Next we give the official solutions by Murray S. Klamkin to the set of "Klamkin Quickies" that opened this Corner.

**SOLUTIONS TO FIVE KLAMKIN QUICKIES**
December 2003

1. Let $m$ and $n$ denote the number of digits in $2^{2003}$ and $5^{2003}$ when expressed in base 10. Then
\[ 10^{m-1} < 2^{2003} < 10^m \quad \text{and} \quad 10^{n-1} < 5^{2003} < 10^n, \]
so that $10^{m+n-2} < 10^{2003} < 10^{m+n}$. Thus, $m + n - 1 = 2003$ and the number of digits is $m + n = 2004.$
2. We must satisfy the Diophantine equation \( m^3 = n^3 + 728 \), with \( n > 1 \). Write the equation as
\[
(m - n)(m^2 + mn + n^2) = 7 \cdot 8 \cdot 13.
\]
Noting that \((m - n)^2 < m^2 + mn + n^2\), we find that \( m - n \) can only be 1, 2, 4, 7, or 8. Since \((m^2 + mn + n^2) - (m - n)^2 = 3mn\), the only solutions are \((m, n) = (9, 1)\), which does not satisfy the condition \( n > 1 \), and \((12, 10)\). Thus, the only possible volume is \( 12^3 = 1728 \).

3. Let the angles subtended at the centre by the sides \( AB, CD \) and \( EF \) be \( 2\alpha, 2\beta, \) and \( 2\gamma \), respectively. Then \( \alpha + \beta + \gamma = 90^\circ \). Let \( R \) be the circumradius. The area of \( ABCDEF \) is
\[
\begin{align*}
[ABCDEF] &= 2(R \sin \alpha)(R \cos \alpha) + 2(R \sin \beta)(R \cos \beta) + 2(R \sin \gamma)(R \cos \gamma) \\
&= R^2(\sin 2\alpha + \sin 2\beta + \sin 2\gamma),
\end{align*}
\]
and similarly, the area of \( BDF \) is
\[
\begin{align*}
[BDI] &= \frac{1}{2} R^2(\sin 2(\alpha + \beta) + \sin 2(\beta + \gamma) + \sin 2(\gamma + \alpha)) \\
&= \frac{1}{2} R^2(\sin 2\gamma + \sin 2\alpha + \sin 2\beta),
\end{align*}
\]
since \( \alpha + \beta + \gamma = 90^\circ \). Thus, we have \( 2[BDI] = [ABCDEF] \).

4. Letting \( S = x^n + x^{n-1} + \cdots + 1 \), the factors are seen to be \( (S + x^{n+1})(S - x^n) \) by expanding.

5. Since \( y(1 + z + x) \geq y + z + x \), etc., the given sum is \( \leq 1 \), with equality if and only if \( x = y = z = 1 \).

Now we have readers' solutions to problems of the 15th Balkan Mathematical Olympiad 1998, given in [2001: 357].

1. Consider the terms of the finite sequence \( \left\lfloor \frac{k^2}{1998} \right\rfloor \), \( k = 1, 2, \ldots, 1997 \), where \( \lfloor x \rfloor \) denotes the integral part of \( x \). How many of the terms of this sequence are different?

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK.

We give Aassila's solution.

Observe that
\[
\left\lfloor \frac{998^2}{1998} \right\rfloor = 498 < 499 = \left\lfloor \frac{999^2}{1998} \right\rfloor.
\]

We have
\[
\frac{(k + 1)^2}{1998} - \frac{k^2}{1998} = \frac{2k + 1}{1998} \begin{cases} < 1, & \text{for } k = 1, 2, \ldots, 998, \\
> 1, & \text{for } k = 999, 1000, \ldots, 1997. \end{cases}
\]
It follows that each integer from 0 to 499 inclusive is a term of the sequence, while each \( k > 999 \) corresponds to a distinct term of the sequence. Thus, the total number of distinct terms is \( 500 + (1997 - 999) = 1498 \).

2. Let \( n \) be an integer, \( n \geq 2 \), and \( 0 < a_1 < a_2 < \cdots < a_{2n+1} \) be real numbers. Prove that the following inequality holds:

\[
\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}.
\]

_Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornstein, Pontoise, France. We give Bornstein's write-up._

Let \( n \geq 2 \) be an integer.

**Lemma.** If \( 0 < a < b < c < d \) are real numbers such that \( a^n + d^n = b^n + c^n \), then \( a + d < b + c \).

**Proof.** Let \( x = b - a \), \( y = c - b \), and \( z = d - c \). From the Binomial Theorem, we have

\[
a^n + (a + x + y + z)^n = a^n + d^n = b^n + c^n = (a + x)^n + (a + x + y + z)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (x + y)^k < 2a^n + \sum_{k=1}^{n} \binom{n}{k} a^{n-k} (2x + y)^k = a^n + (a + 2x + y)^n.
\]

Then \( z < x \); that is \( a + d < b + c \). The lemma is proved.

Let \( 0 < a_1 < a_2 < \cdots < a_{2n+1} \) be real numbers. From the lemma, we deduce that

\[
\sqrt[n]{a_1} + \sqrt[n]{a_3} - \cdots - a_{2n} + a_{2n+1} < \sqrt[n]{a_2} + \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}},
\]

\[
\sqrt[n]{a_3} + \sqrt[n]{a_5} - \cdots - a_{2n} + a_{2n+1} < \sqrt[n]{a_4} + \sqrt[n]{a_3 - a_4 + \cdots - a_{2n} + a_{2n+1}},
\]

\[
\vdots
\]

\[
\sqrt[n]{a_{2n-1}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n-1} - a_{2n} + a_{2n+1}}.
\]

Summing, we get the desired result.
3. Denote by $S$ the set of all points of $\triangle ABC$ except one interior point $T$. Show that $S$ can be represented as a union of disjoint (line) segments.

Solved by Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

**Lemma.** If $MNPQ$ is a trapezoid, with $MN$ parallel to $PQ$, then the set $S$ of all the points interior to, or on the boundary of, $MNPQ$ except the line segment $[MN]$, can be represented as a union of disjoint line segments.

**Proof.** For any point $X$ on the half-open segment $[QM]$, let $Y_X$ be the point on the half-open segment $[PN]$ such that $XY_X$ is parallel to $MN$. Then we have

$$S = \bigcup_{X \in [QM]} [XY_X],$$

which proves the lemma.

Let $T$ be an interior point of the triangle $ABC$. Let $\ell_1$, $\ell_2$, $\ell_3$ be the lines through $T$ parallel to $AB$, $BC$, $CA$, respectively. Let $D$, $E$, $F$ be the respective intersections of $\ell_1$ with $BC$, of $\ell_2$ with $AC$, and of $\ell_3$ with $AB$.

To get the conclusion, it suffices to see that $S$ is the disjoint union of the three trapezoids $TFBD = [TD]$, $TDCE = [TE]$, $TEAF = [TF]$, and to use the lemma in these three cases.

4. Prove that the equation $y^2 = x^5 - 4$ has no integer solutions.

Solved by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France. We give Aassila's write-up.

We have $(x^5)^2 = x^{10} \equiv 0 \bmod 11$ for all $x$ [by Fermat's Theorem]. Hence, $x^5 \equiv -1$, $0$, or $1 \bmod 11$. Thus, $x^5 - 4 \equiv 6$, $7$, or $8 \bmod 11$.

But all squares are congruent to $0$, $1$, $3$, $4$, $5$, or $9 \bmod 11$. Therefore, the equation has no solutions in integers.
Next we turn to readers' solutions to problems of the 1st Mediterranean Mathematical Olympiad, April 22, 1998, given [2001 : 357–358].

1. [Greece]

Let $ABCD$ be a square inscribed in a circle. If $M$ is a point on the arc $AB$ show that $MC \cdot MD > 3\sqrt{3} \cdot MA \cdot MB$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Bradley's answer.

Without loss of generality, let the side of the square be equal to 1. Write $MA = a$, $MB = b$, $MC = c$, $MD = d$.

By Ptolemy's Theorem for $DAMB$ we have

$$b + a\sqrt{2} = d,$$

and for $AMBC$ we have

$$a + b\sqrt{2} = c.$$

Hence,

$$cd = \sqrt{2}a^2 + \sqrt{2}b^2 + 3ab,$$

which, by the AM-GM Inequality, yields

$$cd \geq (3 + 2\sqrt{2})ab > 3\sqrt{3}ab.$$

The last inequality is justified by observing that

$$3 + 2\sqrt{2} > 3\sqrt{3} \iff 17 + 12\sqrt{2} > 27 \iff 12\sqrt{2} > 10,$$

which is clearly true.
2. [Croatia]

(a) Prove that the polynomial \( z^{2n} + z^n + 1 \), \( n \in \mathbb{N} \), is divisible by the polynomial \( z^2 + z + 1 \) if and only if \( n \) is not a multiple of 3.

(b) Find the necessary and sufficient condition that the natural numbers \( p, q \) must satisfy for the polynomial \( z^p + z^q + 1 \) to be divisible by \( z^2 + z + 1 \).

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Pontoise, France. We give Bataille’s solution.

We begin with the more general question (b) and show that the condition which we seek is \( pq \equiv 2 \pmod{3} \).

The roots of \( z^2 + z + 1 \) are \( \omega \) and \( \omega^2 \), where \( \omega = \exp(2\pi i/3) \). Hence, \( z^p + z^q + 1 \) is divisible by \( z^2 + z + 1 \) if and only if \( \omega^p + \omega^q + 1 = 0 \) and \( \omega^{2p} + \omega^{2q} + 1 = 0 \). Suppose that these conditions hold. Then

\[
0 = (\omega^p + \omega^q + 1)^2 = \omega^{2p} + \omega^{2q} + 1 + 2(\omega^{p+q} + \omega^q + \omega^p) = 2(\omega^{p+q} - 1).
\]

Hence, \( \omega^{p+q} = 1 \), which implies that \( p + q \equiv 0 \pmod{3} \). It follows that \( \omega^q = \omega^{-p} = \omega^{2p} \). Then \( \omega^p + \omega^{2p} + 1 = 0 \), which implies that \( p \equiv 1 \) or \( p \equiv 2 \pmod{3} \). Since \( q \equiv -p \pmod{3} \), we get \( pq \equiv 2 \pmod{3} \).

Conversely, if \( pq \equiv 2 \pmod{3} \), say \( p \equiv 1 \) and \( q \equiv 2 \), then

\[
\omega^p + \omega^q + 1 = \omega^{2p} + \omega^{2q} + 1 = \omega^2 + \omega + 1 = 0.
\]

The conclusion follows.

As for (a), the result just obtained provides the condition \( 2n^2 \equiv 2 \); that is, \( n^2 \equiv 1 \pmod{3} \). This is clearly equivalent to \( n \not\equiv 0 \pmod{3} \), which means \( n \) is not a multiple of 3.

3. [Spain]

In a triangle \( ABC \), \( I \) is the incentre and \( D \in (BC) \), \( E \in (CA) \), \( F \in (AB) \) are the points of tangency of the incircle with the sides of the triangle. Let \( M \in (BC) \) be the foot of the interior bisector of \( \angle BIC \) and \( \{P\} = FE \cap AM \). Prove that \( DP \) is the interior bisector of the angle \( \angle FDE \).

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya’s solution.

Let \( X, Y \) be points on \( AM \) such that \( BX \parallel CY \parallel FE \). Since \( \triangle BMX \) and \( \triangle CMY \) are similar, and \( \angle BIM = \angle CIM \), we have

\[
\frac{CY}{BX} = \frac{CM}{BM} = \frac{CI}{BI}.
\]

Since \( EP \parallel CY \), \( FP \parallel BX \) and \( AE = AF \), we have

\[
\frac{EP}{FP} = \frac{EP}{CY} \cdot \frac{BX}{FP} = \frac{AE}{AC} \cdot \frac{AB}{FX} = \frac{AB}{AC} \cdot \frac{CI}{BI}.
\]

(1)
Let $\angle ABC = 2\beta$ and $\angle ACB = 2\gamma$. Then

$$\angle ABI = \angle IBC = \beta \quad \text{and} \quad \angle ACI = \angle ICB = \gamma.$$ 

By the Law of Sines for $\triangle ABC$ and $\triangle IBC$, we get

$$\frac{AB}{AC} = \frac{\sin 2\gamma}{\sin 2\beta} = \frac{2 \sin \gamma \cos \gamma}{2 \sin \beta \cos \beta} = \frac{\sin \gamma \cos \gamma}{\sin \beta \cos \beta},$$

and

$$\frac{CI}{BI} = \frac{\sin \beta}{\sin \gamma}.$$ 

Thus, from (1),

$$\frac{EP}{FP} = \frac{\sin \gamma \cos \gamma \cdot \sin \beta}{\sin \beta \cos \beta \cdot \sin \gamma} = \frac{\cos \gamma}{\cos \beta}. \quad (2)$$

Let $r$ be the radius of the incircle. Then

$$DE = 2r \sin \angle DFE = 2r \sin \angle EDC = 2r \sin(90^\circ - \gamma) = 2r \cos \gamma.$$ 

Similarly, we have $DF = 2r \cos \beta$. Hence,

$$\frac{DE}{DF} = \frac{2r \cos \gamma}{2r \cos \beta} = \frac{\cos \gamma}{\cos \beta}. \quad (3)$$

From (2) and (3) we obtain $\frac{EP}{FP} = \frac{DE}{DF}$.

Therefore, $DP$ is the interior bisector of $\angle FDE$. 
Now we turn to readers' solutions to problems of the Final National Selection Competition 1998 for the Greek Team, given [2001: 358].

1. If $x, y, z > 0$, $k > 2$ and $a = x + ky + kz$, $b = kx + y + kz$, $c = kx + ky + z$, show that

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \geq \frac{3}{2k + 1}.$$

*Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; Michel Bataille, Rouen, France; and Heinz-Jürgen Seiffert, Berlin, Germany. We give Seiffert's generalization.

More generally, if $x_0, x_1, \ldots, x_n > 0 \ (n \geq 1)$, $k > 1$, and

$$a_i = x_i + k \sum_{j = 0 \atop j \neq i}^{n} x_j \ (i = 0, 1, \ldots, n),$$

then

$$\sum_{i = 0}^{n} \frac{x_i}{a_i} \geq \frac{n + 1}{nk + 1}.$$

*Proof. Let $S = \sum_{i = 0}^{n} x_i$. Then, for each $i$ we have $a_i = kS - (k - 1)x_i$; that is,

$$x_i = \frac{kS - a_i}{k - 1}.$$

For all $i, j \in \{0, 1, \ldots, n\}$,

$$\frac{x_i}{a_i} \leq \frac{x_j}{a_j} \iff \frac{kS - a_i}{a_i} \leq \frac{kS - a_j}{a_j} \iff a_j \leq a_i.$$

Hence,

$$\sum_{i = 0}^{n} \sum_{j = 0}^{n} \left( \frac{x_i}{a_i} - \frac{x_j}{a_j} \right) (a_j - a_i) \geq 0,$$

or, equivalently,

$$\left( \sum_{i = 0}^{n} \frac{x_i}{a_i} \right) \left( \sum_{i = 0}^{n} a_i \right) \geq (n + 1)S.$$

The stated inequality now follows by noting that $\sum_{i = 0}^{n} a_i = (nk + 1)S$.

To solve the present proposal, take $n = 2$ and rename $x_0 = x, x_1 = y$, and $x_2 = z$. 
2. Let $ABCD$ be a trapezoid $(AB \parallel CD)$ and $M, N$ be points on the lines $AD$ and $BC$, respectively, such that $MN \parallel AB$. Prove that

$$DC \cdot MA + AB \cdot MD = MN \cdot AD.$$ 

Solved by Rahul Bamotra, student, Sir Winston Churchill High School, Calgary, AB; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Laura Gu, student, Sir Winston Churchill High School, Calgary, AB; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya’s solution.

Let $MN$ meet $AC$ and $BD$ at the points $P$ and $Q$, respectively. Since $PM \parallel AB \parallel CD$, we have

$$PM \parallel CD.$$ Thus, $\frac{MA}{AD} = \frac{MP}{DC};$ that is,

$$DC \cdot MA = MP \cdot AD. \quad (1)$$

Since $MN \parallel AB$ and $MN \parallel DC$, we have

$$\frac{NP}{BA} = \frac{CP}{CA} = \frac{DM}{DA} = \frac{QM}{BA}.$$

Therefore,

$$NP = QM. \quad (2)$$

Since $\frac{QM}{BA} = \frac{DM}{DA}$, then

$$AB \cdot MD = QM \cdot AD.$$ 

Hence, using (2),

$$AB \cdot MD = NP \cdot AD. \quad (3)$$

From (1) and (3) it follows that

$$DC \cdot MA + AB \cdot MD = MP \cdot AD + NP \cdot AD$$

$$= (MP + NP) \cdot AD = MN \cdot AD.$$

3. Prove that if the number $A = \overbrace{111\ldots1}^{n \text{ digits}}$ is prime then the number $n$ must be prime. Is the converse true?

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

[Ed. This problem also appeared as problem #7 of the Chilean Mathematical Olympiads 1994–95, and a solution appears in [2003 : 294].]
4. (a) A polynomial \( P(x) \) with integer coefficients takes the value \(-2\) for seven distinct integer values of \( x \). Prove that it cannot take the value 1996.

(b) Prove that there are irrational numbers \( x, y \) such that the number \( x^y \) is rational.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bradley’s solution.

(a) Suppose \( P(x) \) takes the value \(-2\) for the distinct values \( x_1, x_2, x_3, x_4, x_5, x_6, x_7 \). Then, by the Remainder Theorem, we have
\[
P(x) + 2 = (x - x_1)(x - x_2)(x - x_3) \cdots (x - x_7)Q(x),
\]
where \( Q(x) \) is also a polynomial with integer coefficients.

Suppose now that \( P(k) = 1996 \) for some integer \( k \). Then
\[
(k - x_1)(k - x_2) \cdots (k - x_7)Q(k) = 1998 = 2 \times 27 \times 37.
\]
The factors \((k - x_i)\) are distinct, since the \( x_i \) are distinct. But the maximum number of distinct factors of 1998 whose product is 1998 is six. [Ed. To get the largest number of distinct factors, we could include 1, \(-1\), 3, and \(-3\); the remaining factor 3 would have to be combined with 2, 37, or 3 or \(-3\) in order to maintain distinct factors, leaving us a maximum of six distinct factors. (For example, 1, \(-1\), \(-3\), 9, 2, 37, or 1, \(-1\), 3, \(-3\), 6, 37, etc.).] This contradiction shows that \( P(k) \neq 1996 \) for any integer \( k \).

[Ed. Note that \( P(x) \) may take the value 1996 when \( x \) is not an integer.]

(b) Either \( \sqrt{2} \sqrt{3} \) is rational or it is irrational. If the former, we are done. If the latter, setting \( x = \sqrt{2} \sqrt{3} \) and \( y = \sqrt{2} \) gives \( x^y = 2 \).

5. Let \( I \) be an open interval of width \( \frac{1}{n} \), \( n \in \mathbb{N} \) \( \{0\} \). Determine the maximum number of irreducible fractions \( \frac{a}{b} \) with \( 1 \leq b \leq n \) that lie in \( I \).

Solved by Pierre Bornsztein, Pontoise, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein’s solution.

We prove that the maximum is \( \left\lfloor \frac{n+1}{2} \right\rfloor \), where \( \lfloor x \rfloor \) denotes the integer part of \( x \).

For \( n \in \mathbb{N}^* \), let \( f(n) \) be the maximum number of irreducible fractions \( \frac{a}{b} \) with \( 1 \leq b \leq n \) that may lie in an open interval of width \( \frac{1}{n} \), and let \( g(n) \) be the maximum number of elements a subset \( S \) of \( \{1, 2, \ldots, n\} \) can have if no element of \( S \) is a multiple of another element of \( S \).

Let \( I \) be an open interval of width \( \frac{1}{n} \).
Claim 1. Let \( \frac{a}{b} \) and \( \frac{a'}{b'} \) be distinct, irreducible fractions such that \( 1 \leq b' < b \leq n \) and \( b = kb' \) for some \( k \in \mathbb{N}^* \). Then \( \frac{a}{b} \) and \( \frac{a'}{b'} \) cannot both belong to \( I \).

Proof of Claim 1. Suppose that \( \frac{a}{b}, \frac{a'}{b'} \in I \), with \( \frac{a}{b} \neq \frac{a'}{b'} \). Then
\[
\frac{1}{n} > \left| \frac{a}{b} - \frac{a'}{b'} \right| = \left| \frac{a - ka'}{b} \right| \geq \frac{1}{b} \geq \frac{1}{n},
\]
a contradiction, which proves the claim.

From Claim 1, it follows easily that
\[
f(n) \leq g(n). \tag{1}
\]

Claim 2. We have \( g(n) = \left\lfloor \frac{n+1}{2} \right\rfloor \).

Proof of Claim 2. Let \( k = \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \), and let \( a_1, a_2, \ldots, a_k \) be any \( k \) integers satisfying
\[
1 \leq a_1 < a_2 < \cdots < a_k \leq n.
\]
For each \( i \), let \( a_i = 2^{\alpha_i} \beta_i \), where \( \alpha_i, \beta_i \) are non-negative integers, and \( \beta_i \) is odd.

Suppose that \( n \) is even, say \( n = 2p \). Then \( k = p + 1 \), and there are exactly \( p \) odd integers in \( \{1, 2, \ldots, n\} \). Thus, \( \beta_i = \beta_j \) for some \( i < j \). Since \( a_i < a_j \), we must have \( \alpha_i < \alpha_j \), from which we deduce that \( a_i \) divides \( a_j \).

Suppose that \( n \) is odd, say \( n = 2p + 1 \). Then \( k = p + 2 \), and there are exactly \( p + 1 \) odd integers in \( \{1, 2, \ldots, n\} \). As above, we deduce that there are integers \( i < j \) such that \( a_i \) divides \( a_j \).

In either case, for each choice of \( k \) distinct integers in \( \{1, 2, \ldots, n\} \), there are two of them, say \( a \) and \( b \), such that \( a \) divides \( b \). It follows that
\[
g(n) \leq \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{2}
\]

Conversely, consider the set \( S = \{p + 1, p + 2, \ldots, n\} \), where \( p = \lfloor n/2 \rfloor \). The set \( S \) contains \( \left\lfloor \frac{n+1}{2} \right\rfloor \) elements, and if \( a \) and \( b \) are two distinct elements of \( S \), then neither \( a \) divides \( b \), nor \( b \) divides \( a \). Thus,
\[
g(n) \geq \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{3}
\]
From (2) and (3), Claim 2 is proved.
From Claim 2 and (1), we deduce that
\[ f(n) \leq \left\lfloor \frac{n + 1}{2} \right\rfloor. \quad (4) \]

If \( n = 2p + 1 \), then, for each \( r \in \{1, 2, \ldots, p + 1\} \), the irreducible fraction \( \frac{1}{p + r} \) belongs to the open interval \( I = \left( \frac{1}{2p + 1} - \varepsilon, \frac{2}{2p + 1} - \varepsilon \right) \), where \( \varepsilon = \frac{1}{2(2p + 1)(p + 1)} \). If \( n = 2p \), then, for each \( r \in \{1, 2, \ldots, p\} \), the irreducible fraction \( \frac{1}{p + r} \) belongs to the open interval \( I = \left( \frac{1}{2p} - \varepsilon, \frac{1}{p} - \varepsilon \right) \), where \( \varepsilon = \frac{1}{2p(p + 1)} \). In either case, the open interval \( I \) of width \( \frac{1}{n} \) contains at least \( \left\lfloor \frac{n + 1}{2} \right\rfloor \) irreducible fractions \( \frac{a}{b} \) with \( 1 \leq b \leq n \). It follows that
\[ f(n) \geq \left\lfloor \frac{n + 1}{2} \right\rfloor. \quad (5) \]

From (4) and (5), we have \( f(n) = \left\lfloor \frac{n + 1}{2} \right\rfloor \), as claimed.

The next problem set given in the October 2001 Corner was the 38th National Mathematical Olympiad of Slovenia 1994 Final Round [2001: 359]. Some of our readers have better memories and a more careful systematic approach than I seem to have demonstrated. Pierre Bernshtein points out that this problem set was given previously in the Corner [1998: 132] with solutions appearing in [1999: 208–211] and [1999: 266–269]. Let me thank those who submitted their solutions to some of the problems in response to the 2001 call:

- Robert Bilinski, Outremont, QC (Grade 4: #3)
- Christopher J. Bradley, Clifton College, Bristol, UK (Grade 3: #1, 3, 4; Grade 4: #1, 2, 3)
- Laura Gu, student, Sir Winston Churchill High School, Calgary, AB (Grade 3: #2)
- Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB (Grade 3: #1; Grade 4: #1)
- Toshio Seimiya, Kawasaki, Japan (Grade 3: #4; Grade 4: #4)

That completes the Corner for this issue. Send me your nice solutions and generalizations, as well as contest materials.