Mayhem Solutions

**M63.** Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.

Let $ABC$ be a right-angled triangle with $BC$ as its hypotenuse. From vertex $A$, construct altitude $AD$ and internal angle bisector $AE$ (so $D$ and $E$ are on side $BC$). We are given that $AD = 28$ and $AE = 35$. Determine the area of triangle $ABC$.

**Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.**

Without loss of generality we may assume that $E$ is between $D$ and $C$. Since $AD \perp BC$, we see that $\triangle ABD$ and $\triangle ADE$ are right-angled. Applying the Theorem of Pythagoras to $\triangle ADE$ yields $DE = 21$. Then

$$\angle BAD = \angle BAE - \angle DAE = 45^\circ - \arcsin \frac{3}{5},$$

(1)

since $\angle BAE = \frac{1}{2} \angle BAC = 45^\circ$. Since $\triangle DAB$ and $\triangle ACB$ are similar, we have $\angle BAD = \angle ACB$. Thus,

$$\angle ACB = 45^\circ - \arcsin \frac{3}{5}.$$  

(2)

Then,

$$\sin(\angle ACB) = \sin \left( 45^\circ - \arcsin \frac{3}{5} \right)$$

$$= \sin(45^\circ) \cos(\arcsin \frac{3}{5}) - \sin(\arcsin \frac{3}{5}) \cos(45^\circ)$$

$$= \frac{\sqrt{2}}{2} \cos \left( \arccos \frac{4}{5} \right) - \frac{4}{5} \frac{\sqrt{2}}{2} = \frac{1}{10} \sqrt{2}. $$

(3)

Using (2), we get $\angle AEC = 180^\circ - 45^\circ - \angle ACB = 90^\circ + \arcsin \frac{3}{5}$, whence

$$\sin(\angle AEC) = \cos \left( \arcsin \frac{3}{5} \right) = \cos \left( \arccos \frac{4}{5} \right) = \frac{4}{5}. $$

(4)

Applying (3), (4), and the Law of Sines to $\triangle AEC$, we have

$$AC = 140 \sqrt{2}. $$

(5)

Using (1), we obtain $\angle ABC = 90^\circ - \angle BAD = 45^\circ + \arcsin \frac{3}{5}$. Then

$$\sin(\angle ABC) = \sin \left( 45^\circ + \arcsin \frac{3}{5} \right)$$

$$= \sin(45^\circ) \cos(\arcsin \frac{3}{5}) + \sin(\arcsin \frac{3}{5}) \cos(45^\circ)$$

$$= \frac{\sqrt{2}}{2} \cos \left( \arccos \frac{4}{5} \right) + \frac{3}{5} \frac{\sqrt{2}}{2} = \frac{7}{10} \sqrt{2}. $$

Hence,

$$AB = \frac{28}{\sin(\angle ABC)} = 20 \sqrt{2}. $$

(6)
Finally, using equations (5) and (6), we get

\[
\text{Area of } \triangle ABC = \frac{AB \cdot AC}{2} = 2800.
\]

Also solved by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

**M64. Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.**

For real numbers \( x \), let \( f(x) = [x[x]] - [x[x]] \) (where \([x]\) is the largest integer smaller than or equal to \( x \) and \([x]\) is the smallest integer greater than or equal to \( x \)).

(a) Show that \( f(x) \geq 0 \) for all \( x \geq 0 \), and determine when equality holds.

(b) What is the situation if \( x < 0 \)?

**Solution by the proposer.**

(a) Clearly, \( f(x) = 0 \) if \( x \in \mathbb{N} \). Suppose \( x = k + d \), where \( k \in \mathbb{N} \cup \{0\} \) and \( 0 < d < 1 \). Then

\[
[x[x]] = [(k + d)(k + 1)] = [k^2 + k + dk + d] \geq k^2 + k, \quad (1)
\]

\[
[x[x]] = [(k + d)k] = [k^2 + dk] \leq [k^2 + k] = k^2 + k. \quad (2)
\]

From (1) and (2), we have \( f(x) \geq 0 \).

Now we look for the cases where equality occurs. We have already noted that \( f(x) = 0 \) when \( x \in \mathbb{N} \). Suppose \( x = k + d \), where \( k \in \mathbb{N} \cup \{0\} \) and \( 0 < d < 1 \). If \( f(x) = 0 \), then we must have equality in (1). Hence, \( dk + d < 1 \), or \( d < \frac{1}{k + 1} \). Then \( dk < \frac{k}{k + 1} < 1 \), which implies that \( [k^2 + dk] = k^2 + 1 < k^2 + k \). If \( k \geq 2 \), this is a contradiction. Hence, \( k = 0 \) or \( k = 1 \).

When \( k = 0 \), we have \( x = d \). Thus, \([x[x]] = 0\) and \([x[x]] = [x] = 0\); whence, \( f(x) = 0 \).

When \( k = 1 \), we have \( x = 1 + d \). Thus, \([x] = 1\) and \([x] = 2\). Hence, \([x[x]] = [x] = 2\), while

\[
[x[x]] = [2x] = [2 + 2d] = \begin{cases} 2 & \text{if } 0 < d < \frac{1}{2}, \\ 3 & \text{if } \frac{1}{2} \leq d < 1. \end{cases}
\]

Hence, we have \( f(x) = 0 \) if \( 0 < d < \frac{1}{2} \) and \( f(x) \neq 0 \) if \( \frac{1}{2} \leq d < 1 \).

To summarize, \( f(x) = 0 \) if and only if \( x \in \mathbb{N} \) or \( x \in [0, \frac{1}{2}) \).

(b) For \( x < 0 \), we use the fact that \([-t] = -[t]\) for all \( t \in \mathbb{R} \). Let \( x = -y \) where \( y > 0 \). Then

\[
[x[x]] = [-y[-y]] = [(y([-y]))] = [y[y]], \quad (3)
\]

\[
[x[x]] = [-y[-y]] = [-y([-y])] = [y[y]]. \quad (4)
\]
From (3) and (4), we see that $f(x) \leq 0$. Clearly, equality holds if $x \in \mathbb{Z}$. Suppose $y = k + d$, where $k \in \mathbb{N} \cup \{0\}$ and $0 < d < 1$. Then

$$\lceil y \rceil = \lceil (k + d) \rceil \leq \lceil k^2 + k \rceil = k^2 + k$$

and

$$\lfloor y \rfloor = \lfloor (k + d)(k + 1) \rfloor = k(k + 1) + \lfloor (k + 1)d \rfloor \geq k(k + 1) + 1 > k^2 + k.$$

Hence, equality can not hold.

To summarize, $f(x) \leq 0$ for all $x < 0$, with equality if and only if $x$ is a negative integer.

\textbf{M65. Proposed by the Mayhem Staff.}

I have a number of unit cubes and I arrange them all to form a larger solid cube. I then paint some of the faces of the large cube. When the cube is disassembled it is discovered that 1000 of the cubes have no paint on them. How many faces of the big cube were painted?

\textbf{Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.}

Let us notice that, if a large solid cube is formed by $n^3$ unit cubes, then $(n - 2)^3$ of them are interior and the remaining ones form the faces of the large solid cube. Since some of the faces are painted and there are 1000 cubes with no paint, we see that $n \geq 11$. On the other hand, if $n > 12$, we would have at least 1331 interior cubes, all of which would be unpainted. Thus, $n = 11$ or $n = 12$.

For $n = 11$, there are 331 painted unit cubes, provided we paint 3 faces that have a vertex in common. Then the number of cubes which have no paint on them is $1331 - 331 = 1000$.

For $n = 12$, the cube has 1000 interior cubes. Then, all the faces of the big cube were painted.

Therefore, the number of painted faces was either 3 or 6.

\textbf{M66. Proposé par Václav Konečný, Big Rapids, MI, USA.}

Trouver des entiers positifs $N$ en base 10 tels que $N!$ en base 6 finisse exactement par 99 zéros.

\textbf{Solution de Robert Bilinski, Outremont, QC.}

Un nombre en base 6 va finir avec 99 zéros si le premier coefficient de son expansion en base 6 qui n’est pas zéro est celui de $6^{99}$. Autrement dit, $N!$ doit être un multiple de qui n’est pas un multiple de $6^{99}$ pour avoir exactement 99 zéros. Puisque l’on accumule plus lentement les 3 que les 2, cela arrivera quand $N!$ contiendra 99 fois le facteur 3. En faisant une étude sommaire, on voit que 3! a 1 facteur 3, 6! en a 2, ..., 27! en a 13, 54! en a 26, 81! en a 40, 162! en a 80, et ainsi de suite. On arrête avec 204! qui
a 99 facteurs 3 (et au moins 102 facteurs 2). Ainsi, les nombres qui ont la
propriété voulu sont 204, 205 et 206.

*Résolu aussi par Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine.*

**M67. Proposed by J. Walter Lynch, Athens, GA, USA.**

Find the interval containing \( r \) so that three consecutive terms of the
geometric sequence \( a, ar, ar^2, \ldots \) are the sides of a triangle.

*Solution by Geneviève Lalonde, Massey, ON.*

Since lengths are positive, we know that \( a \) and \( r \) must be positive. We
have two cases:

**Case 1.** \( r \geq 1 \).

In this case \( ar^2 \) is the largest. Thus, in order to have a triangle, we
must have

\[
\begin{align*}
ar^2 &< a + ar, \\
r^2 - r - 1 &< 0, \\
r &< \frac{1 + \sqrt{5}}{2}.
\end{align*}
\]

This, together with our initial assumption, means \( 1 \leq r < \frac{1}{2}(1 + \sqrt{5}) \).

**Case 2.** \( 0 < r < 1 \).

In this case \( a \) is the largest. Thus, in order to have a triangle, we must
have

\[
\begin{align*}
ar^2 + ar &> a, \\
r^2 + r - 1 &> 0, \\
-1 + \sqrt{5} &< r.
\end{align*}
\]

This, together with our initial assumption, means \( \frac{1}{2}(-1 + \sqrt{5}) < r < 1 \).

Putting these two cases together, we get \( \frac{1}{2}(-1 + \sqrt{5}) < r < \frac{1}{2}(1 + \sqrt{5}) \).

**M68. Proposed by the Mayhem Staff.**

You go for a spiralling walk on the Cartesian plane. Starting at \((0, 0)\),
your first five steps are to the points \((1, 0), (1, 1), (0, 1), (-1, 1)\) and \((-1, 0)\).
What point do you arrive at on your \(2002^{\text{th}}\) step?

*Solution by Alfin, grade 11 student, SMU Methodist, Palengau, Indonesia.*
Considering the points on the negative $y$-axis, we see that the $7^{th}$ step is $(0, -1) \text{ and the } 22^{nd} \text{ step is } (0, -2)$. The $n^{th}$ point on the negative $y$-axis is $(0, -n)$, which is step number $(2n + 1)^2 - (n + 1) = 4n^2 + 3n$.

Let us see if the $2002^{nd}$ step is on the negative $y$-axis. We solve:

$$4n^2 + 3n = 2002,$$

$$4n^2 + 3n - 2002 = 0,$$

$$(n - 22)(4n + 91) = 0.$$

Thus, we see that the $22^{nd}$ point on the $y$-axis, namely $(0, -22)$, is the $2002^{nd}$ step.