On log-trigonometric functions

C.-S. Lin

In our previous article [5], some elementary finite sine and cosine series were applied to produce sums for some finite series of integers, by way of differentiation. Naturally, one might wonder what to expect from using integration instead. In this article, we express log-trigonometric functions in terms of infinite cosine series. These are useful in many applications, such as computing log-trigonometric integrals.

Proposition 1 For \( x \neq 0, \pm \pi, \pm 2\pi, \ldots \),

\[
\ln | \sin x | = - \ln 2 - \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k}. \tag{1}
\]

For \( x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \),

\[
\ln | \cos x | = - \ln 2 - \sum_{k=1}^{\infty} (-1)^k \frac{\cos(2kx)}{k}. \tag{2}
\]

Proof. Observe that (2) follows from (1) by replacing \( x \) in (1) by \( x + \frac{\pi}{2} \).

We will prove (1). For this it is enough to consider \( x \) such that \( 0 < x < \frac{\pi}{4} \), because of the symmetry and periodicity of the sine and cosine functions.

Let us recall a familiar finite sine series from the Lemma in [5]. For any integer \( n \geq 1 \), and for \( x \neq 0, \pm 2\pi, \pm 4\pi, \ldots \),

\[
\sum_{k=1}^{n} \sin(kx) = \frac{\cos(\frac{1}{2}x) - \cos((n + \frac{1}{2})x)}{2 \sin(\frac{1}{2}x)} \tag{3}
\]

Rewrite (3) as follows:

\[
\cos((n + \frac{1}{2})x) \csc(\frac{1}{2}x) = \cot(\frac{1}{2}x) - 2 \sum_{k=1}^{n} \sin(kx). \]

Letting \( x = 2t \), where \( t \neq 0, \pm \pi, \pm 2\pi, \ldots \), we have

\[
\cos((2n + 1)t) \csc t = \cot t - 2 \sum_{k=1}^{n} \sin(2kt). \tag{4}
\]

For \( 0 < x < \frac{\pi}{2} \), we integrate the left side of (4) by parts from \( x \) to \( \frac{\pi}{2} \), with \( u = \csc t \) and \( dv = \cos((2n + 1)t) \, dt \):
\[
\int_x^y \cos((2n+1)t) \csc t \, dt = \frac{1}{2n+1} \left( \sin((2n+1)t) \csc t \right)_{x}^{y} - \int_x^y \sin((2n+1)t) \frac{d}{dt} \csc t \, dt
\]

\[
= \frac{1}{2n+1} \left( (2n+1)t \csc t \right)_{x}^{y} - \int_x^y \sin((2n+1)t) \frac{d}{dt} \csc t \, dt
\]

Since \( |\sin((2n+1)x)| \leq 1 \) for all \( x \), we have

\[
\left| \int_x^y \cos((2n+1)t) \csc t \, dt \right| \leq \frac{1}{2n+1} \left( 1 + \csc x + \int_x^y \frac{d}{dt} \csc t \, dt \right) = \frac{1}{2n+1} \left( 1 + \csc x - \left[ \csc t \right]_{x}^{y} \right) = \frac{2 \csc x}{2n+1},
\]

which approaches 0 as \( n \to \infty \).

Now define \( h(n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \). Thus, \( h(n) \) is the sum of the first \( n \) terms of the alternating harmonic series, which converges by the Alternating Series Test. Integrating the right side of (4) from \( \frac{\pi}{4} \) to \( \frac{\pi}{2} \) yields

\[
\left[ \ln(\sin t) + \sum_{k=1}^{n} \frac{\cos(2kt)}{k} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \begin{cases} \frac{1}{2} \ln 2 - h(n) + \frac{1}{2} h\left( \frac{n-1}{2} \right) & \text{if } n \text{ is odd}, \\ \frac{1}{2} \ln 2 - h(n) + \frac{1}{2} h\left( \frac{n}{2} \right) & \text{if } n \text{ is even}. \end{cases}
\]

This must approach 0 as \( n \to \infty \). Therefore,

\[
\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}. \tag{5}
\]

For \( 0 < x < \frac{\pi}{2} \), the integral of the right side of (4) from \( x \) to \( \frac{\pi}{2} \) is

\[
\left[ \ln(\sin t) + \sum_{k=1}^{n} \frac{\cos(2kt)}{k} \right]_{x}^{\frac{\pi}{2}} = -\ln(\sin x) - h(n) - \sum_{k=1}^{n} \frac{\cos(2kx)}{k},
\]

which must approach 0 as \( n \to \infty \). Thus, using (5), we obtain (1).

Remarks.

1. Formula (3) may be obtained by summing the geometric series

\[
\sum_{k=1}^{n} e^{ikx} \quad \text{to get} \quad \sum_{k=1}^{n} e^{ikx} = \frac{e^{i(n+1)x} - e^{ix}}{e^{ix} - 1},
\]

and then taking imaginary parts.
2. Integrating term-by-term in the series in (1) and (2), we can evaluate the following improper integrals:

\[
\int_0^\pi \ln(\sin x) \, dx = \int_0^\pi \ln(\cos x) \, dx = -\frac{\pi}{2} \ln 2,
\]

\[
\int_0^\pi \ln(\sin x) \, dx = -\pi \ln 2.
\]

While these results are correct, the term-by-term integration of the series in (1) and (2) is not easily justified. (Convergence of the series is not uniform on the relevant intervals.) The second formula above is used to prove Jensen’s formula in elementary Complex Analysis. The evaluation of this improper integral relies heavily on contour integration in [1, p. 159], and is done using the symmetric method in [6, p. 275]. The integral also appears in [2].

3. Trigonometric series for \(\ln(\tan x)\), \(\ln(\cot x)\), \(\ln(\sec x)\), and \(\ln(\csc x)\) can be derived easily from (1) and (2). We leave the details to the reader.

4. It is the author’s belief that our proof of (5) is new. The normal way of getting \(\ln 2\) is by Mercator’s log series [3, p. 367],

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \leq 1.
\]

5. The Fourier series (1) is proved differently in [4, Theorem 216].

References

C.-S. Lin
Department of Mathematics
Bishop’s University
Lennoxville, PQ
J1M 1Z7, Canada
plin@ubishops.ca