SKOLIAD No. 73

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Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by 1 April 2004. A copy of MATHEMATICAL MAYHEM Vol. 6 will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The first item in this issue is the 2003 W.J. Blundon Mathematics Contest. My thanks go out to Don Rideout of Memorial University for forwarding the material to me.

The Twentieth W.J. Blundon Mathematics Contest (*)&

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 19, 2003

1. Solve: \( \log_2(9 - 2^x) = 3 - x \).

2. Show that \((\sqrt{5} + 2)^{\frac{1}{3}} - (\sqrt{5} - 2)^{\frac{1}{3}}\) is rational, and find its value.

3. If \(a^3 + b^3 = 4\) and \(ab = \frac{2}{3}\), where \(a\) and \(b\) are real, find \(a + b\).

4. Find \(x, y,\) and \(z\) such that when any one of them is added to the product of the other two, the result is 2.

5. If \(a, b,\) and \(c\) are the three zeros of \(P(x) = x^3 - x^2 + x - 2\), find \(a + b + c\) and \(a^2 + b^2 + c^2\).
6. If \( \sin x + \cos x = \sqrt{\frac{2 + \sqrt{3}}{2}} \), with \( 0 < x < \frac{\pi}{2} \), find \( x \).

7. Prove that two consecutive odd positive integers cannot have a common factor other than 1.

8. Triangle \( ABC \) has vertices \( A(3, 1) \), \( B(5, 7) \), and \( C(1, y) \). Find all \( y \) so that angle \( C \) is a right angle.

9. In the diagram to the right, \( PQ = 8 \), \( TS = 12 \), and \( QS = 20 \). Find \( QR \) so that \( \angle PRT \) is a right angle.

10. A square of side 2 is placed with one side on a tangent to a circle of radius 5 so that the square lies outside the circle, and one vertex of the square lies on the circle. A line is drawn from the centre of the circle through the vertex of the square that is not on the tangent and not on the circle. This line cuts the tangent at a point \( T \). If the tangent meets the circle at \( S \), find the length of the line segment \( TS \).

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Next we look at the solutions to the 1997 Mandelbrot Competition, Round 1, individual and team tests from [2003 : 65-67].

**The Mandelbrot Competition**  
**Division B Round One Individual Test**  
**November 1997**

1. (*) What angle less than \( 180^\circ \) is formed by the hands of a clock at 2:30 pm? (Express the answer in degrees.)  

   \( \text{Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.} \)

   If the hour hand is pointing at the two and the minute hand at six, the angle between them is \( 30^\circ \times (6 - 2) = 120^\circ \). But the hour hand is actually pointing at the mid-point between two and three, which means we have to subtract \( 30^\circ / 2 = 15^\circ \). Therefore, the angle is \( 105^\circ \).

2. (*) If \( x = \sqrt{\frac{6}{7}} \), then evaluate \( \left(x + \frac{1}{x}\right)^2 \).  

   \( \text{(1 point)} \)
Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

\[
(x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2} = \frac{6}{7} + 2 + \frac{7}{6} = \frac{169}{42}.
\]

3. (a) How many possible values are there for the sum \(a + b + c\) if \(a, b,\) and \(c\) are positive integers and \(abc = 50^3\) (2 points)

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Without loss of generality, let \(a \geq b \geq c\). When \(abc = 50^3\), the only possible values for \((a, b, c)\) are \((50, 1, 1), (25, 2, 1), (10, 5, 1),\) and \((5, 5, 2)\). Since their sums are distinct, there are four possible values for the sum.

4. List the numbers \(\sqrt{2}, \sqrt[3]{3},\) and \(\sqrt{5}\) in order from greatest to least. (2 points)

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Comparing the size of \(\sqrt{2}, \sqrt[3]{3},\) and \(\sqrt{5}\) is equivalent to comparing \((\sqrt{2})^{30}, (\sqrt[3]{3})^{30},\) and \((\sqrt{5})^{30}\), since they are all positive.

First, note that \((\sqrt{2})^{30} = 2^{15} = 8^5\) and \((\sqrt[3]{3})^{30} = 3^{10} = 9^5\). Since \(9^5 > 8^5\), it follows that \(\sqrt[3]{3} > \sqrt{2}\). Similarly, \((\sqrt{2})^{30} = 2^{15} = 32^3\) and \((\sqrt[5]{5})^{30} = 5^6 = 25^3\). Since \(32^3 > 25^3\), it follows that \(\sqrt{2} > \sqrt[5]{5}\).

Thus, the order is \(\sqrt[3]{3} > \sqrt{2} > \sqrt[5]{5}\).

5. Compute \(3^A\) where \(A = \frac{(\log 1 - \log 4)(\log 9 - \log 2)}{(\log 1 - \log 9)(\log 8 - \log 4)}\). All logarithms are base three. (2 points)

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Simplifying, using the Laws of Logarithms, yields

\[
A = \frac{(0 - 2 \log 2)(2 - \log 2)}{(0 - 2)(\log \frac{2}{3})} = \frac{-2 \log 2(2 - \log 2)}{-2 \log 2} = 2 - \log 2.
\]

Therefore,

\[
\log 2 = 2 - A,
\]
\[
2 = 3^{2-A},
\]
\[
2 \cdot 3^A = 3^2,
\]
\[
3^A = \frac{9}{2}.
\]
6. Joe and Andy are playing a simple game on a circular board with \( n \) spaces. First Joe advances five spaces from the starting space, then Andy advances seven, then Joe advances five, then Andy advances seven, and so on. The first player to land back on the original space wins. If \( n \) is a random two-digit number, what is the probability that Joe wins? (3 points)

**Official solution.**

If \( n \) is not divisible by either 5 or 7, then Joe will finish when he has advanced \( 5n \) spaces, while Andy will finish when he has advanced \( 7n \) spaces. This will require \( n \) moves for each player. Since Joe goes first, he will finish first.

If \( n \) is divisible by 5 but not 7, Joe will win, since he will finish in only \( n/5 \) moves. Similarly, if \( n \) is divisible by 7 but not 5, then Andy will win, finishing in \( n/7 \) moves.

Finally, if \( n \) is divisible by both 5 and 7, then Andy will win, since he will finish in \( n/7 \) moves versus Joe's \( n/5 \).

Hence, Andy will win if and only if \( n \) is divisible by 7, or 13 times out of the 90 two-digit numbers. Thus, the probability that Joe wins is \( 77/90 \).

7. In the diagram, \( M \) is the mid-point of \( AB \) and \( Y \) is the mid-point of \( AC \). Hence, \( Q \) is a trisection point of \( CM \); we call the other trisection point \( P \) and extend \( BP \) to meet \( AC \) at \( X \). Evaluate \( (CX + AY)/XY \). (3 points)

![Diagram of a triangle with points A, B, C, M, Q, P, Y and X, with lines connecting them to form triangles and segments.]

**Official solution.**

Applying Menelaus' Theorem to \( \triangle ACM \) as intersected by the line \( BP \), we have

\[
\frac{AX}{XC} \cdot \frac{CP}{PM} \cdot \frac{MB}{BA} = 1.
\]

Since \( CP/PM = 1/2 \) and \( MB/BA = 1/2 \), we have \( AX = 4XC \). Thus, \( XC = \frac{1}{5}AC \). Since \( YC = \frac{1}{2}AC \), we have \( XY = \frac{3}{10}AC \), which yields \( (CX + AY)/XY = 7/3 \).

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.
The Mandelbrot Competition
Division B Round One Team Test
November 1997

**Facts:** The Weighted Power Mean Inequality for two positive variables states that if \( x_1, x_2, w_1 \) and \( w_2 \) are positive real numbers, and \( m \) and \( n \) are non-zero integers with \( m > n \), then

\[
\left( \frac{w_1 x_1^m + w_2 x_2^m}{w_1 + w_2} \right)^\frac{1}{m} \geq \left( \frac{w_1 x_1^n + w_2 x_2^n}{w_1 + w_2} \right)^\frac{1}{n},
\]

which is quite a mouthful. The positive variables are \( x_1 \) and \( x_2 \). The two sides are equal if and only if \( x_1 = x_2 \). The numbers \( w_1 \) and \( w_2 \) "weight" the variables in different proportions. Try \( w_1 = w_2 = 1 \) to see the standard Power Mean Inequality; then compare with \( w_1 = 1 \) and \( w_2 = 2 \), which emphasizes \( x_2 \). Finally, the non-zero integers \( m \) and \( n \) vary the powers. For example, use \( m = 1 \) and \( n = -1 \) to obtain the Arithmetic Mean–Harmonic Mean Inequality.

**Setup:** On the planet Garth a certain laser has the curious property that when reflected off a special mirror it always continues in a direction perpendicular to the original path. Some popular Garthian children’s games are based on this phenomenon. The ones described below involve a player situated in the corner of a rectangular mirrored hallway of width one plog (about 7.3 metres), as shown below. The player directs the laser beam an angle of \( \theta \) away from the left wall, hitting the far wall a distance \( d \) from the end of the hall on the first bounce.

![Diagram showing the first, second, and third bounces of a laser beam on a Garthian hallway](image)

**Problems:** *(Please, no calculus-based solutions.)*

**Part i:** (4 points) Show that the laser’s path length up to the second bounce is \( \frac{1}{\cos \theta} + \frac{1}{\sin \theta} \).
Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

From the diagram, we get \( \cos \theta = \frac{1}{x_2} \), or \( x_1 = \frac{1}{\cos \theta} \). Similarly, \( \sin \theta = \frac{1}{x_2} \), or \( x_2 = \frac{1}{\sin \theta} \). Thus, \( x_1 + x_2 = \frac{1}{\sin \theta} + \frac{1}{\cos \theta} \).

Part ii: (4 points) The object of one of the simpler games is to have the shortest path length after two bounces. Use the standard Power Mean Inequality with \( m = 2 \), \( n = -1 \), \( x_1 = \cos \theta \), and \( x_2 = \sin \theta \) to prove that the shortest possible path length is \( 2\sqrt{2} \). Invoke the equality condition to show that we need \( \tan \theta = 1 \) (that is, \( d = 1 \)) to obtain this minimum.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Substituting \( m = 2 \) and \( n = -1 \) into the standard Power Mean Inequality, we get

\[
\sqrt[2]{\frac{\cos^2 \theta + \sin^2 \theta}{2}} \geq \left( \frac{\frac{1}{\cos \theta} + \frac{1}{\sin \theta}}{2} \right)^{-1}.
\]

Since \( \cos^2 \theta + \sin^2 \theta = 1 \), this simplifies to

\[
\sqrt{\frac{1}{2}} \geq \frac{2}{\frac{1}{\cos \theta} + \frac{1}{\sin \theta}},
\]

\[
\frac{1}{\cos \theta} + \frac{1}{\sin \theta} \geq \frac{2}{\sqrt{\frac{1}{2}}} = 2\sqrt{2}.
\]

Thus, the minimum value for \( \frac{1}{\cos \theta} + \frac{1}{\sin \theta} \) is \( 2\sqrt{2} \). For this to occur, \( x_1 = x_2 \); that is, \( \sin \theta = \cos \theta \), or \( \tan \theta = 1 \).

Part iii: (4 points) Show that the total path length after three bounces (a more challenging version) is \( 2 \left( \frac{1}{\cos \theta} \right) + \frac{1}{\sin \theta} \). To minimize this, we employ a clever strategy. Begin by finding numbers \( w_1 \) and \( \lambda_1 \) such that \( w_1 \lambda_1^2 = 1 \) and \( w_1/\lambda_1 = 2 \).
Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Let \( x_3 \) be the length of the third bounce. Then \( \cos \theta = \frac{1}{x_3} \), or
\[
x_3 = \frac{1}{\cos \theta}.
\]
Hence, the total length is
\[
2 \left( \frac{1}{\cos \theta} \right) + \frac{1}{\sin \theta}.
\]
Solving \( w_1 \lambda_1^2 = 1 \) and \( w_1 = 2\lambda_1 \) yields \( \lambda_1^2 = \frac{1}{2} \) Thus, \( \lambda_1 = \frac{1}{\sqrt{2}} \) and
\[
w_1 = \frac{2}{\sqrt{2}}.
\]

Part iv: (4 points) Now apply the Weighted Power Mean Inequality with \( m = 2 \) and \( n = -1 \) as before, using \( w_1, x_1 = \lambda_1 \cos \theta \), and \( x_2 = \sin \theta \), to prove that the minimum path length is \((1 + 2^{2/3})^{3/2}\). What value of \( d \) should the player aim for?

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

By the Weighted Power Mean Inequality, we get
\[
\left( \frac{\frac{2}{\sqrt{2}} \cdot \left( \frac{1}{\sqrt{2}} \right)^2 \cdot \cos^2 \theta + \sin^2 \theta}{2 + \frac{2}{\sqrt{2}}} \right)^{\frac{1}{2}} \geq \left( \frac{\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{2} \cos \theta + \frac{1}{\sin \theta}}{2 + \frac{2}{\sqrt{2}}} \right)^{-1}.
\]
Since \( \sin^2 \theta + \cos^2 \theta = 1 \), we get
\[
\left( \frac{\sqrt{2}}{2 + \sqrt{2}} \right)^{\frac{1}{2}} \geq \frac{2 + \frac{\sqrt{2}}{2}}{\cos \theta + \frac{1}{\sin \theta}} = \frac{2^\frac{1}{2} + 1}{\cos \theta + \frac{1}{\sin \theta}}.
\]
Then
\[
\frac{2}{\cos \theta} + \frac{1}{\sin \theta} \geq (2^\frac{1}{2} + 1) \cdot (2^\frac{1}{2} + 1)^{\frac{1}{2}}
\]
\[
= (2^\frac{1}{2} + 1)^{\frac{3}{2}}.
\]
Therefore, the minimum path length is \((2^\frac{1}{2} + 1)^{\frac{3}{2}}\). Equality occurs when \( \frac{1}{\sqrt{2}} \cos \theta = \sin \theta \); that is, \( d = \tan \theta = \frac{1}{\sqrt{2}} \). Thus, the player should aim for \( d = \frac{1}{\sqrt{2}} \).

Part v: (5 points) Use this technique to find the shortest path length after five bounces.

Official solution.

The total path length is \( 3 \left( \frac{1}{\cos \theta} \right) + 2 \left( \frac{1}{\sin \theta} \right) \). Thus, we set \( m = 2 \), \( n = -1 \), \( x_1 = \lambda_1 \cos \theta \), and \( x_2 = \sin \theta \) as before. However, this time we use \( w_2 = 2 \) because of the 2 in front of \( \frac{1}{\sin \theta} \) in the expression for the path length. Incorporating these into the Weighted Power Mean Inequality, we
arrive at

\[
\left( \frac{w_1 \lambda_1^2 \cos^2 \theta + 2 \sin^2 \theta}{w_1 + 2} \right)^{\frac{1}{2}} \geq \left( \frac{w_1 \lambda_1^{-1} (\cos \theta)^{-1} + 2(\sin^2 \theta)^{-1}}{w_1 + 2} \right)^{-1}.
\]

To guarantee nice simplifications, we need \( w_1 \lambda_1^2 = 2 \). To obtain our expression for path length, we need \( w_1 \lambda_1^{-1} = 3 \). Solving these equations produces \( w_1 = \sqrt{18} \) and \( \lambda_1 = \frac{2}{3} \). Using these values in the above inequality yields

\[
\sqrt{\frac{2}{\sqrt{18} + 2}} \geq \frac{\sqrt{18} + 2}{3 \left( \frac{1}{\cos \theta} \right) + 2 \left( \frac{1}{\sin \theta} \right)}.
\]

Multiplying through as usual reveals the desired minimum path length:

\[
\text{path length} \geq \left( 2 + \sqrt{18} \right)^{3/2} / \sqrt{2},
\]

which can be written in the more interesting form

\[
\text{path length} \geq \left( 2^{\frac{3}{2}} + 3^{\frac{3}{2}} \right)^{\frac{3}{2}}.
\]

The five-bounce minimum is attained when \( \sqrt{\frac{2}{3}} \cos \theta = \sin \theta \). Therefore, we must aim for \( d = \sqrt{\frac{2}{3}} \).

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

That ends another Skoliad. Continue to send me your contests and solutions.