Three-Pile Nim with Blocking

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1. Introduction

Nim, also known as Bouton's Nim, is a two-player counter-pickup game that is well known in combinatorial game theory. In this paper we develop a winning strategy for a more complicated variation of Nim, in which exactly one move can be blocked at each stage of the game. Remarkably, the winning strategy for the more complicated version is much simpler than for ordinary (Bouton's) Nim.

Specifically, we explore a three-pile game with two players, a moving player and a blocking player, whose roles alternate between moves. As in ordinary Nim, a move consists of the removal of any number of counters from any single pile. Before each move, including the first move, the blocking player must eliminate exactly one of the moving player's possible moves. For example, if the moving player is confronted with piles of size 6, 10, and 10, the blocking player could forbid the removal of 7 counters from the first of the two 10-counter piles. A forbidden move is forgotten as soon as the next move is made. The winner is the last player to make an allowed move.

2. Bouton's Nim

Before developing the strategy for our game, let us review the strategy for playing Bouton's Nim. The general ideas actually apply to all "last player wins" combinatorial games. The idea is to partition the set $\mathcal{P}$ of all possible positions into two subsets $\mathcal{S}$ and $\mathcal{U}$, where the positions in $\mathcal{S}$ are "safe" to move to and the positions in $\mathcal{U}$ are "unsafe" to move to.

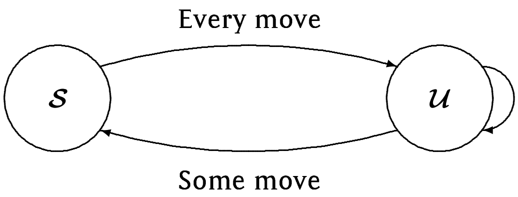
We say that a position v is *accessible* from a position u, and we write $u \longmapsto v$, if there is a move from u to v. Suppose there are two subsets $\mathcal{S}$ and $\mathcal{U}$ which partition $\mathcal{P}$ (that is, $\mathcal{S} \cup \mathcal{U} = \mathcal{P}$ and $\mathcal{S} \cap \mathcal{U} = \phi$) and which possess the following three properties:

(1) From each position v in S, every position accessible from v belongs to $\mathcal{U}$.

(2) From each position u in $\mathcal{U}$, there is at least one position v in $\mathcal{S}$ which is accessible from u.

(3) All terminal positions belong to $\mathcal{S}$.

The sets $\mathcal{S}$ and $\mathcal{U}$ describe a winning strategy. A moving player faced with a position in $\mathcal{U}$ simply moves to a position in $\mathcal{S}$. That player can continue to move to positions in $\mathcal{S}$, ultimately winning the game. Such a strategy is depicted in the diagram below.



In three pile Nim, the members of the set $\mathcal{S}$ can be described as follows. A position in which the piles are of sizes a, b and c is denoted by (a, b, c). Associate with each such position the binary representation of the three integers a, b, c, and align these representations vertically as though we were adding them. If the number of 1's in each column is even, we say that the binary configuration is balanced, and the corresponding position belongs to $\mathcal{S}$. In other words, we take the sum in each column modulo 2. Note that $\mathcal{S}$ and its complement $\mathcal{P} - \mathcal{S}$ satisfy the three properties above.

Consider the game $(13, 15, 17)$.

$$
\begin{array}{rcccccc}
13 & = & & 1 & 1 & 0 & 1 \\
15 & = & & 1 & 1 & 1 & 1 \\
17 & = & \boxed{1} & 0 & 0 & \boxed{0} & \boxed{1} \\
\hline
\text{Mod 2 Sum} & = & 1 & 0 & 0 & 1 & 1
\end{array}
$$

Notice that the first, fourth, and fifth columns have 1's in the bottom row, indicating that these columns have an odd number of 1's. Also note that the three entries in the row with the 17 that are boxed need to be changed so that the columns they occupy become balanced. This can be done by replacing the pile of 17 counters with one having 2 counters. The only winning move is $(13, 15, 17) \longmapsto (13, 15, 2)$. The reason this move is unique is that the 1 in the leftmost column can be eliminated only by a move from the pile with 17 counters. The result can be depicted as:

$$
\begin{array}{rccccc}
13 & = & 1 & 1 & 0 & 1 \\
15 & = & 1 & 1 & 1 & 1 \\
2 & = & & & 1 & 0 \\
\hline
\text{Mod 2 Sum} & = & 0 & 0 & 0 & 0
\end{array}
$$

3. Blocking Nim

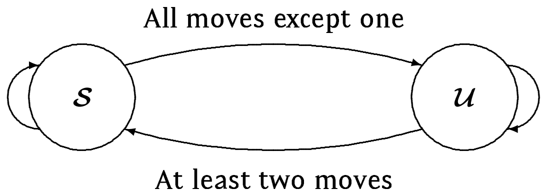
Now we consider a game where one move is blocked at each stage. Remarkably, this apparently more complicated game yields a strategy that does not require binary arithmetic. A solution is a partition $(\mathcal{S}, \mathcal{U})$ of the set $\mathcal{P}$ of positions with the following properties:

(a) Every terminal position belongs to $\mathcal{S}$.

(b) For each position u in $\mathcal{U}$, there are at least two moves from u to positions in $\mathcal{S}$.

(c) For each position v in $\mathcal{S}$, there is at most one position of $\mathcal{S}$ accessible from v.

The figure below shows how the winning strategy for the blocking game differs from that of the ordinary game.



When denoting a position by (a, b, c), we will generally require $a \leq b \leq c$. In the proof, however, we do not always adhere to this convention because the arithmetic makes it difficult to compare the sizes of the piles once a move has been made. Using this notation, there are two terminal positions: $(0, 0, 0)$ and $(0, 0, 1)$. The position $(0, 0, 1)$ is terminal because the move $(0, 0, 1) \longmapsto (0, 0, 0)$ must be blocked.

Theorem 1 . Let $\mathcal{S}$ denote the set of all positions of the form (a, a, a), where $a \geq 0$, together with the positions (a, b, c) such that $a + b + 1 = c$, and let $\mathcal{U} = \mathcal{P} - \mathcal{S}$. Then the partition $(\mathcal{S},\mathcal{U})$ of $\mathcal{P}$ satisfies conditions (a), (b), and (c) above.

Proof: We can write $\mathcal{S}$ as the union of three sets:

$$\begin{aligned} \mathcal{S} &= \{(a, a, a) |\ a \in N\} \cup \{(a, a, c) |\ 2a + 1 = c\} \\ &\qquad \cup \{(a, b, c) |\ a < b \text{ and } a + b + 1 = c\} . \end{aligned}$$

We can write $\mathcal{U}$ as:

$$\begin{aligned} \mathcal{U} &= \{(a, b, c) |\ a = b < c \text{ and } c \neq 2a + 1\} \\ &\qquad \cup \{(a, b, c) |\ a < b \leq c \text{ and } a + b + 1 \neq c\} . \end{aligned}$$

To see condition (a), note that $(0, 0, 0)$ and $(0, 0, 1)$ both belong to $\mathcal{S}$.

Let us show next that property (b) holds. Suppose that (a, b, c) belongs to $\mathcal{U}$. If $a = b < c$ and $c \neq 2a + 1$, then we have two cases to consider: either (i) $2a + 1 > c$ or (ii) $2a + 1 < c$. In case (i), there are two moves to $(c - a - 1, a, c)$, which is a member of $\mathcal{S}$. That is, either of the piles with a counters can be reduced to $c - a - 1$ counters where $c - a - 1 \geq 0$ since $a < c$. In case (ii), there are two moves to positions in $\mathcal{S}$, namely $(a, a, c) \longmapsto (a, a, 2a + 1)$ and $(a, a, c) \longmapsto (a, a, a)$.

On the other hand, if $a < b \leq c$ and $c \neq a + b + 1$, we again consider two cases: (i) $a+b+1 > c$ and (ii) $a+b+1 < c$. If $c > a+b+1$, there are two members of $\mathcal{S}$ we could move to, $(a, b, a + b + 1)$ and $(a, b - a - 1, b)$. The latter position is available because $b - a \geq 1$. In case (ii), the move $(a, b, c) \longmapsto (c-a-1, a, c) \in \mathcal{S}$ is always possible because $0 \leq c-a-1 < b$. Also, the move $(a, b, c) \longmapsto (c - b - 1, b, c) \in \mathcal{S}$ is possible when $b < c$ because $0 \leq c - b - 1 < a$. When $b = c$, there are always two moves $(a, b, b) \longmapsto (a, b - a - 1, b) \in \mathcal{S}$, since $a < b$, and we can reduce either pile with b counters to $b - a - 1$.

To prove property (c), let (a, b, c) belong to $\mathcal{S}$. If $a = b = c$, there is no move to another member of $\mathcal{S}$. If the position is of the form (a, a, b) with $2a + 1 = b$, there is only one move to another position of $\mathcal{S}$, namely (a, a, a), because any reduction in a pile of size a results in a position (e, a, b) that does not satisfy $e + a + 1 = b$. And finally, if (a, b, c) satisfies $a < b$ and $a + b + 1 = c$, then there is no move to a position of the form (a, a, a). There is at most one move to a position for which the sum of the first two smaller pile sizes is 1 less than the third. It would involve taking counters from the largest of the three piles. ∎

4. Open Questions

We do not know how to extend this result to games with more than three piles or to games in which the blocking player can block more than one move. There is another version in which instead of blocking a single move, the blocking player is allowed to block a single position. Thus, for example the move from $(2, 2, 2)$ to $(1, 2, 2)$ could be prohibited. We can solve the three-pile game but we cannot extend this result.

5. References

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