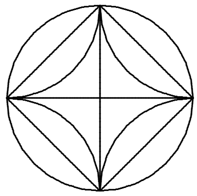
Mayhem Solutions

M57. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Quatre points sont egalement espacés autour d'un cercle ayant un rayon r. Le cercle est donc divisé par 4 arcs égaux. Renversez les arcs en laissant le point du bout en place. Trouvez l'aire de la figure ainsi obtenue.

Solution de Robert Bilinski, Outremont, QC.



Puisque les quatre points sur le cercle sont également espacés, le quadrilatère formé par les quatre points est un carré. On remarque que la différence entre les aires du cercle et du carré est la même qu'entre le carré et l'étoile formée par le renversement des arcs de cercle. Le cercle a pour aire πr^2.

Le carré est formé de quatre triangles isocèles rectangles de côtés égaux r et d'hypoténuse $\sqrt{2}r$. Puisque l'hypoténuse des triangles est le côté du carré, son aire est $\left(\sqrt{2}r\right)^2$, soit $2r^2$. La différence entre l'aire du cercle et l'aire du carré est $(\pi - 2)r^2$. Donc l'aire de l'étoile est $(4 - \pi)r^2$.

M58. *Proposed by the Mayhem Staff.*

Find all positive integers x and y which satisfy the equation $x^y = y^x$.

Solution by Mihály Bencze, Brasov, Romania.

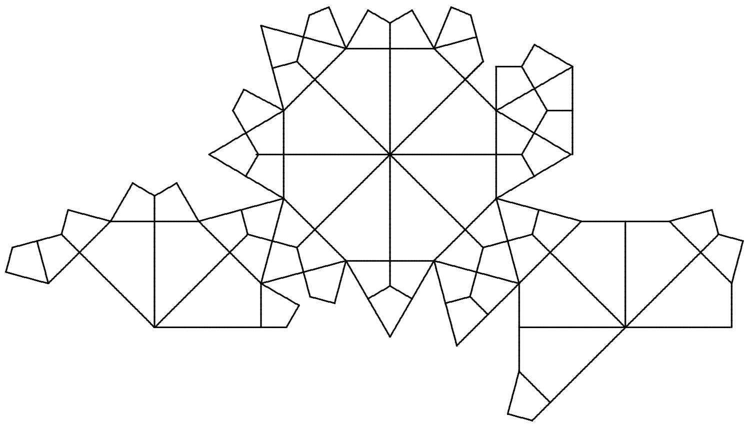
The equation is trivially true if $x = y$. We will search for solutions where $x \neq y$.

The original equation implies $\frac{\ln x}{x} = \frac{\ln y}{y}$. If we let $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1 - \ln x}{x^2}$. Therefore, $f(x)$ is increasing on the interval $(0, e)$ and decreasing on $(e, +\infty)$. Hence, if x, $y \in (0, e)$ and $x > y$, then $f(x) > f(y)$; similarly, if x, $y \in (e, +\infty)$ and $x > y$, then $f(x) < f(y)$. Thus, if $x > y$, we must have $y \in (0, e)$ and $x \in (e, +\infty)$. Checking $y = 1$ and $y = 2$ (the only possible values of y), we find that $(x, y) = (4, 2)$ is a solution with $x \neq y$. Therefore, all possible solutions are:

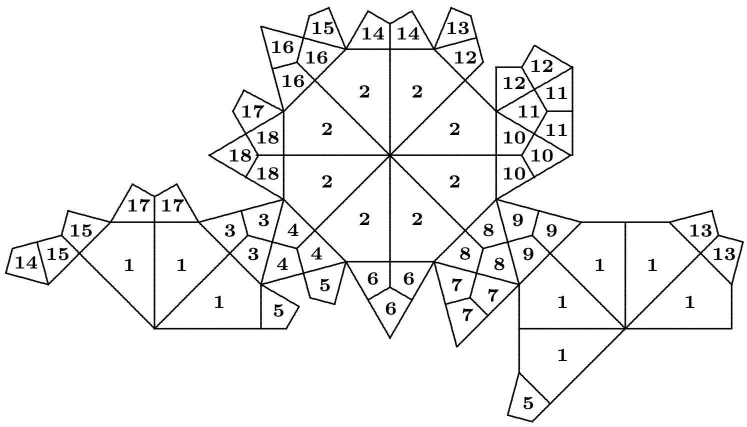
1. $x = y$;
2. $x = 2$, $y = 4$;
3. $x = 4$, $y = 2$.

M59. *Proposed by Izidor Hafner, Tržaška 25, Ljubljana, Slovenia.*

The diagram below represents the net of a polyhedron in which the faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.



Solution by Robert Bilinski, Outremont, QC.



M60. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all positive integers for which $\left\lfloor \sum\limits_{k=1}^{n} \sqrt{k} \right\rfloor = n$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Solution by the proposer.

Let $S_n = \left\lfloor \sum_{k=1}^{n} \sqrt{k} \right\rfloor$. We note that $S_1 = 1$, $S_2 = 2$, $S_3 = 4$. For $n > 3$, $S_n > S_{n-1} + 2$. Thus, $S_n = n$ only for $n = 1, 2$.

M61. *Proposed by theMayhem Staff.*

You are given 54 weights which weigh 1^2, 2^2, 3^2, …, 54^2. Group these into three sets of equal weight.

Solution by Geneviève Lalonde, Massey, ON.

Note that summing 9 consecutive squares n^2, $(n+1)^2$, …, $(n+8)^2$ yields $9n^2+72n+204 = 3(3n^2+24n+68)$. Among these nine squares, we cannot make 3 sets each totalling $3n^2+24n+68$ because $(n+8)^2$ cannot be grouped with two of the other squares to give the desired total (as can be easily checked). If we take

$$\text{Set 1}\left\{\begin{array}{c}(n+1)^2\\(n+3)^2\\(n+8)^2\end{array}\right. ,\qquad \text{Set 2}\left\{\begin{array}{c}n^2\\(n+5)^2\\(n+7)^2\end{array}\right. ,\qquad \text{Set 3}\left\{\begin{array}{c}(n+2)^2\\(n+4)^2\\(n+6)^2\end{array}\right. ,$$

the first two sets each total $3n^2+24n+74$ and the last totals $3n^2+24n+56$.

Therefore, we can break our 54 weights into 6 groups of 9 and use our sets above within each group of 9, making sure that each of our 3 sets contains two of the subsets that total only $3n^2+24n+56$. There are many solutions, one of which is:

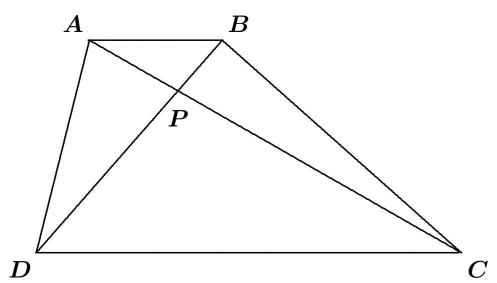
$$\begin{array}{ll}\text{Set 1:} & \{1^2, 6^2, 8^2, 11^2, 13^2, 18^2, 20^2, 22^2, 27^2, 29^2,\\ & \quad 31^2, 36^2, 39^2, 41^2, 43^2, 48^2, 50^2, 52^2\},\\ \text{Set 2:} & \{2^2, 4^2, 9^2, 10^2, 15^2, 17^2, 21^2, 23^2, 25^2, 30^2,\\ & \quad 32^2, 34^2, 38^2, 40^2, 45^2, 47^2, 49^2, 54^2\},\\ \text{Set 3:} & \{3^2, 5^2, 7^2, 12^2, 14^2, 16^2, 19^2, 24^2, 26^2, 28^2,\\ & \quad 33^2, 35^2, 37^2, 42^2, 44^2, 46^2, 51^2, 53^2\}.\end{array}$$

Each of these groups sums to 17 985.

M62. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Let $ABCD$ be a trapezoid where sides AB and CD are parallel and the diagonals AC and BD intersect at point P. Suppose $AB = 50$, $CD = 160$, and the area of triangle PAD is 2000. Determine the area of the trapezoid.

Solution by Geneviève Lalonde, Massey, ON.



From $AB \parallel CD$, we get $\angle PAB = \angle PCD$ and $\angle PBA = \angle PDC$. Thus, $\triangle PAB$ and $\triangle PCD$ are similar. If we name the heights of P from AB and CD as h_1 and h_2, respectively, we get

$$\frac{h_1}{h_2} = \frac{AB}{CD} = \frac{5}{16}.$$

Then $h_1 = 5h$ and $h_2 = 16h$, for some real number h.

Using the notation $[ABC]$ to represent the area of the figure ABC, we have $[ADC] = \frac{1}{2}(160)(21h) = 1680h$. We also have

$$[ADC] = [ADP] + [PDC] = 2000 + \tfrac{1}{2}(160)(16h) = 2000 + 1280h.$$

Setting these two expressions equal, we get $h = 5$; whence, the height of the trapezoid is $21h = 105$. Therefore, $[ABCD] = \frac{1}{2}(50 + 160)(105) = 11\,025$.