SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of LI ZHOU, Polk Community College, Winter Haven, FL, USA from the list of solvers of 2763 and 2764.

2508. [2000 : 46; 2001 : 58–61] *Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan.*

(Corrected) In problem 2408 [1999 : 49; 2000 : 55] we defined a point P to be *Cevic* with respect to $\triangle ABC$ if the vertices D, E, F of its pedal triangle determine concurrent cevians; more precisely, D, E, F are the feet of perpendiculars from P to the respective sides BC, CA, AB, while AD, BE, CF are concurrent.

1. Show that a point D on the line BC can determine 0, 1, 2, or infinitely many positions for E on AC for which P is Cevic.
2. Describe the possible locations of E if D divides the segment BC in the ratio $\lambda : 1 - \lambda$ (when P is Cevic and λ is an arbitrary real number).

II. *A solution of O. Giering, Technische Universität München, München, Germany.*

By coincidence, while ***CRUX with MAYHEM*** readers were working on problem 2508, O. Giering was independently investigating what he calls *concurrence problems for triangles*: Kopunktalitätsprobleme bei Dreiecken, *Sitzungsber. Abt. II, Österreich Akad. Wiss., Math.-Naturwiss. Kl.* **209**, (2000), 3–18. Among the numerous results of that paper he shows that for a given triangle ABC, the locus of Cevic points is a cubic passing through the vertices, the orthocenter H, and the circumcentre O. Moreover, the cubic is centrally symmetric about O, while its asymptotes are the perpendicular bisectors of the sides. Giering also studies the mapping that associates with each Cevic point the intersection point of the three cevians; this mapping is easily seen to fix A, B, C, H, and to take O to the centroid G. Not so easily seen, the image of the cubic under this mapping is another cubic. Some of these results (and others not mentioned here) were discovered or proved with the help of a computer. Indeed, a major theme of the author is that the computer allows us today to extend the confines of "elementary geometry" beyond the study of lines and circles. Those ***CRUX with MAYHEM*** readers who do not have access to Giering's paper (and do not have a computer handy) can explore the special case of the problem where the triangle is isosceles: the cubic degenerates into a hyperbola together with the triangle's line of symmetry.

2766. [2002 : 397] *Proposed by K.R.S. Sastry, Bangalore, India.*

In a Heron triangle, the sides a, b, c satisfy the equation $b = a(a-c)$. Prove that the triangle is isosceles. (A Heron triangle has integer sides and integer area.)

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let a, b, c be the sides of *any* integer-sided triangle where $b = a(a-c)$. Let $a - c = d$. Then $b = ad$ and $c = a - d$. From the Triangle Inequality $a + c > b$. Hence,

$$\begin{aligned} ad &< 2a - d\,, \\ d &< \frac{2a}{a+1} \;<\; 2\,. \end{aligned}$$

Since d is an integer, $d = 1$, and $a = b$. Thus, the triangle is isosceles.

Such triangles exist with integer area. For example, $a = b = 17$ and $c = 16$.

Also solved by PIERRE BORNSZTEIN, Pontoise, France; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Both Bornsztein and the proposer noted that there are infinitely many such triangles.

2768. [2002 : 398] *Proposed by Mohammed Aassila, Strasbourg, France.*

Let x_1, x_2, ..., x_n be n positive real numbers. Prove that

$$\frac{x_1}{\sqrt{x_1x_2+x_2^2}} + \frac{x_2}{\sqrt{x_2x_3+x_3^2}} + \cdots + \frac{x_n}{\sqrt{x_nx_1+x_1^2}} \;\geq\; \frac{n}{\sqrt{2}}\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Define $f(t) = \dfrac{e^t}{\sqrt{e^t+1}}$ for $t \in \mathbb{R}$. Then $f'(t) = e^t\left(\frac{1}{2}e^t+1\right)(e^t+1)^{-\frac{3}{2}}$ and $f''(t) = e^t\left(\frac{1}{4}e^{2t}+\frac{1}{2}e^t+1\right)(e^t+1)^{-\frac{5}{2}} > 0$. So f is strictly convex. Therefore,

$$\begin{aligned} &\frac{x_1}{\sqrt{x_1x_2+x_2^2}} + \frac{x_2}{\sqrt{x_2x_3+x_3^2}} + \cdots + \frac{x_n}{\sqrt{x_nx_1+x_1^2}} \\ &\qquad = f\left(\ln\frac{x_1}{x_2}\right) + f\left(\ln\frac{x_2}{x_3}\right) + \cdots + f\left(\ln\frac{x_n}{x_1}\right) \\ &\qquad \geq nf\left(\frac{\ln\frac{x_1}{x_2} + \ln\frac{x_2}{x_3} + \cdots + \ln\frac{x_n}{x_1}}{n}\right) \\ &\qquad = nf(0) \;=\; \frac{n}{\sqrt{2}}\,, \end{aligned}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL L. DAYAO, Ateneo de Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. There was one incomplete solution.

2769. [2002 : 398] Corrected in [2002 : 532] *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*

In $\triangle ABC$, suppose that $\cos B - \cos C = \cos A - \cos B \geq 0$. Prove that

$$(b^2+c^2)\cos A - (a^2+b^2)\cos C \;\leq\; (c^2-a^2)\sec B\,.$$

Solution by Michel Bataille, Rouen, France.

By hypothesis, $\cos A \geq \cos B \geq \cos C$, so that $A \leq B \leq C$. It follows that $B < \frac{\pi}{2}$; in particular, $\cos B > 0$.

By the Law of Sines, the proposed inequality is equivalent to

$$\begin{aligned} &2(\sin^2 B + \sin^2 C)\cos A\cos B \\ &\quad -2(\sin^2 A + \sin^2 B)\cos C\cos B \\ &\qquad\qquad \leq\; 2(\sin^2 C - \sin^2 A)\,. \end{aligned} \tag{1}$$

Let $X = \cos B - \cos C = \cos A - \cos B$. Then

$$\begin{aligned} 2\cos A\cos B &= \cos^2 A + \cos^2 B - X^2\,, \\ 2\cos C\cos B &= \cos^2 C + \cos^2 B - X^2\,. \end{aligned}$$

Substituting in the left-hand side L of (1) yields

$$\begin{aligned} L &= (\sin^2 A - \sin^2 C)X^2 + \sin(B+A)\sin(B-A) \\ &\qquad + \sin(C+A)\sin(C-A) + \sin(C+B)\sin(C-B)\,, \\ &= (\sin^2 A - \sin^2 C)X^2 \\ &\qquad + \tfrac{1}{2}(\cos 2A - \cos 2B + \cos 2A - \cos 2C + \cos 2B - \cos 2C)\,, \\ &= (\sin^2 A - \sin^2 C)X^2 + 2(\sin^2 C - \sin^2 A)\,. \end{aligned}$$

Thus, (1) is equivalent to $\sin^2 A - \sin^2 C \leq 0$. Now by the observation above, $\sin^2 A - \sin^2 C = \cos^2 C - \cos^2 A = (\cos C - \cos A)(2\cos B) \leq 0$.

This completes the proof.

Also solved by EVANGELINE P. BAUTISTA and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2770. [2002 : 398] *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, MD, USA.*

In $\triangle ABC$, suppose that $a \le b \le c$ and $\angle ABC \ne \frac{\pi}{2}$. Prove that

$$2 + \sec B \;\le\; \left(1 + \tfrac{b}{a}\right)\left(1 + \tfrac{b}{c}\right) .$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

We are given that $a \le b \le c$, which implies that $\angle ABC < \frac{\pi}{2}$. The proof is accomplished by using the following successive transformations of the desired inequality:

$$\begin{aligned}
\sec B &\le \left(1 + \frac{b}{a}\right)\left(1 + \frac{b}{c}\right) - 2 ,\\
\frac{1}{\cos B} &\le \frac{ab + ac + b^2 + bc}{ac} - 2 ,\\
\cos B &\ge \frac{ac}{ab + b^2 + bc - ac} ,\\
\frac{a^2 + c^2 - b^2}{2ac} &\ge \frac{ac}{ab + b^2 + bc - ac} ,\\
\frac{(a+b+c)(a-b+c)}{2ac} - 1 &\ge \frac{ac}{b(a+b+c) - ac} ,\\
\frac{(a+b+c)(a-b+c)}{2ac} &\ge \frac{b(a+b+c)}{b(a+b+c) - ac} ,\\
(a-b+c)[b(a+b+c) - ac] &\ge 2abc ,\\
(a-b+c)b(a+b+c) - a^2c - abc - ac^2 &\ge 0 ,\\
(a+b+c)(ab - b^2 + bc - ac) &\ge 0 ,\\
(a+b+c)(b-a)(c-b) &\ge 0 .
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; WINFER TABARES and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2771★. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all pairs of positive integers a and b such that

$$(a+b)^b \;=\; a^b + b^a .$$

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain (modified slightly by the editor).

Clearly, $(a,1)$ is a solution for all positive integers a. We show that these are the only solution pairs.

Assuming that $b>1$, we have

$$a^b+b^a \;=\; (a+b)^b \;=\; \sum_{k=0}^{b}\binom{b}{k}a^{b-k}b^k > a^b+b^b\,,$$

and thus, $a>b>1$. Let $d=\gcd(a,b)$. Set $a_1=\frac{a}{d}$, $b_1=\frac{b}{d}$. Then we have $a_1>b_1$ and $\gcd(a_1,b_1)=1$, and the given equation becomes

$$\begin{aligned} d^b(a_1+b_1)^b &= d^b a_1^b + d^a b_1^a\,,\\ \text{or}\qquad (a_1+b_1)^b &= a_1^b + d^{a-b}b_1^a\,. \end{aligned} \tag{1}$$

If $d>2$, then (1) has no solutions in positive integers by the Fermat-Wiles Theorem.

If $d=2$, then (1) becomes

$$(a_1+b_1)^{2b_1} \;=\; a_1^{2b_1} + (2^{a_1-b_1}b_1^{a_1})^2\,, \tag{2}$$

which implies that a_1+b_1 and a_1 have the same parity. Thus, b_1 must be even. Let $b_1=2b_2$. Then (2) becomes

$$\left((a_1+b_1)^{b_2}\right)^4 \;=\; (a_1^{b_2})^4 + (2^{a_1-b_1}b_1^{a_1})^2\,.$$

But it is well known that the equation $x^4-y^4=z^2$ has no non-trivial integer solutions. [*Ed:* See, for example, *Number Theory with Computer Applications* by Ramanujachary Kumanduri and Cristina Romero, p. 352.]

If $d=1$, then $(a,b)=1$ and the given equation can be written as

$$\begin{aligned} a^b+\binom{b}{1}a^{b-1}b+\cdots+\binom{b}{b-1}ab^{b-1}+b^b &= a^b+b^a\,,\\ \text{or}\quad a^{b-1}b^2+\frac{b(b-1)}{2}a^{b-2}b^2+\cdots+ab^b+b^b &= b^a\,. \end{aligned} \tag{3}$$

Suppose first that $b>2$. Then $a>3$, and (3) becomes

$$a^{b-1}+\frac{b(b-1)}{2}a^{b-2}+\cdots+ab^{b-2}+b^{b-2} \;=\; b^{a-2}\,. \tag{4}$$

If b has an odd prime divisor p, then $p\,\Big|\,\frac{b(b-1)}{2}$. Hence, (4) implies that $p|a^{b-1}$ and thus, $p|a$. However, this contradicts $(a,b)=1$. Therefore, $b=2^k$ where $k\in\mathbb{N}$. Since we are assuming that $b>2$, we have $k>1$

and $\frac{b(b-1)}{2} = 2^{k-1}(2^k - 1)$, which is even. Then (4) implies that a is even which again is a contradiction.

Hence, $b = 2$, and the given equation becomes $(a+2)^2 = a^2 + 2^a$, or $a + 1 = 2^{a-2}$. By simple induction on n, it is easily seen that $n \leq 2^{n-3}$ for all integers $n \geq 6$, and that $n \neq 2^{n-3}$ for $1 \leq n \leq 5$. Hence, $a + 1 = 2^{a-2}$ has no solutions in integers. Our proof is now complete.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina. A partial solution was submitted by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Guersenzvaig considered a similar problem and showed that for positive integers a, b, and c, the Diophantine equation $(a+b)^b = a^b + b^{ac}$ holds if and only if either $b = 1$ or $(a,b,c) = (1,2,3)$. His proof also used the Fermat-Wiles Theorem.

2772. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\, f\big(yf(x) - 1\big) \;=\; x^2 f(y) - f(x) \text{ for all real } x \text{ and } y.$$

Solution by D. Kipp Johnson, Beaverton, OR, USA.

First we note that the constant function $f(x) = 0$ is a solution of the given equation.

Let $f(x)$ be a solution such that $f(x) \neq 0$ for some x. We show that this implies $f(x) = x$, thereby establishing that the given equation has only two solutions: $f(x) = 0$ and $f(x) = x$.

Letting $x = 0$ in the given equation gives

$$f(0)\left[f\big(yf(0) - 1\big) + 1\right] \;=\; 0\,.$$

Suppose $f(0) \neq 0$. Since $yf(0) - 1$ can take any value x (just replace y by $(x+1)/f(0)$), we obtain a possible candidate for a solution, namely, the constant function $f(x) = -1$. A quick check shows that this function is not a solution of the given equation. Therefore, $f(0) = 0$.

Now, suppose that $f(x) = 0$ for some $x \neq 0$. Then the original equation gives $0 = x^2 f(y)$, which can only happen if $f(y) = 0$ for all y, a contradiction, because we have already assumed that f is not zero everywhere. Thus, we can conclude that $f(x) = 0$ if and only if $x = 0$.

Letting $x = y = 1$ in the original equation gives $f(1)\, f\big(f(1) - 1\big) = 0$, and since $f(1) \neq 0$, we must have $f\big(f(1) - 1\big) = 0$, implying $f(1) - 1 = 0$. Hence, $f(1) = 1$.

When $x = 1$, the original equation then becomes

$$f(y-1) \;=\; f(y) - 1\,. \tag{1}$$

Now, in the original equation, take $y = 1$ and use the equality (1) to obtain

$$\begin{aligned} f(x)\,[f(f(x) - 1)] &= x^2 - f(x)\,, \\ f(x)\,[f(f(x)) - 1] &= x^2 - f(x)\,, \\ f(x)\,f(f(x)) - f(x) &= x^2 - f(x)\,. \end{aligned}$$

Thus,

$$f(x)\,f(f(x)) \;=\; x^2\,. \tag{2}$$

Now, replace x by $x - 1$ in (2), apply (1) three times, and finally apply (2):

$$\begin{aligned} f(x-1)\,f(f(x-1)) &= (x-1)^2\,, \\ (f(x)-1)\,[f(f(x)-1)] &= (x-1)^2\,, \\ (f(x)-1)\,[f(f(x))-1] &= x^2 - 2x + 1\,, \\ f(x)\,f(f(x)) - f(x) - f(f(x)) + 1 &= x^2 - 2x + 1\,, \\ x^2 - f(x) - f(f(x)) + 1 &= x^2 - 2x + 1\,. \end{aligned}$$

Therefore,

$$f(x) + f(f(x)) \;=\; 2x\,. \tag{3}$$

Now, (2) and (3) form a system of two equations in the unknowns $f(x)$ and $f(f(x))$. Eliminating $f(f(x))$ gives

$$[x - f(x)]^2 \;=\; 0\,,$$

so that $f(x) = x$, as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; PAUL DAYAO, Ateneo the Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incomplete solution submitted.

2773. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China and Wu Kang, South China Normal University, Guang Zhou City, Guang Dong Province, China.*

Find all sequences of integers $x_1, x_2, \ldots, x_n, \ldots$, such that ij divides $x_i + x_j$ for any two distinct positive integers i and j.

Solution by D. Kipp Johnson, Beaverton, OR, USA.

The only sequence is $x_n = 0$ for all n. By hypothesis, $1 \cdot n$ divides $x_1 + x_n$, and $2 \cdot n$ divides $x_2 + x_n$ for all positive integers $n > 2$. This implies that n divides the difference $x_1 + x_n - (x_2 + x_n) = x_1 - x_2$ for all such n. But the only integer divisible by an infinite set of integers is 0. Thus, $x_1 - x_2 = 0$. A similar argument shows that $x_k = x_{k+1}$ for all positive k, and all the terms of the sequence are equal.

Now $1 \cdot n$ divides $x_1 + x_n = 2x_1$ for all $n > 1$, which means that $x_1 = 0$, and all the terms are therefore zero.

Also solved by REY A. BARCELON and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL L. DAYAO, Ateneo de Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2774. [2002 : 399] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Let x be a real number such that $0 < x \leq \frac{2}{9}\pi$. Prove that

$$(\sin x)^{\sin x} < \cos x .$$

(This is a generalization of Problem 10261 in the American Mathematical Monthly [1992 : 872, 1994 : 690]).

Solution by Kee-Wai Lau, Hong Kong, China.

For $0 < t < 1$, let $f(t) = 2t \ln t - \ln(1 - t^2)$. Then by simple computations, $f''(t) = \frac{2}{t} + \frac{2(1+t^2)}{(1-t^2)^2} > 0$ and therefore, f is convex on $(0, 1)$. Since $\lim\limits_{t \to 0^+} f(t) = 0$ by l'Hospital's Rule, and $f\left(\sin \frac{2\pi}{9}\right) \approx -0.035 < 0$, we see that $f(t) < 0$ for $0 < t < \sin \frac{2\pi}{9}$. Setting $t = \sin x$ we conclude that $2 \sin x \ln(\sin x) - \ln(\cos^2 x) < 0$ or $(\sin x)^{\sin x} < \cos x$ for $0 < x < \frac{2\pi}{9}$, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one partly incorrect solution.

Both Guersenvaig and Janous showed that the given inequality is, in fact, true for all x such that $0 < x < c$, where c is a constant with value slightly larger than $\frac{2\pi}{9} \approx 0.69813$. Guersenvaig gave $c \approx \frac{2.0392\pi}{9} \approx 0.711815108$ while Janous estimated (by "numeric methods") that $c \approx 0.71182794$.

2775. [2002 : 455] *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

In $\triangle ABC$, let M be the mid-point of BC. Prove that

$$\cos\left(\frac{B-C}{2}\right) \geq \sin(\angle AMB) \geq 8 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) .$$

Solution by Michel Bataille, Rouen, France.

(a) Proof of $\cos\left(\frac{B-C}{2}\right) \geq \sin(\angle AMB)$.

Without loss of generality, suppose that $B \geq C$ (hence, $b \geq c$). Denote $AH = h$, $AD = d$, and $AM = m$ (the altitude, internal bisector and median from A to BC, respectively). Since $\angle BAH = \frac{\pi}{2} - B = \frac{A+C-B}{2} \leq \frac{A}{2}$ and $BD = \frac{ac}{b+c} \leq \frac{ac}{2c} = BM$, going along BC from B, we meet the points H, D, and M in that order. It follows that $\angle HAD = \frac{A}{2} - (\frac{\pi}{2} - B) = \frac{B-C}{2}$. Thus, $\cos\left(\frac{B-C}{2}\right) = \frac{h}{d}$, while $\sin(\angle AMB) = \frac{h}{m}$. Since $d \leq m$ (because $HD \leq HM$), we obtain $\cos\left(\frac{B-C}{2}\right) \geq \sin(\angle AMB)$.

There is equality if and only if $d = m$; that is, when $\triangle ABC$ is isosceles with $B = C$.

(b) Proof of $\sin(\angle AMB) \geq 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$.

The area of $\triangle AMB$ is $\frac{1}{2}\cdot\frac{a}{2}\cdot m\sin(\angle AMB)$ as well as $\frac{1}{2}\cdot\frac{a}{2}\cdot c\sin B$. Hence, $\sin(\angle AMB) = \frac{c\sin B}{m} = \frac{2R\sin B\sin C}{m}$. Thus, the proposed inequality is successively equivalent to

$$\begin{aligned} R\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) &\geq m\sin\left(\frac{A}{2}\right) \\ 2R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) &\geq m\sin A \\ \frac{R}{2}(\sin A + \sin B + \sin C) &\geq m\sin A \\ Rs &\geq am\,, \end{aligned}$$

where R and s denote the circumradius and the semiperimeter of $\triangle ABC$, respectively. This completes the proof since the last inequality is known: two proofs (by Li Zhou and Mihàly Bencze) can be found in *Mathematics Magazine* Vol. 75 No 2, April 2002, pp. 148–9 as solutions to problem 1620.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

2776. [2002 : 456] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, we have

(a) $a < b < c$,

(b) D is a point on the segment AC such that $CD = a$,

(c) E is a point on the segment AB such that $BE = a$,

(d) F is a point on CB produced such that $BF = b - a$,

(e) G is a point on the segment AB such that $AG = b$,

(f) H is a point on AC produced such that $AH = c$,

(g) K is a point on BC produced such that $BK = c$,

(h) M_1, M_2, and M_3, are the circumcentres of $\triangle ADE$, $\triangle BFG$ and $\triangle CHK$, respectively.

We denote the circumcentre by O, the circumradius by R, the incentre by I and the inradius by r. Prove that

(1) $DE \parallel FG \parallel HK$.

(2) $\triangle ADE$, $\triangle BFG$ and $\triangle CHK$ have equal circumradii, ρ.

(3) Show that $\rho = OI$.

(4) $DE : FG : HK = a : b : c$.

(5) Show that the circle centre I, radius R, passes through M_1, M_2, and M_3.

Editor's comment. Several readers added two further properties to our list:

(6) $DE, FG, HK \perp OI$.

(7) $M_1 I \perp BC$, $M_2 I \perp CA$, $M_3 I \perp AB$.

I. *Solution to the original five parts by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

(1) and (4). Since $\frac{DC}{AC} = \frac{a}{b} = \frac{BC}{FC}$, we have $DB \| AF$. Hence, $\angle DBE = \angle FAG$. Also, $\triangle BDE \sim \triangle AFG$ since $\frac{BD}{AF} = \frac{DC}{AC} = \frac{a}{b} = \frac{BE}{AG}$. Consequently, $DE \| FG$ and $\frac{DE}{FG} = \frac{a}{b}$. Similarly, $FG \| HK$ and $\frac{FG}{HK} = \frac{b}{c}$.

(2). By the Law of Sines, the circumradius of $\triangle ADE$ is

$$\frac{DE}{2\sin A} = \frac{a}{b} \cdot \frac{FG}{2\sin A} = \frac{\sin A}{\sin B} \cdot \frac{FG}{2\sin A} = \frac{FG}{2\sin B},$$

which is the circumradius of $\triangle BFG$. Similarly, $\triangle CHK$ also has the same circumradius ϱ.

(3). Let L and M be the mid-points of AB and AC, respectively. Suppose that the incircle of $\triangle ABC$ is tangent to AB at Z and to AC at Y (so that $IZ \perp AB$ and $IY \perp AC$). Locate S and T so that IZ and IY are the perpendicular bisectors of OS and OT, respectively. Then $OS \| AB$ and $OS = 2LZ = 2AZ - 2AL = (b + c - a) - c = b - a = AD$. Similarly, $OT \| AC$ and $OT = c - a = AE$. Thus, $\triangle OST \cong \triangle ADE$, whence $OI = AM_1 = \varrho$. (This argument provides an alternative proof to (2) without the use of trigonometry, since $BM_2 = CM_3 = OI$ by analogous arguments.)

(5) Let D', E', and M_1', be the reflections of D, E, and M_1 in the line AI. Then $\triangle OST$ is a translation of $\triangle AD'E'$. Hence, (recalling that I is the circumcentre of $\triangle OST$ because it was defined in (3) to be on the perpendicular bisectors of the sides), $R = OA = IM_1' = IM_1$. Similarly, $IM_2 = IM_3 = R$.

II. *Solution to parts* (6) *and* (7) *by Toshio Seimiya, Kawasaki, Japan.*

(6). Put $a + b + c = 2s$ and let the incircle be tangent to BC, CA, AB at X, Y, Z, respectively. Since $CD = CB$ and $CY = CX$, we get $DY = CD - CY = CB - CX = BX = s - b$. Similarly, we have $EZ = CX = s - c$. Since $IY \perp AC$, $IZ \perp AB$, and $IY = IZ$, we have

$$\begin{aligned} ID^2 - IE^2 &= (IY^2 + DY^2) - (IZ^2 + EZ^2) \;=\; DY^2 - EZ^2 \\ &= (s-b)^2 - (s-c)^2 \;=\; (2s-b-c)(c-b) \\ &= a(c-b)\,. \end{aligned} \qquad (1)$$

The powers of D and of E with respect to the circumcircle of triangle ABC are $AD \cdot DC = OA^2 - OD^2$ and $AE \cdot EC = OA^2 - OE^2$. It follows that

$$\begin{aligned} OD^2 - OE^2 &= AE \cdot EB - AD \cdot DC \\ &= (c-a)a - (b-a)a \;=\; a(c-b)\,. \end{aligned} \qquad (2)$$

From equations (1) and (2), we have $ID^2 - IE^2 = OD^2 - OE^2$, so that $IO \perp DE$. In the same way, we get $IO \perp FG$ and $IO \perp HK$. Of course, this argument also provides an alternative treatment of part (1).

(7). Using the powers of B and C with respect to circle ADE, we obtain $BE \cdot BA = BM_1^2 - AM_1^2$ and $CD \cdot CA = CM_1^2 - AM_1^2$. We therefore have

$$\begin{aligned} BM_1^2 - CM_1^2 &= (BM_1^2 - AM_1^2) - (CM_1^2 - AM_1^2) \\ &= BE \cdot BA - CD \cdot CA \;=\; ac - ab = a(c-b)\,. \end{aligned} \qquad (3)$$

Since $IX \perp BC$, we have

$$BI^2 - CI^2 \;=\; BX^2 - CX^2 \;=\; (s-b)^2 - (s-c)^2 \;=\; a(c-b)\,. \qquad (4)$$

Thus, from equations (3) and (4) we have $BM_1^2 - CM_1^2 = BI^2 - CI^2$, so that $M_1I \perp BC$. Likewise, $M_2I \perp CA$ and $M_3I \perp AB$.

*Also solved by *MICHEL BATAILLE, Rouen, France; *MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK* (*omitted part* (5))*; JOEL SCHLOSBERG, student, New York University, NY, USA; *PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer. An asterisk indicates that the solver included a proof of parts* (6) *or* (7).

Woo wondered how a configuration with so many noteworthy properties had not been discovered long ago. He added the suggestion that readers with suitable computer graphics might want to draw the figure so that the vertices of ABC are free to move around the circumcircle while it and the incircle remain fixed. The circles with centres M_i move in orbit about the incircle while triangle ABC changes its size and shape.

Comment based on the solution of Benito, Ciaurri, and Fernández. It follows from part (5) that the common ratio in part (4) is $\sqrt{1-\frac{2r}{R}}$:

$$\frac{DE}{a} = \frac{FG}{b} = \frac{HK}{c} = \frac{OI}{R} = \frac{\sqrt{R^2-2Rr}}{R} = \sqrt{1-\frac{2r}{R}}\,.$$

2777. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

For $x \geq 0$, let $y(x)$ represent the only real root of the equation $y^3+26xy=27$. Prove that the function $x \mapsto y(x)$ is strictly decreasing on $[0,1]$. Also, find the value of $\int_0^1 y^2(x)\,\ln y(x)\,dx$.

Solution by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.

We shall use the notation $y=f(x)$. We use the inverse function to show that the integral has the value $(243\cdot\ln 3-128)/52 \approx 2.67$.

For $y \neq 0$, we solve the equation $y^3+26xy=27$ for x to get

$$x = g(y) = \frac{1}{26}\left(\frac{27-y^3}{y}\right)\,.$$

Since $x=g(y)<0$ for $y<0$ and we are only interested in $x \in [0,1]$, we look at values of y with $0<y$. Now $g'(y) = -\frac{27+2y^3}{26y^2} < 0$ for $0<y$, so that g is strictly decreasing (hence one-to-one) and differentiable (with non-zero derivative) on this interval. By inspection the ordered pairs $(1,1)$ and $(3,0)$ are on the graph of $x=g(y)$. Hence, the only part of the curve we care about is $1 \leq y \leq 3$. Restricting $g(y)$ to the interval $[1,3]$ gives a one-to-one, onto function $g:[1,3]\to[0,1]$. Therefore, the inverse $y=f(x)$, $f:[0,1]\to[1,3]$, exists and is also strictly decreasing, onto, and differentiable, with derivative being

$$f'(x) = \frac{1}{g'(f(x))} = -\frac{26[f(x)]^2}{27+2[f(x)]^3}\,.$$

In the integral, make the (one-to-one) substitution $y=f(x)$, so that

$$dy = f'(x)\,dx = -\frac{26[f(x)]^2}{27+2[f(x)]^3}\,dx\,,$$

which implies

$$-\frac{1}{26}(27+2y^3)\,dy = [f(x)]^2\,dx\,.$$

This substitution transforms the integral into

$$I = \int_0^1 [f(x)]^2\ln[f(x)]\,dx = \frac{1}{26}\int_1^3 (27+2y^3)\ln y\,dy\,.$$

Using tables or integration by parts, we find

$$\begin{aligned} I &= \frac{1}{26}\left\{\left(27y+\frac{y^4}{2}\right)\ln y - 27y - \frac{y^4}{8}\right\}\Bigg|_1^3 \\ &= \frac{1}{52}\{243\cdot\ln 3 - 128\}\,. \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; ROBERT BILINSKI, Outremont, QC (first part only); PAUL BRACKEN, Concordia University, Montréal, QC; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANGEL JOVAL ROQUET, La Seu d'Urgell, Spain; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Woo observed that one requires neither calculus nor any elaborate argument to show that $y(x)$ is strictly decreasing on $[0,1]$. The value $x = \frac{27-y^3}{26y}$ clearly decreases as y increases through its positive values because the numerator decreases while the denominator increases.

Since the problem stated that y is the only real root of the given equation for x a fixed non-negative number, there was no need for us to verify it. Nevertheless, many solvers did verify it, perhaps because uniqueness is critical for the integration. Descartes' Rule of Signs does the job simply. Alternatively, many solvers simply restricted the domain and range to the first quadrant as in the featured solution. The situation is different when x is a sufficiently large negative number; in that case the given cubic equation will be satisfied by two negative values of y in addition to its one positive value.

2778. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $z \neq 1$ is a complex number such that $z^n = 1$ $(n \geq 1)$. Prove that

$$\big|nz - (n+z)\big| \;\leq\; \frac{(n+1)(2n+1)}{6}\,|z-1|^2\,.$$

Preliminary comment. The original proposal from Bencze was to show that

$$|nz - (n+2)| \;\leq\; \frac{(n+1)(2n+1)}{6}|z-1|^2\,.$$

The editor mistakenly turned the first 2 into a z. Happily, the modified problem is still of interest, although it has lost some of its intuitive meaning.

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.

Differentiating the familiar identity

$$\sum_{k=0}^{n} x^k \;=\; \frac{x^{n+1}-1}{x-1}$$

with respect to x, we get

$$\sum_{k=1}^{n} kx^{k-1} \;=\; \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}\,.$$

Multiplying both sides by x and differentiating again, we arrive at

$$\sum_{k=1}^{n} k^2x^{k-1} \;=\; g(x)\,,$$

where

$$g(x) \;=\; \frac{n^2x^{n+2}-(2n^2+2n-1)x^{n+1}+(n+1)^2x^n-x-1}{(x-1)^3}\,.$$

Taking $x = z$ and using $|z| = 1$ (which we were given), we obtain

$$|g(z)| \;\le\; \sum_{k=1}^{n} k^2|z|^{k-1} \;=\; \frac{n(n+1)(2n+1)}{6}\,. \tag{1}$$

On the other side, taking into account that $z^n = 1$, $z \neq 1$, we get

$$g(z) \;=\; \frac{n(nz^2-2(n+1)z+n+2)}{(z-1)^3} \;=\; \frac{n(nz-(n+2))}{(z-1)^2}\,. \tag{2}$$

From (1) and (2) we therefore conclude that

$$|nz-(n+2)| \;\le\; \frac{(n+1)(2n+1)}{6}|z-1|^2\,.$$

This was the inequality that the proposer had intended for us to verify. For the inequality as it appears in our problem, it remains to show that

$$|nz-(n+z)| \;<\; |nz-(n+2)|\,,$$

where $|z| = 1$ and n is a positive integer. This is a routine calculation, comparing the square of both sides and using $z\overline{z} = |z|^2 = 1 \ge \mathrm{Re}(z)$. Alternatively, a sketch of the circles $|nz-(n+z)|$ and $|nz-(n+2)|$ (for $z = e^{it}$, $0 \le t < 2\pi$) makes the inequality clear. Note that the inequality is strict.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Janous proved the stronger inequality:

$$|nz-(n+z)| \;\le\; \frac{n^2}{3}|z-1|^2\,.$$

He pointed out that even his inequality is far from best possible, and conjectured that the best factor of $|z-1|^2$ on the right side would be

$$\frac{\sqrt{4n(n-1)\sin^2\left(\frac{\pi}{n}\right)+1}}{4\sin^2\left(\frac{\pi}{n}\right)}\,.$$

2779. [2002 : 457] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $\alpha \in \mathbb{R}^*$ and $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function for which $f(x) \neq \alpha x \;\forall\; x \in \mathbb{R}$. Prove that there is a sequence $\{x_n\}$ for which $\lim\limits_{n\to\infty} f'(x_n) = \alpha$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Put $g(x) = f(x) - \alpha x$. Then g is differentiable and $g(x) \neq 0$ for all $x \in \mathbb{R}$. It is clear that we need to show there exists a sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} g'(x_n) = 0$.

If $g'(c) = 0$ for some c, then we are done by taking $x_n \equiv c$. Now suppose that $g'(x) \neq 0$ for all $x \in \mathbb{R}$. We have four cases: (1) $g > 0$ and $g' > 0$; (2) $g > 0$ and $g' < 0$; (3) $g < 0$ and $g' > 0$; (4) $g < 0$ and $g' < 0$. In all cases, either $y_n = g(n)$ or $y_n = g(-n)$ is a bounded monotone sequence; thus, $\{y_n\}$ converges. By the Mean Value Theorem, there exists x_n in $(n, n+1)$ or $(-n-1, -n)$ such that $g'(x_n) = y_{n+1} - y_n$ or $g'(x_n) = y_n - y_{n+1}$. Since $\lim\limits_{n\to\infty} |y_{n+1} - y_n| = 0$, we have $\lim\limits_{n\to\infty} g'(x_n) = 0$.

[*Ed.* Most solvers noted that the result holds for $\alpha = 0$. Also, implicit in this solution, and explicitly stated in most solutions, is the fact that the Intermediate Value Theorem holds for derivatives.]

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; HÉCTOR P. PÉREZ and NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer. There were 2 incorrect solutions.