SKOLIAD No. 72

Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are looking for solutions especially from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 April 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 6** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The first item this issue comes from the 2003 Maritime Mathematics Competition written on March 25, 2003. My thanks go out to David Horrocks of the University of Prince Edward Island for forwarding the material to me. We especially invite students in grade 10 (or equivalent) or earlier to send solutions.

2003 Maritime Mathematics Competition
Concours de Mathématiques des Maritimes 2003

1. Si un homme partage un certain nombre de bonbons entre ses enfants, chaque enfant en reçoit quinze et il en reste un. S'il partage le même nombre de bonbons entre ses enfants et deux de leurs amis, chaque enfant en reçoit onze et il en reste trois. De combien de bonbons s'agit-il ?

When a father distributes a number of candies among his children, each child receives 15 candies and there is one left over. If, however, two friends join the group and the candies are redistributed, then each child receives 11 candies and there are three left over. What is the total number of candies?

2. Pour chaque entier strictement positif n, posons

$$f(n) = \left(4(1)^2 - 1\right) \times \left(4(2)^2 - 1\right) \times \cdots \times \left(4n^2 - 1\right).$$

Par example, $f(1) = 3$ et $f(2) = 3 \times 15 = 45$.

Trouver toutes les valeurs de n pour lesquelles $f(n)$ est un carré parfait.

For any positive integer n, define

$$f(n) \;=\; \left(4(1)^2-1\right)\times\left(4(2)^2-1\right)\times\cdots\times\left(4n^2-1\right).$$

For example, $f(1)=3$ and $f(2)=3\times 15=45$.

Find all values of n for which $f(n)$ is a perfect square.

3. Une échelle longue de dix mètres est placée contre un mur vertical. Si le point milieu de l'échelle est deux fois plus distant du sol que du mur, à quelle hauteur l'échelle s'appuie-t-elle contre le mur ?

. . . .

A 10-metre ladder rests against a vertical wall. The mid-point of the ladder is twice as far from the ground as it is from the wall. What height on the wall does the ladder reach?

4. Trouver un nombre à six chiffres dont le premier chiffre est 1 et qui devient trois fois plus grand si le premier chiffre est déplacé à l'autre bout pour devenir le chiffre des unités.

. . . .

Find a 6-digit number with first (that is, leftmost) digit 1 such that if the first digit is transferred to the right, then the number so obtained is three times the original number.

5. Évaluer

$$\sqrt[3]{5+2\sqrt{13}}+\sqrt[3]{5-2\sqrt{13}}\,.$$

. . . .

Evaluate

$$\sqrt[3]{5+2\sqrt{13}}+\sqrt[3]{5-2\sqrt{13}}\,.$$

6. Trouver toutes les paires d'entiers positifs (x,y) telles que

$$x^2-11y! \;=\; 2003\,.$$

(Par définition, $1!=1$, $2!=1\cdot 2=2$, $3!=1\cdot 2\cdot 3=6$, etc.)

. . . .

Find all pairs of positive integers (x,y) such that

$$x^2-11y! \;=\; 2003\,.$$

(Note that $1!=1$, $2!=(1)(2)=2$, $3!=(1)(2)(3)=6$, etc.)

Next we present the solutions to the 2002 Manitoba Mathematical Contest that appeared in [2003 : 3].

MANITOBA MATHEMATICAL CONTEST, 2002

For students in Senior 4

9:00 a.m. – 11:00 a.m. Wednesday, February 20, 2002

Sponsored by

The Actuaries' Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society, and The University of Manitoba

1. (a) (∗) Solve the equation $x^4 - 3x^2 + 2 = 0$.

(b) (∗) Solve the equation $\frac{4}{(x-3)^2} - \frac{4}{(x-3)} + 1 = 0$.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

(a)

$$\begin{aligned} x^4 - 3x^2 + 2 &= 0\,, \\ (x^2-1)(x^2-2) &= 0\,. \end{aligned}$$

Thus, $x^2 - 1 = 0$ or $x^2 - 2 = 0$, which means that $x = \pm 1$ or $x = \pm\sqrt{2}$.

(b) From the equation

$$\frac{4}{(x-3)^2} - \frac{4}{x-3} + 1 \;=\; 0\,,$$

we know that $x \neq 3$. Multiplying both sides by $(x-3)^2$, we obtain

$$\begin{aligned} 4 - 4(x-3) + (x-3)^2 &= 0\,, \\ 4 - 4x + 12 + x^2 - 6x + 9 &= 0\,, \\ x^2 - 10x + 25 &= 0\,, \\ (x-5)^2 &= 0\,. \end{aligned}$$

Thus, $x = 5$ is the only root.

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

2. (a) (∗) Solve the equation $9x^3 - 9x^2 - 4x + 4 = 0$.

(b) (∗) Thirty-six students took a final exam. The average score of those who passed was 60, the average score of those who failed was 42 and the average of all the scores was 53. How many students did not pass the exam?

Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

(a) The equation factors to

$$\begin{aligned} (x-1)(9x^2-4) &= 0\,, \\ (x-1)(3x+2)(3x-2) &= 0\,. \end{aligned}$$

Thus, the solutions are $x = 1, \pm\frac{2}{3}$.

(b) Let p and f be the number of students who passed and failed the exam, respectively. Then

$$p + f = 36, \quad \text{or} \quad p = 36 - f,$$

and

$$\frac{60p + 42f}{36} = 53, \quad \text{or} \quad 10p + 7f = 318.$$

Substituting the first equation into the second yields

$$\begin{aligned} 10(36 - f) + 7f &= 318, \\ 3f &= 42, \end{aligned}$$

or $f = 14$. Therefore, 14 students failed the exam.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

3. (a) ($*$) The area of a rectangle is 3 and its perimeter is 7. What is the length of the diagonal of this rectangle?

(b) ($*$) In this problem O is the origin, A is the point $(3, 4)$ and B is a point in the first quadrant on the line joining O and A. If the length of AB is 6 what are the coordinates of B?

(a) *Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.*

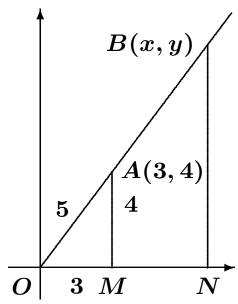
Let l and w represent the length and width of the rectangle. We know that $lw = 3$ and $2(l + w) = 7$, or $l + w = \frac{7}{2}$. Squaring the second equation, the result is $l^2 + w^2 + 2lw = \frac{49}{4}$. Subtracting twice the first equation from this gives $l^2 + w^2 = \frac{49}{4} - 6 = \frac{25}{4}$. Thus, by the Pythagorean Theorem, the length of the diagonal is $\sqrt{\frac{25}{4}} = \frac{5}{2}$.

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

(b) *Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.*

Triangle AMO is a right-angled 3-4-5 triangle. Also, triangles AMO and BNO are similar, since they are right-angled and share an angle at O. If we let (x, y) be the coordinates of B, we get

$$\begin{aligned} \frac{OB}{OA} &= \frac{NO}{MO} = \frac{BN}{AM}, \\ \frac{11}{5} &= \frac{x}{3} = \frac{y}{4}. \end{aligned}$$



Thus, $x = \frac{33}{5}$ and $y = \frac{44}{5}$, and $B = \left(\frac{33}{5}, \frac{44}{5}\right)$.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

4. (a) (∗) Solve the equation $\sqrt{3-x}+\sqrt{12-4x}=\sqrt{x-1}$.

(b) (∗) If p, q and r are the three roots of the equation $x^3-7x^2+3x+1=0$, find the value of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$.

Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

(a)

$$\begin{aligned}\sqrt{3-x}+\sqrt{12-4x} &= \sqrt{x-1}\,,\\ \sqrt{3-x}+2\sqrt{3-x} &= \sqrt{x-1}\,,\\ 3\sqrt{3-x} &= \sqrt{x-1}\,,\\ 9(3-x) &= x-1\,,\\ 27-9x &= x-1\,,\\ x &= \frac{14}{5}\,.\end{aligned}$$

Substituting this value into the original equation verifies that it is a solution.

(b) We have

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = \frac{pq+pr+pq}{pqr}\,.$$

For any polynomial equation of the form $ax^3+bx^2+cx+d=0$, where p, q, and r are the roots, we have $qr+pr+pq=c/a$ and $pqr=-d/a$. Thus, for $x^3-7x^2+3x+1=0$, we have

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = \frac{pq+pr+pq}{pqr} = \frac{c/a}{-d/a} = \frac{3}{-1} = -3\,.$$

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

5. (a) (∗) If a and b are real numbers such that $\sqrt{a}-\sqrt{b}=\sqrt{2}$ and $a-b=10$, find a and b.

(b) (∗) If k is a real number such that $3\left(2^{k+3}\right)-2^{2k}=128$, what are the possible values of k?

(a) *Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.*

Squaring the first equation yields $a+b-2\sqrt{ab}=2$. The second equation is equivalent to $a=10+b$. Putting these two together gives

$$\begin{aligned}10+b+b-2\sqrt{b(10+b)} &= 2\,,\\ 2b+8 &= 2\sqrt{b(10+b)}\,,\\ b+4 &= \sqrt{b(10+b)}\,.\end{aligned}$$

Squaring both sides gives $b^2+8b+16=b^2+10b$, which simplifies to $b=8$. This implies that $a=18$. Substituting into the original equations verifies that $(a,b)=(18,8)$ is a solution.

(b) *Official Solution*.

From the original equation, we get

$$\begin{aligned} 3\left(2^{k+3}\right)-2^{2k} &= 128\,, \\ 3\left(2^{k}\right)\left(2^{3}\right)-\left(2^{k}\right)^{2} &= 128\,, \\ \left(2^{k}\right)^{2}-24\left(2^{k}\right)+128 &= 0\,, \\ \left(2^{k}-8\right)\left(2^{k}-16\right) &= 0\,. \end{aligned}$$

It follows that $2^k = 8$ or $2^k = 16$, and $k = 3$ or $k = 4$.

Also solved by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON; and Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

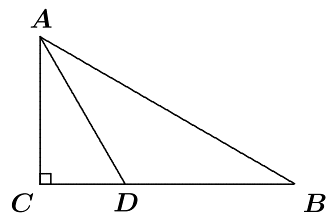
6. (a) In triangle ABC, $\angle BAC = 60^\circ$, $\angle ACB = 90^\circ$, and D is on BC. If AD bisects $\angle BAC$, prove that $DB = 2CD$.

(b) In triangle ABC, $AC = BC = 5$, and $AB = 8$. What is the radius of the circle which passes through A, B, and C?

Solution by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON.

(a) In $\triangle BAC$ we have

$$AC = AB\cos 60^\circ\,.$$



That is, $\frac{AB}{AC} = 2$. By the Angle Bisector Theorem,

$$\frac{DB}{DC} = \frac{AB}{AC} = 2\,.$$

Therefore, $DB = 2CD$.

(b) Drop a perpendicular from C to meet AB at D. By symmetry, D is the mid-point of AB. Thus, $AD = DB = 4$. Also by symmetry, $\angle ADC = \angle BDC = 90^\circ$. Thus, since $\triangle CDA$ is right-angled, $\sin A = \frac{3}{5}$. By the Sine Rule,

$$\frac{a}{\sin A} = 2R\,,$$

where R is the radius we seek and $a = BC = 5$. Then

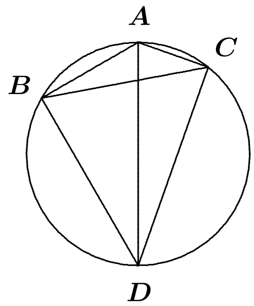
$$R = \frac{5}{2\left(\frac{3}{5}\right)} = \frac{25}{6}\,.$$

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

7. A, B, and C are points on a circle of radius 3. In triangle ABC, $\angle ACB = 30^\circ$ and $AC = 2$. Find BC.

Solution by Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

Let D be the point on the circle such that AD is a diameter. The length AD is then 6. Since $\angle ADB$ subtends the same arc as $\angle ACB$, we have $\angle ADB = 30^\circ$. Since AD is a diameter, we have $\angle ABD = 90^\circ$. Then $\angle BAD = 60^\circ$. Therefore,



$$\begin{aligned} AB : BD : AD &= 1 : \sqrt{3} : 2 \\ &= 3 : 3\sqrt{3} : 6 \,. \end{aligned}$$

Thus, $AB = 3$ and $BD = 3\sqrt{3}$.

Now, $\angle ACD = 90^\circ$, since AD is a diameter. Using the Pythagorean Theorem in $\triangle ADC$ yields

$$CD^2 = 6^2 - 2^2 = 32\,;$$

that is, $CD = \sqrt{32} = 4\sqrt{2}$. Thus, by Ptolemy's Theorem, we have

$$\begin{aligned} AB \cdot CD + BD \cdot AC &= AD \cdot BC\,, \\ 12\sqrt{2} + 6\sqrt{3} &= 6 \cdot BC\,, \\ BC &= 2\sqrt{2} + \sqrt{3}\,. \end{aligned}$$

Also solved by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON; and Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

8. If x and y are real numbers, prove that $x^3y + xy^3 \le x^4 + y^4$.

Solution by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON.

Without loss of generality, assume that $x \le y$; then, $x^3 \le y^3$. Let $a_n \in \{x^3,\, y^3\}$ and $b_n \in \{x,\, y\}$ for $n \in \{1,\, 2\}$. By the Rearrangement Inequality, $S = a_1b_1 + a_2b_2$ is maximized when the sequences are similarly ordered, and minimized when the sequences are oppositely ordered. In our case,

$$(x^3)(y) + (y^3)(x) \le S \le (x^3)(x) + (y^3)(y)\,.$$

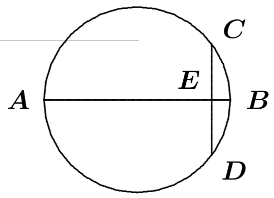
Therefore, $x^3y + xy^3 \le x^4 + y^4$, for all $x,\, y \in \mathbb{R}$.

Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

9. A, B, C, and D are points on a circle. AB is the diameter. CD is perpendicular to AB and meets AB at E. If AB and CD are integers and $AE - EB = \sqrt{7}$, find AE.

Solution by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON.

Let diameter $AB = d$ and chord $CD = c$, where $c, d \in \mathbb{Z}$. Since chord CD is perpendicular to diameter AB, we know that $CE = ED = c/2$. Since $AE + EB = d$ and $AE - EB = \sqrt{7}$, we get $2EB + \sqrt{7} = d$. Thus, $EB = \frac{d - \sqrt{7}}{2}$. Also, $AE = EB + \sqrt{7} = \frac{d + \sqrt{7}}{2}$.



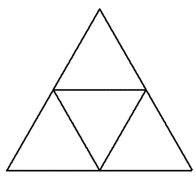
The Intersecting Chord Theorem tells us $(CE)(ED) = (AE)(EB)$; that is, $c^2 = d^2 - 7$. Since c and d are integers, c^2 and d^2 are perfect squares with a difference of 7. The only solution is $c = 3$ and $d = 4$, which gives $AE = \frac{4 + \sqrt{7}}{2}$.

Also solved by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

10. Nine points, no three of which lie on the same straight line, are located inside an equilateral triangle of side 4. Prove that some three of these points are vertices of a triangle whose area is not greater than $\sqrt{3}$.

Solution by Maximilian Butler, grade 12 student, Math Challenge Program, University of Western Ontario, London, ON.

Divide the given triangle into four smaller equilateral triangles of side length 2, as in the diagram. Each of the smaller equilateral triangles has area $\frac{1}{2}(2)(2)\sin 60^\circ = \sqrt{3}$. By the Pigeonhole Principle, at least $\lceil \frac{9}{4} \rceil = 3$ points must be in one of the smaller triangles. These 3 points then form a triangle whose area is no more than $\sqrt{3}$.



Also solved by Sarah Hogarth, grade 11 student, home school, Math Challenge Program, University of Western Ontario, London, ON; and Jennifer Park, grade 9 student, Bluevale C.I., Waterloo, ON.

That brings us to the end of another issue of Skoliad. This month a copy of **MATHEMATICAL MAYHEM Vol. 1** goes to Jennifer Park. Congratulations Jennifer! Continue sending in contests and solutions.