Mayhem Solutions

M44. Proposed by K.R.S. Sastry, Bangalore, India.

$ABCD$ is a Heron parallelogram (in which the sides, the diagonals and the area are natural numbers). The diagonals $AC$ and $BD$ have measures 85 and 41, respectively. Determine the measures of the sides $AB$ and $BC$.

Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

Let $AB$ and $BC$ be $x$ and $y$, with $x, y \in \mathbb{N}$. By the Law of Cosines

\[
\frac{x^2 + y^2 - BD^2}{2xy} = \cos(\angle BAD) = -\cos(\angle ADC) = -\left(\frac{x^2 + y^2 - AC^2}{2xy}\right),
\]

and so $x^2 + y^2 = \frac{AC^2 + BD^2}{2} = 4453$. Since $n^2 \equiv 0, 1, 4, 5, 6, 9 \pmod{10}$ for any $n \in \mathbb{Z}$, one of $x^2, y^2$ must be congruent to 4 (mod 10) and the other congruent to 9 (mod 10). A check of the possibilities shows that the equation $x^2 + y^2 = 4453$ is satisfied only when $\{x, y\} = \{63, 22\}$ or $\{58, 33\}$. If $\{x, y\} = \{63, 22\}$, then $\cos(\angle BAD) = 1$. Thus, we reject $\{63, 22\}$. On the other hand, if $\{x, y\} = \{58, 33\}$, then $\cos(\angle BAD) = \frac{21}{29}$ and $\sin(\angle BAD) = \frac{20}{29}$, making the area $\frac{20}{29} \cdot 58 \cdot 33 \in \mathbb{N}$. Hence, the only solutions are $(AB, BC) = (58, 33)$ or $(33, 58)$.

M45. Proposed by a Canadian Customs officer, Pearson International Airport, Toronto, ON.

A 10 metre long ladder is leaning upright against a wall, touching the edge of a cubic box. The box itself is put against the wall and measures 2 cubic metres. What is the height of the top of the ladder from the ground?

Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

Let $x = \sqrt{2}$, the length of the box. Let $h$ and $y$ be as shown in the diagram.
Then, by similar triangles, \( \frac{h - x}{x} = \frac{x}{y} \). Then \( y, x, h - x \) form a geometric progression. Let \( x = yr \) and \( h - x = xr \). Then, since \( h^2 + (x + y)^2 = 100 \), we have

\[
(x + xr)^2 + \left( \frac{x + \frac{x}{r}}{r} \right)^2 = 100
\]

\[
r^2 + 2r + 1 + \frac{2}{r} + \frac{1}{r^2} = \frac{100}{x^2}
\]

\[
\left( r + 1 + \frac{1}{r} \right)^2 = \frac{100}{x^2} + 1.
\]

Thus, since \( r > 0 \), we have \( r + 1 + \frac{1}{r} = \sqrt{\frac{100}{x^2} + 1} \); that is,

\[
r^2 + r\left( 1 - \sqrt{\frac{100}{x^2} + 1} \right) + 1 = 0.
\]

From here and from \( h = x(r + 1) \), we find the height to be

\[
h = \frac{\sqrt{100 + x^2} + x \pm \sqrt{(\sqrt{100 + x^2} + x)(\sqrt{100 + x^2} - 3x)}}{2};
\]

that is, about 9.895 m or 1.444 m.

Twelve incorrect solutions were received.

Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

The lattice polygons in the upper row of the figure are characterized by a common property, the lower ones by the reverse. Which property is it?
Solution by the proposer.
Let the lattice point denoted by the penta-star be the origin. Then all vertices of the upper polygons are visible points from the origin; that is, their coordinates are coprime. The vertices of the lower polygons are all invisible.

M47. Proposed by Bill Sands, University of Calgary, Calgary, AB.
(a) Find all monic quadratic polynomials \( x^2 + ax + b \) with integer roots, where \( 1, a, b \) is an arithmetic progression.
(b) Prove that there are no real numbers \( a, b, c \) such that \( 1, a, b, c \) is an arithmetic progression and \( x^3 + ax^2 + bx + c \) has all real roots.

(a) Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.
Since \( 1, a, b \) is an arithmetic progression, we have \( 1 + b = 2a \), and \( x^2 + ax + b = x^2 + ax + 2a - 1 \). If we let \( \alpha \geq \beta \) be the integer roots of this polynomial, then \( \alpha + \beta = -a \) and \( \alpha \beta = 2a - 1 \). Then \( 2 \left[ -\left( \alpha + \beta \right) \right] - 1 = \alpha \beta \), from which \( 3 = \alpha \beta + 2\alpha + 2\beta + 4 = (\alpha + 2)(\beta + 2) \). Therefore, either

(1) \( \alpha + 2 = 3 \) and \( \beta + 2 = 1 \), or
(2) \( \alpha + 2 = -1 \) and \( \beta + 2 = -3 \).

In the first case, \( (\alpha, \beta) = (1, -1) \), and hence \( a = 0 \); in the second case, \( (\alpha, \beta) = (-3, -5) \), and hence \( a = 8 \). Thus, the monic quadratic polynomials are \( x^2 - 1 \) and \( x^2 + 8x + 15 \).

(b) Solution by the proposer.
If the roots of \( x^3 + ax^2 + bx + c \) are \( r, s, \) and \( t \), then
\[
\begin{align*}
\alpha &= -r - s - t, \\
b &= rs + rt + st, \\
c &= -rst.
\end{align*}
\]
Hence, if \( 1, a, b, c \) is an arithmetic progression, then
\[
-2r - 2s - 2t = rs + rt + st + 1 \quad \text{and} \quad 2rs + 2rt + 2st = -rst - r - s - t.
\]
Solving the first equation for \( t \), we get
\[
t = -\frac{2r + 2s + rs + 1}{r + s + 2}.
\]
Plugging this into the second equation gives
\[
4rs + 3r^2 + 3s^2 + r^2s^2 + 2r^2s + 2rs^2 + 2r + 2s + 1 = 0,
\]
which we can write as
\[
2(r^2 + s^2) + (r + 1)^2(s + 1)^2 = 0.
\]
This obviously has no real solution.

Part (b) was also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.
M48. Proposed by J. Walter Lynch, Athens, GA, USA.
Tell how to make a single stopper that will stop a square hole, a round hole, and a triangular hole, and will pass through each.

If one wanted to give a hint, he might point out that a pyramid will stop a square hole and a triangular hole, and a cylinder will stop a square hole and a round hole.

Solution by the proposer.
Start with a right circular cylinder with equal height and diameter. This will already stop the round hole and the square hole. Now set the cylinder on a table with one of the circular surfaces on top and cut a wedge off of each of two opposite sides. Do this by placing a knife along a diameter on the top of the cylinder and cutting down to the outer edge at the bottom of the cylinder. Replace the knife on the top of the cylinder and cut off the symmetric wedge.

Now, in addition to the round hole and the square hole, the stopper will stop the triangular hole.

M49. K.R.S. Sastry, Bangalore, India.
The figure shows a Heron pentagon in which the sides, the diagonals and the area are natural numbers.
(a) $AB = AE = 15$, $AC = AD = 20$ and $BCDE$ is a rectangle. Find the length of $BD$.

(b) Give a set of general expressions for the sides, the diagonals and the area to generate an infinite family of such Heron pentagons $ABCDE$ as in the figure.
(a) Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

Introduce variables $h$, $x$, and $y$, as shown in the diagram below. Then we have $h^2 + x^2 = 15^2$ and $(h + y)^2 + x^2 = 20^2$, which implies that $y(2h + y) = 175 = 5^2 \cdot 7 > y^2$. Thus, since $x > 0$, we must have $y = 7$. $h = 9$, and $x = 12$. This yields $BD^2 = 4x^2 + y^2 = 625$, giving $BD = 25$. This solution produces an area of 276 satisfying the condition that the area is an integer. Therefore, the solution is $BD = 25$.

(b) Solution by the proposer.

Set

$$
AB = AE = (m^2 - n^2)(m^2 + n^2),
$$
$$
BC = DE = 6m^2n^2 - m^4 - n^4,
$$
$$
CD = BE = 4mn(m^2 - n^2),
$$
$$
AC = AD = 2mn(m^2 + n^2),
$$
$$
BD = CE = (m^2 + n^2)^2.
$$

Then the area is $[ABCDE] = 2mn(m^2 - n^2)(10m^2n^2 - m^4 - n^4)$, where $\gcd(m, n) = 1$ and $n < m < (\sqrt{2} + 1)n$.

Part (b) was also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON, whose infinite family of solutions is a subset of the proposer’s set.

M50. Proposed by the Mayhem Staff.

This question is a bit of a variation of a well known and used problem. There are forms of the question where you want to use four 4’s and some operations to make as long a list of values as possible. Thus

$$
\frac{4 + 4}{4 + 4} = 1, \quad 4 + 4 - \sqrt{4} - 4 = 2,
$$

and so on. It is popular to use the digits of the year in such a problem (although, we will have to deal with a couple of zeros for a while).

The problem is to make as many numbers as possible using up to five $\pi$’s. Thus, some acceptable results would be:

$$
\frac{\pi + \pi + \pi}{\pi} = 3, \quad \left[\sqrt{\pi\pi} - \pi + \frac{\pi}{\pi}\right]! = 6.
$$
Solutions by Sabrina Liao, student, York Mills C.I., North York, ON; Adrian Florea, student, École secondaire St-Luc, Montréal, QC; Peng Liu, student, Glebe C.I., Ottawa, ON; James Meredith, Hudson H.S., Hudson Heights, QC; Rébecca Millette, student, École secondaire Dorval-Jean XXIII, Dorval, QC; Maxime Pelletier, student, Collège Sainte-Anne de Lachine, Lachine, QC; Jing Qin, student, École Émile-Legault, Saint-Laurent, QC; Danny Quan, student, Collège Jean de Brébeuf, Montréal, QC; Diana Rapeanu, student, Collège Notre-Dame Sacré-Coeur, Montréal, QC; Sarah Shaker, student, École secondaire Félix-Leduc, Pointe-Claire, QC; and Bob Wang, student, Merivale H.S., Nepean, ON.

1 = \frac{\pi}{\pi} \quad 2 = \frac{\pi + \pi}{\pi} \quad 3 = \frac{\pi + \pi + \pi}{\pi}

4 = [\pi] - \frac{\pi + \pi}{\pi} \quad 5 = \frac{\pi + \pi}{\pi} + [\pi] \quad 6 = \left(\frac{\pi + \pi + \pi}{\pi}\right)!

7 = [\pi \times \pi] - \frac{\pi + \pi}{\pi} \quad 8 = [\sqrt{\pi\pi}] + \frac{\pi + \pi}{\pi} \quad 9 = [\pi + \pi + \pi]

10 = [\pi \times \pi] + \frac{\pi}{\pi} \quad 11 = \left[\frac{\pi\pi}{\pi}\right]

12 = [\pi + \pi + \pi + \pi] \quad 13 = \left[\left(\pi + \frac{\pi}{\pi}\right) \times \pi\right]

14 = \left([\sqrt{\pi + \pi}] \times \pi\right) \quad 15 = [\pi]! + [\pi + \pi + \pi]

Above is a taste of some of the wonderful solutions sent in. Jing Qin sent in a list from 1 to 100 with only a couple of omissions, while Bob Wang sent in 51 solutions. Thus, for this issue’s prizes, we award a subscription of CRUX with MAYHEM to Jing Qin, and a copy of ATOM (A Taste Of Mathematics), volume 2, to Bob Wang. Continue sending us your solutions and problem proposals.