Pólya's Paragon

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This month I would like to spend some time examining some difficult easy problems. That may seem rather contradictory. The three problems I have selected are taken from past grade 7 and 8 mathematics contests. This is what will make them easier than some of the problems I have posed in the past. On the other hand, they are the last questions from their respective contests and will therefore require a little more insight and creativity on the part of the reader. I should also note that, when attempting these problems, we must put on our "grade 8 hats" and forget all the mathematics we have come to learn since that time. I find it remarkable how much mathematics I take for granted. It is a worthwhile exercise to return to the basics.

One of my favourite aspects of grade 7 and 8 problems is that they can be used as a teaching tool. After years of doing mathematics contests, the tricks and techniques seem commonplace. I have to remember, however, that at some point even I saw them for the first time.

The following problem has a solution that is related to a particular set of numbers that a grade 7 student may not be familiar with. Naturally, this pattern is rather intriguing and can be used as a platform for many more problems in the future.

Gauss, Grade 7, #25, 2001

A triangle can be formed having side lengths 4, 5, and 8. It is impossible, however, to construct a triangle having side lengths 4, 5, and 9. Ron has 8 sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. The shortest possible length of the longest of the eight sticks is

(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

Solution: Since we want the shortest possible length for the longest stick, it makes sense to start with the smallest sticks, each having length 1. Although a grade 8 student may not recognize it as such, the triangle inequality is involved in the question. It is clear that if sticks of length a and b are in our set, there cannot be any other stick of length less than $a+b$, since these three would then form a triangle. Using this information, we systematically add more sticks, taking the smallest possible length at each step. This gives us sticks with lengths 1, 1, 2, 3, 5, 8, 13, and 21. Is it much of a coincidence that these numbers form the Fibonacci sequence? Although this is not a formal proof that 21 is the shortest possible, a simple proof by contradiction could be used to show it is indeed a lower bound. This is left as an exercise.

I have found that the most prevalent formulas when solving problems in a contest setting are the formulas for the sums of arithmetic and geometric sequences. These cropped up in at least one question, it seems to me, from every contest that I can remember writing. Even though Gauss is said to have found one of these formulas at a ridiculously young age, most of our grade 7 and 8 students do not know them. How would we then approach the following problem?

Gauss, Grade 8, #21, 1999:

The sum of seven consecutive integers is always:

(A) odd (B) a multiple of 7 (C) even
(D) a multiple of 4 (E) a multiple of 3

Solution 1: If we jump right to the formula for the sum of an arithmetic sequence, we can see that the sum starting at a is $7(a-1)+28$ which is always a multiple of 7. Easy enough, right? What if we did not know the formula for the sum of an arithmetic sequence?

Solution 2: It is easy for a grade 8 student to check the first couple of values obtained by computing these sums. For instance, $1+2+3+4+5+6+7=28$ and $2+3+4+5+6+7+8=35$. Now, since we know that one of the 5 possible answers must be correct we can now eliminate ones that are not satisfied by the values we have computed. Looking at 28, we can eliminate (A) and (E). Then, looking at 35 we can eliminate (C) and (D). That leaves only one possible answer, namely (B).

To quote Sir Arthur Conan Doyle, "*When you have eliminated the impossible, that which remains, however improbable, must be the truth.*"

This technique, although not appropriate when trying to solve problems in general, can be very useful and time-saving when attacking multiple choice questions. In my experience, this particular problem takes an average high school student longer than a grade 7 student. First, they remember that a formula exists; then, they spend quite a bit of time trying to remember or re-derive it before they can solve the problem!

For my last problem this month, I have chosen one of my favourites. It exemplifies the idea of using a problem as a teaching tool, as well as appealing to the more limited mathematical knowledge of a grade 7 student.

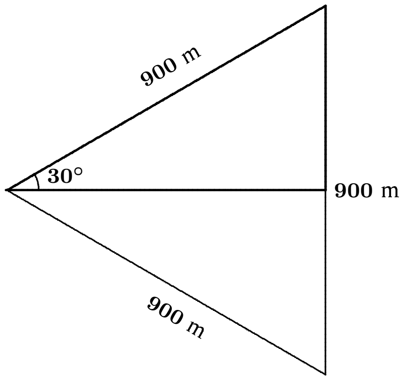
Variant on a Gauss, Grade 7, problem:

A particular ski lift takes passengers from the base of a mountain to its peak at a rate of 1.5 m/s. The lift travels at a 30 degree incline and it takes 10 minutes to reach the top. What is the height of the mountain?

Solution 1: We begin with a simple calculation: Ten minutes is the same as 600 seconds; hence, we have travelled a total of $(1.5)(600)=900$ metres. Fitting this to a diagram, we get a $30°$-$60°$-$90°$ triangle with 900 m along its hypotenuse. Some simple trigonometry will tell us that the height of the mountain is 450 m.

But wait! How many grade 7 students do you know who have a working understanding of trigonometry? It certainly is not common. How do you think a grade 7 student can solve this problem? The following solution is my best guess.

Solution 2: First, we recognize that a $30°$-$60°$-$90°$ triangle is really half of an equilateral triangle (we reflect along the side opposite the 60-degree angle).



Each side of this triangle is exactly 900 m. Now we see that the two $30°$-$60°$-$90°$ triangles are identical (congruent). Therefore, 900 m is twice the length that we are looking for. Thus, the height of the mountain is 450 m.

This, in my opinion, is one of the best forms of problem solving. It is not simply that we can calculate what the answer should be, but that we can calculate it without some of the "higher" mathematics that many of us take for granted. In some ways, knowing too much can slow us down and prevent us from being creative when the need arises.

I am always impressed when a student can solve the problem I have just shown here. If you remember back to when you first learned the $30°$-$60°$-$90°$ triangle and the 1-$\sqrt{3}$-2 side lengths, it was probably proven to you by taking an equilateral triangle, splitting it in two, and applying the Pythagorean theorem. By solving this problem, a grade 7 student has demonstrated that she can independently develop the same result years before it is taught in her curriculum. In my mind, this is problem-solving at its best.