High, Low, High, Low, It's Off To Work We Go

Zhe Li and Paul Belcher

In the British Mathematical Olympiad Round 1 in 1995 there was the following question:

5. The seven dwarfs walk to work each day in single file, with heights alternating up-down-up- or down-up-down-. For how long can they continue with a new order every day? What if Snow White always comes too?

We will generalize this problem. Let $w(n)$, $n \in \mathbb{N}^+$, represent the number of ways that n dwarves of different heights can walk to work with their heights alternating up, down, up, ..., or down, up, down, ... Directly listing and counting gives $w(1) = 1$, $w(2) = 2$, $w(3) = 4$, $w(4) = 10$, and $w(5) = 32$.

The heights of the dwarves can be represented, from smallest to largest simply as $1, 2, 3, \ldots, n$. (Since we are interested only in the relative heights, any increasing sequence will do here.) If we let $u(n)$ be the number of arrangements of the heights that go up, down, up, ..., and $d(n)$ be the number of arrangements that go down, up, down, ..., then we notice that $u(n) = d(n)$, for $n \geq 2$. We can prove this as follows.

Let $a_1, a_2, \ldots, a_n$ be an arrangement that goes up, down, up, ... To this we apply the transformation $a_i \to (n+1) - a_i$. This will give a unique new arrangement that goes down, up, down, ... To visualize this, let the height of the tunnel in which the dwarves are walking to work, singing, be $n+1$. Then consider another set of dwarves upside down with their feet on the roof of the tunnel, directly above the first set, as shown in Figure 1.

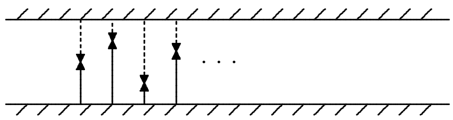

Figure 1

We now attempt to find a recurrence relation for $w(n)$. Let the tallest dwarf, of height n, who will be called Max, be in the i^{th} position. There are $i-1$ dwarves before Max, and they could be selected in $\binom{n-1}{i-1}$ ways. Their order would have to go ..., up, down, and finally up to Max. Apart from the cases where $i = 1$ or $i = 2$, the number of ways of correctly arranging these $i-1$ dwarves is $d(i-1) = \frac{1}{2}w(i-1)$, as we can see by considering the dwarves in reverse order. Following Max, the $n-i$ dwarves must have an order going down from Max, then up, down, ... Apart from the cases where $i = n - 1$ or $i = n$, the number of ways of correctly arranging these $n - i$ dwarves is $d(n-i) = \frac{1}{2}w(n-i)$.

We have a problem with $n = 1$, since $w(1) = u(1) = d(1) = 1$. To deal with this, we shall define a new sequence $W(n)$ by $W(1) = W(0) = 2$ and $W(n) = w(n)$ if $n \geq 2$. We consider $u(0) = d(0) = 1$. Then we have the recurrence relation

$$W(n) \;=\; \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{W(i-1)}{2} \frac{W(n-i)}{2}\,, \quad n \;\geq\; 2\,, \qquad (1)$$

with $W(0) = 2$, $W(1) = 2$.

From this recurrence relation we can build up our sequence as follows: $W(2) = 2$, $W(3) = 4$, $W(4) = 10$, $W(5) = 32$, $W(6) = 122$, $W(7) = 544$. This agrees with our directly computed list and answers the question for 7 dwarves: they can walk to work in 544 ways with an alternating height pattern. If Snow White wants to take part as well, then calculating one step further gives $W(8) = 2770$.

If we let $t(n) = [W(n)]/n!$, then $t(n) \to 0$ as $n \to \infty$, as we will now show. Suppose we take an arrangement of dwarves having the required property and exchange two successive dwarves. The new arrangement will not have the required property if $n \geq 3$. There are $n - 1$ ways of exchanging two dwarves, producing $(n-1)W(n)$ arrangements which are all different. Thus, the number of arrangements not having the desired property is at least $(n-1)W(n)$. Hence, $n! \geq (n-1)W(n) + W(n) = nW(n)$. Therefore,

$$\frac{W(n)}{n!} \;\leq\; \frac{1}{n}\,, \quad n \;\geq\; 3\,,$$

giving the result. Therefore, if the dwarves just arrange themselves randomly, then the probability that the arrangement has the desired property tends to zero as n tends to infinity.

We now attempt to find a non-recursive formula for $W(n)$ by means of a generating function for $t(n)$. From (1), replacing $i - 1$ by i, we have

$$W(n) \;=\; \frac{1}{4} \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-1-i)!} W(i)W(n-1-i)\,, \quad n \;\geq\; 2\,.$$

Hence,

$$nt(n) \;=\; \frac{1}{4} \sum_{i=0}^{n-1} t(i)t(n-1-i)\,, \quad n \geq 2\,, \qquad (2)$$

with $t(0) = 2$ and $t(1) = 2$. Consider the generating function $h(x)$:

$$h(x) \;=\; t(0) + t(1)x + t(2)x^2 + \cdots + t(n)x^n + \cdots\,.$$

In $\frac{1}{4}[h(x)]^2$, the first term is 1 and the coefficient of x^{n-1}, for $n \geq 2$, is $\frac{1}{4}\sum_{i=0}^{n-1} t(i)t(n-1-i) = nt(n)$, using (2). Also,

$$h'(x) = t(1) + 2t(2)x + \cdots + nt(n)x^{n-1} + \cdots .$$

Therefore, $h'(x) = \frac{1}{4}[h(x)]^2 + 1$. Letting $h(x) = y$, we have the differential equation $\frac{dy}{dx} = \frac{1}{4}y^2 + 1$, which has solution $\arctan\left(\frac{y}{2}\right) = \frac{x}{2} + c$. When $x = 0$, we get $y = h(0) = t(0) = 2$, and also $c = \arctan(1) = \frac{\pi}{4}$. Thus, $y = h(x) = 2\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$.

By considering the Taylor expansion of $h(x)$, we find that $t(n)$, the coefficient of x^n, is given by $t(n) = \frac{2}{n!}D^n \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\Big|_{x=0}$. Thus,

$$W(n) = 2D^n \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\Big|_{x=0} = \frac{1}{2^{n-1}}D^n \tan x\Big|_{x=\frac{\pi}{4}} .$$

The table below verifies this result for small values of n.

n	$D^n \tan(x)$	$D^n \tan(x)$ at $x = \frac{\pi}{4}$	$\frac{1}{2^{n-1}}D^n \tan(x)$ at $x = \frac{\pi}{4}$
0	$\tan x$	1	2
1	$1 + \tan^2 x$	2	2
2	$2\tan x(1 + \tan^2 x)$	4	2
3	$2(3\tan^2 x + 1)(1 + \tan^2 x)$	16	4
4	$8\tan x(3\tan^2 x + 2)(1 + \tan^2 x)$	80	10
5	$8(1 + \tan^2 x)$ $\times(15\tan^4 x + 15\tan^2 x + 2)$	512	32
6	$16\tan x(1 + \tan^2 x)$ $\times(45\tan^4 x + 60\tan^2 x + 17)$	3904	122

Our conclusion is that the number of ways, $w(n)$, of arranging n objects of different magnitudes in order so that the differences in magnitude alternate between positive and negative is given by

$$w(n) = \frac{1}{2^{n-1}}D^n \tan x\Big|_{x=\frac{\pi}{4}} ,$$

$n \geq 2$, with $w(1) = 1$.

Zhe Li is a 20-year-old student from China at Atlantic College who investigated this problem for his extended essay in Maths in the International Baccalaureate. Paul Belcher is Head of Maths at Atlantic College and was supervisor for the extended essay.

United World College of the Atlantic
St. Donat's Castle
Llantwit Major
Vale of Glamorgan
CF61 1WF, UK