Mayhem Solutions

M38. *Proposed by the Mayhem staff.*

Find all values of n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square.

Solution by Andrew Mao, grade 10 *student, H.B. Beal S.S., London, ON.*

We can check that

$$\begin{aligned} 1! &= 1 &\equiv& \; 1 \pmod 5 \\ 1! + 2! &= 3 &\equiv& \; 3 \pmod 5 \\ 1! + 2! + 3! &= 9 &\equiv& \; 4 \pmod 5 \\ 1! + 2! + 3! + 4! &= 33 &\equiv& \; 3 \pmod 5 \end{aligned}$$

If $n > 4$, then

$$1! + 2! + \cdots + n! \;=\; (1! + 2! + 3! + 4!) + (5! + 6! + \cdots + n!)\,.$$

But $k! \equiv 0 \pmod 5$ when $k \geq 5$. Thus, the sum in the second bracket is divisible by 5. Therefore,

$$1! + 2! + \cdots + n! \;\equiv\; 1! + 2! + 3! + 4! \;\equiv\; 3 \pmod 5\,.$$

Then $1! + 2! + \cdots + n!$ is not a square, since, for any natural number m, we have $m^2 \equiv 0$, $m^2 \equiv 1$, or $m^2 \equiv 4 \pmod 5$. Therefore, $n = 1$ and $n = 3$ are the only solutions.

Also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Antonio Lei, year 12 *student, Colchester Royal Grammar School, Colchester, UK.*

M39. *Proposed by the Mayhem staff.*

Given x is a positive real number and

$$x \;=\; 2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \cfrac{1}{2002 + \frac{1}{x}}}}}\,,$$

find x.

Solution by Alfian, grade 11 *student, SMU Methodist* 1*, Palearbang, Indonesia.*

Let us consider the more general problem:

$$x \;=\; A + \cfrac{1}{A + \cfrac{1}{A + \cfrac{1}{A + \cfrac{1}{A + \frac{1}{x}}}}}\,,$$

for some real number A.

Notice that we can rewrite

$$A + \frac{1}{x} \;=\; \frac{Ax+1}{x}\,.$$

Then the next level becomes

$$A + \frac{1}{A+\frac{1}{x}} \;=\; A + \frac{1}{\frac{Ax+1}{x}} \;=\; A + \frac{x}{Ax+1} \;=\; \frac{A^2x+x+A}{Ax+1}\,.$$

Continuing the process, we end up with

$$x \;=\; \frac{A^5x+4A^3x+3Ax+A^4+3A^2+1}{A^4x+3A^2x+x+A^3+2A}\,,$$

which leads to the quadratic equation

$$(A^4+3A^2+1)x^2 - A(A^4+3A^2+1)x - (A^4+3A^2+1) \;=\; 0\,.$$

Thus, as long as $A^4+3A^2+1 \neq 0$ (which is true for all real A), we have $x^2 - Ax - 1 = 0$. Hence,

$$x \;=\; \frac{A \pm \sqrt{A^2+4}}{2}\,.$$

Since x is given to be positive, we must choose

$$x \;=\; \frac{A + \sqrt{A^2+4}}{2}\,.$$

Now in our particular case, $A = 2002$, and $x = 1001 + \sqrt{1001^2+1}$.

Also solved by Austrian 2002 IMO team; Kevin Chung, OAC student, Earl Haig S.S., North York, ON; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

M40. *Proposed by Louis-François Préville-Ratelle, student, Cégep Régional de Lanaudière à L'Assomption, Joliette, QC.*

Suppose a and b are two divisors of the integer n, with $a < b$. Prove:

$$\left\lfloor \frac{n}{a+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{b} \right\rfloor \;=\; \left\lfloor \frac{n}{\frac{n}{b}+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{\frac{n}{a}} \right\rfloor\,.$$

Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

For example, if $n = 24$, $a = 3$, and $b = 6$, this says:

$$\left\lfloor \frac{24}{4} \right\rfloor + \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor \;=\; \left\lfloor \frac{24}{5} \right\rfloor + \left\lfloor \frac{24}{6} \right\rfloor + \left\lfloor \frac{24}{7} \right\rfloor + \left\lfloor \frac{24}{8} \right\rfloor\,,$$

which evaluates to the identity $6+4+4 = 4+4+3+3$.

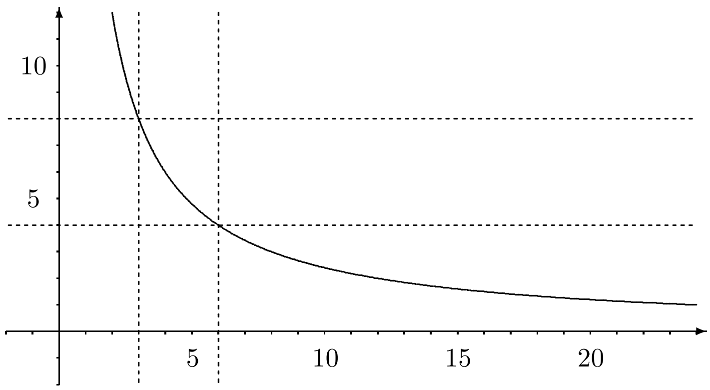
Solution by the proposer.
We want to show that

$$\left\lfloor \frac{n}{a+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{b} \right\rfloor \qquad (1)$$

equals

$$\left\lfloor \frac{n}{\frac{n}{b}+1} \right\rfloor + \cdots + \left\lfloor \frac{n}{\frac{n}{a}} \right\rfloor . \qquad (2)$$

Consider the graph of the function $y = \dfrac{n}{x}$. Draw vertical lines at $x = a$ and $x = b$, and horizontal lines at $y = \dfrac{n}{b}$ and at $y = \dfrac{n}{a}$. (See diagram below for $n = 24$, $a = 3$, and $b = 6$.)



Then the number (1) is exactly the number of integer points lying below the graph, strictly to the right of the first vertical line and to the left of (or on) the second vertical line. Similarly, the number (2) is exactly the number of integer points lying to the left of the graph, below (or on) the upper vertical line and strictly above the lower vertical line.

To show that (1) and (2) are equal, we can forget the points which were counted in both (1) and (2). We need to show that the number of points counted in (1) but not in (2) is the same as the number of points counted in (2) but not in (1). This is relatively easy, because both of those regions are simply rectangular.

The points counted in (1) but not in (2) are on or below $y = \frac{n}{b}$, and between $x = a + 1$ and $x = b$. There are $\left(\frac{n}{b}\right) \times (b - (a + 1) + 1) = n\left(1 - \frac{a}{b}\right)$ of these. Similarly, the points counted in (2) but not in (1) are on or to the left of $x = a$, above $y = \frac{n}{b}$ and below or on $y = \frac{n}{a}$. There are $a \times \left(\frac{n}{a} - \left(\frac{n}{b} + 1\right) + 1\right) = n\left(1 - \frac{a}{b}\right)$ of these.

Thus, the two numbers are the same, and the equality is established.

M41. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find the number of orders of wins and losses that can occur in a World Series. For example if the series ends after five games there are eight possible orders: ANNNN NANNN NNANN NNNAN NAAAA ANAAA AANAA AAANA where A is for an American League win and N is for a National League win. Note that the series ends as soon as one team wins four games.

Solution by Sabrina Liao, student, York Mills C.I., North York, ON.

Since the series ends when a team wins 4 games, the series could end after 4, 5, 6, or 7 games.

- 4 game series: 2 possible orders: NNNN AAAA.
- 5 game series: 8 possible orders: NAAAA ANAAA AANAA AAANA and 4 more with the N and A reversed.
- 6 game series: If N wins, then the order must end with N, with 2 A's and 3 N's in the other 5 positions. Using a similar argument for when A wins, we find that the number of orders is $2 \times \binom{5}{2} = 20$.
- 7 game series: By the same reasoning, there will be $2 \times \binom{6}{3} = 40$ orders.

Thus, there altogether $2 + 8 + 20 + 40 = 70$ orders of wins and losses.

Also solved by George Adler, student, Gloucester H.S., Gloucester, ON; Steven Béliveau, student, Ecolé d'éducation internationale de Laval, Laval, QC; Robert Bilinski, Outremont, QC; Kevin Chung, OAC student, Earl Haig S.S., North York, ON; Jean-Philippe Lemieux, student, Ecolé secondaire Dorval-Jean XXIII, Dorval, QC; Peng Liu, student, Glebe C.I., Ottawa, ON; James Meredith, Hudson H.S., Hudson Heights, QC; Rébecca Millette, student, Ecolé secondaire Dorval-Jean XXIII, Dorval, QC; Alexandra Ortan, student, Ecolé Joseph-François-Perrault, Montréal, QC; Maxime Pelletier, student, Collège Sainte-Anne de Lachine, Lachine, QC; Jing Qin, student, Ecolé Emilé-Legault, Saint-Laurent, QC; Diana Rapeanu, student, Collège Notre-Dame Sacré-Coeur, Montréal, QC; Sarah Shaker, student, Ecolé secondaire Félix-Leclerc, Pointe-Claire, QC; Siwen Sun, student, Collège Sainte-Louis, Lachine, QC; Bob Wang, student, Merivale H.S., Nepean, ON; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Nicolas Wionzek, student, Almonte and District H.S., Almonte, ON. Four incorrect solutions were received.

The proposer notes that of the 70 possible orders, 43 have occurred. He provides the following list, of when the orders occurred, from most to least frequent, for your enjoyment.

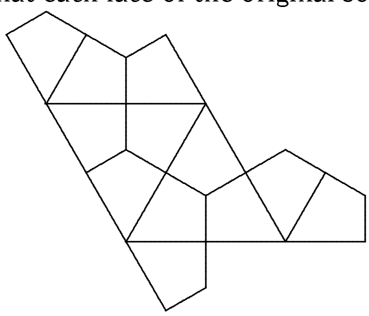
AAAA: *1914, 1927, 1928, 1932, 1938, 1939, 1950, 1966, 1989, 1998, 1999;*
ANAAA: *1913, 1941, 1943, 1949, 1961, 1974, 1984;*
NNNN: *1907, 1922, 1954, 1963, 1976, 1990;*
NAAANA: *1911, 1935, 1936, 1948, 1992;*
AANAA: *1916, 1929, 2000;*
NNANN: *1908, 1933, 1988;*
ANAANA: *1918, 1977, 1993;*
NAAAA: *1915, 1983;*
NNAAAA: *1978, 1996;*
AANNNAA: *1987, 1991;*
NNANAAA: *1958, 1985;*
ANANANN: *1940, 1946;*
ANNANAN: *1931, 1975;*
NANANAN: *1909, 1997;*
AAANA: *1910, 1937, 1970;*
AANNAA: *1917, 1930, 1953;*
AANNNAN: *1955, 1965, 1971;*
ANNNN: *1942, 1969;*
NANAAA: *1923, 1951;*
NANANAA: *1924, 1952;*
ANAANNN: *1925, 1979;*
ANNAANN: *1926, 1982;*
NAANNAN: *1960, 1964;*

NANNN: *1905;*	ANANAA: *1906;*	AANNNN: *1981;*
NNAANN: *1980;*	ANANNN: *1944;*	ANNNAN: *1959;*
NNANAN: *1995;*	NNAAANA: *1956;*	AANANNA: *1972;*
AANNANA: *1947;*	ANAANNA: *1912;*	ANANANA: *1962;*
ANANNAA: *1973;*	NANAANA: *1945;*	NANNAAA: *1968;*
AANNANN: *1986;*	ANANNAN: *1957;*	NANAANN: *1934;*
NANNAAN: *1967;*	NNAAANN: *2001.*	

The proposer notes that the years 1903, 1919, 1920, and 1921 are not included, because in these years the winner won five games.

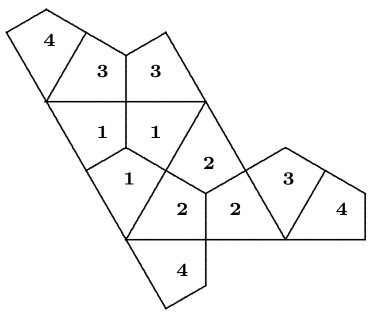
M42. *Proposed by Izidor Hafner, Tržaška 25, Ljubljana, Slovenia.*

The diagram below represents the net of a polyhedron. The faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.



Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

The original polyhedron is a regular tetrahedron. The faces are numbered as below.



M43. *Proposed by the Mayhem staff.*

Prove that

$$\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{2002}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{2002}$$

is an integer.

Solution by Austrian 2002 *IMO team.*

Let a_n be the sequence with

$$a_n=\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{n}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{n}.$$

Thus, $\frac{7+\sqrt{29}}{2}$ and $\frac{7-\sqrt{29}}{2}$ are the two roots of the characteristic equation, $x^2+px+q=0$, of a_n. Therefore, we have

$$p = -\left(\frac{7+\sqrt{29}}{2}+\frac{7-\sqrt{29}}{2}\right) = -7\,,$$
$$q = \frac{7+\sqrt{29}}{2}\times\frac{7-\sqrt{29}}{2} = 5\,.$$

Thus, the characteristic equation is $x^2 = 7x-5$. Therefore, the recurrence relation is $a_{n+2} = 7a_{n+1}-5a_n$. Now we only need determine the first two values, which are $a_0 = 1$ and $a_1 = 1$.

Because a_0 and a_1 are integers and the coefficients of the recursion formula are integers, all values of the sequence are integers.

Also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON. Seven incomplete or incorrect solutions were received.

The mail gremlins were at work last month; we received solutions from Andrew Mao, grade 10 student, H.B. Beal S.S., London, ON for M30, M32, M33, M34, M35, M36, and M37. Sorry about that Andrew. This issue's Mayhem Taunt winner is ...Andrew Mao! Andrew will receive a subscription to ***Crux Mathematicorum*** for 2003.