MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **~~Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans~~ Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

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The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, 2191 Saturn Crescent, Orleans, Ontario, K4A 3T6** or emailed to

mayhem-editors@cms.math.ca

Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 October 2003*. Solutions received after this time will be considered only if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

M88. *Proposed by the Mayhem Staff.*

A set S consists of six numbers. When we take all possible subsets of S containing 5 elements, the sums of the elements of these subsets are 87, 92, 98, 99, 104, and 110, respectively. Determine the six numbers in S.

. . . .

Un ensemble S contient six nombres. Si l'on prend tous les sous-ensembles de S ne contenant que 5 éléments, les sommes des éléments de ces sous-ensembles sont respectivement 87, 92, 98, 99, 104, et 110. Déterminer les six nombres dans S.

M89. *Proposed by the Mayhem Staff.*

Find all positive integers x for which $x(x+60)$ is a perfect square.

. . . .

Trouver tous les entiers positifs x pour lesquels $x(x+60)$ est un carré parfait.

M90. *Proposed by the Mayhem Staff.*

Determine the largest positive integer n for which 2002^n is a factor of $2002!$. What happens if 2002 is replaced by 2003 or 2004?

. . . .

Trouver le plus grand entier positif n tel que 2002^n soit un facteur de $2002!$. Qu'arrive-t-il si l'on remplace 2002 par 2003 ou 2004 ?

M91. *Proposed by Robert Morewood, Burnaby South Secondary School, Burnaby, BC.*

Let k be a four-digit integer. Determine all possible values of k for which k^{2003} ends in the four digits 2003. What happens if 2003 is replaced by 2002 or 2004?

. . . .

Soit k un nombre de quatre chiffres. Trouver toutes les valeurs possibles de k pour lesquelles le nombre k^{2003} se termine par 2003. Qu'arrive-t-il si l'on remplace 2003 par 2002 ou 2004 ?

M92. *Proposed by the Mayhem Staff.*

A 3×3 magic square consists of nine distinct values, such that each of the rows, columns, and diagonals have a constant sum. Below is an example of a 3×3 magic square.

Suppose that a 3×3 magic square has a constant sum of T. Let the middle entry of this square be E. Prove that $T = 3E$.

2	9	4
7	5	3
6	1	8

Un carré magique de 3×3 est formé de neuf valeurs distinctes, telles que la somme des éléments de chacune des lignes, des colonnes et des diagonales donne la même constante. Voir l'exemple ci-dessus.

Soit T la somme constante d'un carré magique 3×3. Si l'on désigne l'élément du centre par E, montrer que $T = 3E$.

M93. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

In triangle ABC, suppose that $\tan A$, $\tan B$, $\tan C$ are in harmonic progression. Show that a^2, b^2, c^2 form an arithmetic progression.

[*Note:* x, y, z are in harmonic progression if $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ form an arithmetic progression.]

Dans un triangle ABC, on suppose que $\tan A$, $\tan B$, $\tan C$ sont en progression harmonique. Montrer que a^2, b^2, c^2 forment une progression arithmétique .

[*Note :* x, y, z sont en progression harmonique si $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ forment une progression arithmétique.]

Mayhem Solutions

M38. *Proposed by the Mayhem staff.*

Find all values of n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square.

Solution by Andrew Mao, grade 10 *student, H.B. Beal S.S., London, ON.*

We can check that

$$\begin{aligned} 1! &= 1 &\equiv& \ 1 \pmod 5 \\ 1! + 2! &= 3 &\equiv& \ 3 \pmod 5 \\ 1! + 2! + 3! &= 9 &\equiv& \ 4 \pmod 5 \\ 1! + 2! + 3! + 4! &= 33 &\equiv& \ 3 \pmod 5 \end{aligned}$$

If $n > 4$, then

$$1! + 2! + \cdots + n! = (1! + 2! + 3! + 4!) + (5! + 6! + \cdots + n!) .$$

But $k! \equiv 0 \pmod 5$ when $k \geq 5$. Thus, the sum in the second bracket is divisible by 5. Therefore,

$$1! + 2! + \cdots + n! \equiv 1! + 2! + 3! + 4! \equiv 3 \pmod 5 .$$

Then $1! + 2! + \cdots + n!$ is not a square, since, for any natural number m, we have $m^2 \equiv 0$, $m^2 \equiv 1$, or $m^2 \equiv 4 \pmod 5$. Therefore, $n = 1$ and $n = 3$ are the only solutions.

Also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Antonio Lei, year 12 *student, Colchester Royal Grammar School, Colchester, UK.*

M39. *Proposed by the Mayhem staff.*

Given x is a positive real number and

$$x = 2002 + \cfrac{1}{2002 + \cfrac{1}{2002+\cfrac{1}{2002+\cfrac{1}{2002+\frac{1}{x}}}}},$$

find x.

Solution by Alfian, grade 11 student, SMU Methodist 1, Palearbang, Indonesia.

Let us consider the more general problem:

$$x = A + \cfrac{1}{A + \cfrac{1}{A+\cfrac{1}{A+\cfrac{1}{A+\frac{1}{x}}}}},$$

for some real number A.

Notice that we can rewrite

$$A + \frac{1}{x} = \frac{Ax+1}{x}.$$

Then the next level becomes

$$A + \frac{1}{A+\frac{1}{x}} = A + \frac{1}{\frac{Ax+1}{x}} = A + \frac{x}{Ax+1} = \frac{A^2x+x+A}{Ax+1}.$$

Continuing the process, we end up with

$$x = \frac{A^5x + 4A^3x + 3Ax + A^4 + 3A^2 + 1}{A^4x + 3A^2x + x + A^3 + 2A},$$

which leads to the quadratic equation

$$(A^4 + 3A^2 + 1)x^2 - A(A^4 + 3A^2 + 1)x - (A^4 + 3A^2 + 1) = 0.$$

Thus, as long as $A^4 + 3A^2 + 1 \neq 0$ (which is true for all real A), we have $x^2 - Ax - 1 = 0$. Hence,

$$x = \frac{A \pm \sqrt{A^2+4}}{2}.$$

Since x is given to be positive, we must choose

$$x = \frac{A + \sqrt{A^2+4}}{2}.$$

Now in our particular case, $A = 2002$, and $x = 1001 + \sqrt{1001^2+1}$.

Also solved by Austrian 2002 IMO team; Kevin Chung, OAC student, Earl Haig S.S., North York, ON; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

M40. *Proposed by Louis-François Préville-Ratelle, student, Cégep Régional de Lanaudière à L'Assomption, Joliette, QC.*

Suppose a and b are two divisors of the integer n, with $a < b$. Prove:

$$\left\lfloor\frac{n}{a+1}\right\rfloor + \cdots + \left\lfloor\frac{n}{b}\right\rfloor \;=\; \left\lfloor\frac{n}{\frac{n}{b}+1}\right\rfloor + \cdots + \left\lfloor\frac{n}{\frac{n}{a}}\right\rfloor \,.$$

Here, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

For example, if $n = 24$, $a = 3$, and $b = 6$, this says:

$$\left\lfloor\frac{24}{4}\right\rfloor + \left\lfloor\frac{24}{5}\right\rfloor + \left\lfloor\frac{24}{6}\right\rfloor \;=\; \left\lfloor\frac{24}{5}\right\rfloor + \left\lfloor\frac{24}{6}\right\rfloor + \left\lfloor\frac{24}{7}\right\rfloor + \left\lfloor\frac{24}{8}\right\rfloor \,,$$

which evaluates to the identity $6 + 4 + 4 = 4 + 4 + 3 + 3$.

Solution by the proposer.

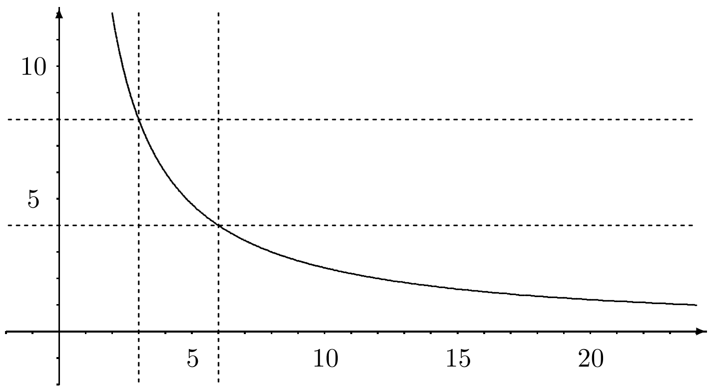
We want to show that

$$\left\lfloor\frac{n}{a+1}\right\rfloor + \cdots + \left\lfloor\frac{n}{b}\right\rfloor \qquad (1)$$

equals

$$\left\lfloor\frac{n}{\frac{n}{b}+1}\right\rfloor + \cdots + \left\lfloor\frac{n}{\frac{n}{a}}\right\rfloor \,. \qquad (2)$$

Consider the graph of the function $y = \frac{n}{x}$. Draw vertical lines at $x = a$ and $x = b$, and horizontal lines at $y = \frac{n}{b}$ and at $y = \frac{n}{a}$. (See diagram below for $n = 24$, $a = 3$, and $b = 6$.)



Then the number (1) is exactly the number of integer points lying below the graph, strictly to the right of the first vertical line and to the left of (or on) the second vertical line. Similarly, the number (2) is exactly the number of integer points lying to the left of the graph, below (or on) the upper vertical line and strictly above the lower vertical line.

To show that (1) and (2) are equal, we can forget the points which were counted in both (1) and (2). We need to show that the number of points counted in (1) but not in (2) is the same as the number of points counted in (2) but not in (1). This is relatively easy, because both of those regions are simply rectangular.

The points counted in (1) but not in (2) are on or below $y = \frac{n}{b}$, and between $x = a + 1$ and $x = b$. There are $\left(\frac{n}{b}\right) \times (b - (a+1) + 1) = n\left(1 - \frac{a}{b}\right)$ of these. Similarly, the points counted in (2) but not in (1) are on or to the left of $x = a$, above $y = \frac{n}{b}$ and below or on $y = \frac{n}{a}$. There are $a \times \left(\frac{n}{a} - \left(\frac{n}{b} + 1\right) + 1\right) = n\left(1 - \frac{a}{b}\right)$ of these.

Thus, the two numbers are the same, and the equality is established.

M41. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find the number of orders of wins and losses that can occur in a World Series. For example if the series ends after five games there are eight possible orders: ANNNN NANNN NNANN NNNAN NAAAA ANAAA AANAA AAANA where A is for an American League win and N is for a National League win. Note that the series ends as soon as one team wins four games.

Solution by Sabrina Liao, student, York Mills C.I., North York, ON.

Since the series ends when a team wins 4 games, the series could end after 4, 5, 6, or 7 games.

- 4 game series: 2 possible orders: NNNN AAAA.
- 5 game series: 8 possible orders: NAAAA ANAAA AANAA AAANA and 4 more with the N and A reversed.
- 6 game series: If N wins, then the order must end with N, with 2 A's and 3 N's in the other 5 positions. Using a similar argument for when A wins, we find that the number of orders is $2 \times \binom{5}{2} = 20$.
- 7 game series: By the same reasoning, there will be $2 \times \binom{6}{3} = 40$ orders.

Thus, there altogether $2 + 8 + 20 + 40 = 70$ orders of wins and losses.

Also solved by George Adler, student, Gloucester H.S., Gloucester, ON; Steven Béliveau, student, Ecole d'éducation internationale de Laval, Laval, QC; Robert Bilinski, Outremont, QC; Kevin Chung, OAC student, Earl Haig S.S., North York, ON; Jean-Philippe Lemieux, student, Ecole secondaire Dorval-Jean XXIII, Dorval, QC; Peng Liu, student, Glebe C.I., Ottawa, ON; James Meredith, Hudson H.S., Hudson Heights, QC; Rébecca Millette, student, Ecole secondaire Dorval-Jean XXIII, Dorval, QC; Alexandra Ortan, student, Ecole Joseph-François-Perrault, Montréal, QC; Maxime Pelletier, student, Collège Sainte-Anne de Lachine, Lachine, QC; Jing Qin, student, Ecole Emile-Legault, Saint-Laurent, QC; Diana Rapeanu, student, Collège Notre-Dame Sacré-Coeur, Montréal, QC; Sarah Shaker, student, Ecole secondaire Félix-Leclerc, Pointe-Claire, QC; Siwen Sun, student, Collège Sainte-Louis, Lachine, QC; Bob Wang, student, Merivale H.S., Nepean, ON; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Nicolas Wionzek, student, Almonte and District H.S., Almonte, ON. Four incorrect solutions were received.

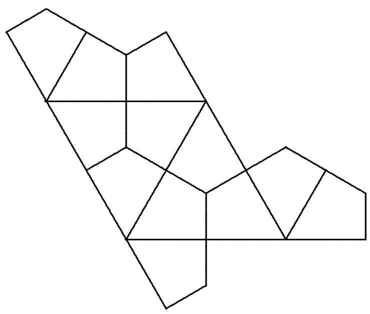
The proposer notes that of the 70 possible orders, 43 have occurred. He provides the following list, of when the orders occurred, from most to least frequent, for your enjoyment.

AAAA: *1914, 1927, 1928, 1932, 1938, 1939, 1950, 1966, 1989, 1998, 1999;*
ANAAA: *1913, 1941, 1943, 1949, 1961, 1974, 1984;*
NNNN: *1907, 1922, 1954, 1963, 1976, 1990;*
NAAANA: *1911, 1935, 1936, 1948, 1992;*
AANAA: *1916, 1929, 2000;*
NNANN: *1908, 1933, 1988;*
ANAANA: *1918, 1977, 1993;*
NAAAA: *1915, 1983;*
NNAAAA: *1978, 1996;*
AANNNAA: *1987, 1991;*
NNANAAA: *1958, 1985;*
ANANANN: *1940, 1946;*
ANNANAN: *1931, 1975;*
NANANAN: *1909, 1997;*
AAANA: *1910, 1937, 1970;*
AANNAA: *1917, 1930, 1953;*
AANNNAN: *1955, 1965, 1971;*
ANNNN: *1942, 1969;*
NANAAA: *1923, 1951;*
NANANAA: *1924, 1952;*
ANAANNN: *1925, 1979;*
ANNAANN: *1926, 1982;*
NAANNAN: *1960, 1964;*
NANNN: *1905;*
NNAANN: *1980;*
NNANAN: *1995;*
AANNANA: *1947;*
ANANNAA: *1973;*
AANNANN: *1986;*
NANNAAN: *1967;*
ANANAA: *1906;*
ANANNN: *1944;*
NNAAANA: *1956;*
ANAANNA: *1912;*
NANAANA: *1945;*
ANANNAN: *1957;*
NNAAANN: *2001.*
AANNNN: *1981;*
ANNNAN: *1959;*
AANANNA: *1972;*
ANANANA: *1962;*
NANNAAA: *1968;*
NANAANN: *1934;*

The proposer notes that the years 1903, 1919, 1920, and 1921 are not included, because in these years the winner won five games.

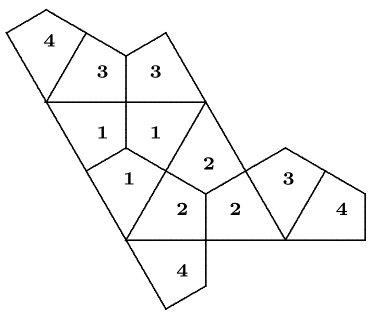
M42. *Proposed by Izidor Hafner, Tržaška 25, Ljubljana, Slovenia.*

The diagram below represents the net of a polyhedron. The faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.



Solution by Kevin Chung, OAC student, Earl Haig S.S., North York, ON.

The original polyhedron is a regular tetrahedron. The faces are numbered as below.



M43. *Proposed by the Mayhem staff.*
Prove that

$$\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{2002}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{2002}$$

is an integer.

Solution by Austrian 2002 *IMO team.*
Let a_n be the sequence with

$$a_n=\frac{29-5\sqrt{29}}{58}\left(\frac{7+\sqrt{29}}{2}\right)^{n}+\frac{29+5\sqrt{29}}{58}\left(\frac{7-\sqrt{29}}{2}\right)^{n}.$$

Thus, $\frac{7+\sqrt{29}}{2}$ and $\frac{7-\sqrt{29}}{2}$ are the two roots of the characteristic equation, $x^2+px+q=0$, of a_n. Therefore, we have

$$\begin{aligned} p &= -\left(\frac{7+\sqrt{29}}{2}+\frac{7-\sqrt{29}}{2}\right) = -7\,, \\ q &= \frac{7+\sqrt{29}}{2}\times\frac{7-\sqrt{29}}{2} = 5\,. \end{aligned}$$

Thus, the characteristic equation is $x^2=7x-5$. Therefore, the recurrence relation is $a_{n+2}=7a_{n+1}-5a_n$. Now we only need determine the first two values, which are $a_0=1$ and $a_1=1$.

Because a_0 and a_1 are integers and the coefficients of the recursion formula are integers, all values of the sequence are integers.

Also solved by Kevin Chung, OAC student, Earl Haig S.S., North York, ON. Seven incomplete or incorrect solutions were received.

The mail gremlins were at work last month; we received solutions from Andrew Mao, grade 10 student, H.B. Beal S.S., London, ON for M30, M32, M33, M34, M35, M36, and M37. Sorry about that Andrew. This issue's Mayhem Taunt winner is . . . Andrew Mao! Andrew will receive a subscription to ***Crux Mathematicorum*** for 2003.

High, Low, High, Low, It's Off To Work We Go

Zhe Li and Paul Belcher

In the British Mathematical Olympiad Round 1 in 1995 there was the following question:

5. The seven dwarfs walk to work each day in single file, with heights alternating up-down-up- or down-up-down-. For how long can they continue with a new order every day? What if Snow White always comes too?

We will generalize this problem. Let $w(n)$, $n \in \mathbb{N}^+$, represent the number of ways that n dwarves of different heights can walk to work with their heights alternating up, down, up, ..., or down, up, down, ... Directly listing and counting gives $w(1) = 1$, $w(2) = 2$, $w(3) = 4$, $w(4) = 10$, and $w(5) = 32$.

The heights of the dwarves can be represented, from smallest to largest simply as $1, 2, 3, \ldots, n$. (Since we are interested only in the relative heights, any increasing sequence will do here.) If we let $u(n)$ be the number of arrangements of the heights that go up, down, up, ..., and $d(n)$ be the number of arrangements that go down, up, down, ..., then we notice that $u(n) = d(n)$, for $n \geq 2$. We can prove this as follows.

Let $a_1, a_2, \ldots, a_n$ be an arrangement that goes up, down, up, ... To this we apply the transformation $a_i \to (n+1) - a_i$. This will give a unique new arrangement that goes down, up, down, ... To visualize this, let the height of the tunnel in which the dwarves are walking to work, singing, be $n+1$. Then consider another set of dwarves upside down with their feet on the roof of the tunnel, directly above the first set, as shown in Figure 1.

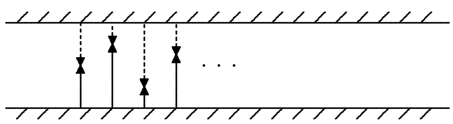


Figure 1

We now attempt to find a recurrence relation for $w(n)$. Let the tallest dwarf, of height n, who will be called Max, be in the i^{th} position. There are $i-1$ dwarves before Max, and they could be selected in $\binom{n-1}{i-1}$ ways. Their order would have to go ..., up, down, and finally up to Max. Apart from the cases where $i = 1$ or $i = 2$, the number of ways of correctly arranging these $i-1$ dwarves is $d(i-1) = \frac{1}{2}w(i-1)$, as we can see by considering the dwarves in reverse order. Following Max, the $n-i$ dwarves must have an order going down from Max, then up, down, ... Apart from the cases where $i = n-1$ or $i = n$, the number of ways of correctly arranging these $n-i$ dwarves is $d(n-i) = \frac{1}{2}w(n-i)$.

We have a problem with $n = 1$, since $w(1) = u(1) = d(1) = 1$. To deal with this, we shall define a new sequence $W(n)$ by $W(1) = W(0) = 2$ and $W(n) = w(n)$ if $n \geq 2$. We consider $u(0) = d(0) = 1$. Then we have the recurrence relation

$$W(n) \;=\; \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{W(i-1)}{2} \frac{W(n-i)}{2}\,, \quad n \;\geq\; 2\,, \tag{1}$$

with $W(0) = 2$, $W(1) = 2$.

From this recurrence relation we can build up our sequence as follows: $W(2) = 2$, $W(3) = 4$, $W(4) = 10$, $W(5) = 32$, $W(6) = 122$, $W(7) = 544$. This agrees with our directly computed list and answers the question for 7 dwarves: they can walk to work in 544 ways with an alternating height pattern. If Snow White wants to take part as well, then calculating one step further gives $W(8) = 2770$.

If we let $t(n) = [W(n)]/n!$, then $t(n) \to 0$ as $n \to \infty$, as we will now show. Suppose we take an arrangement of dwarves having the required property and exchange two successive dwarves. The new arrangement will not have the required property if $n \geq 3$. There are $n-1$ ways of exchanging two dwarves, producing $(n-1)W(n)$ arrangements which are all different. Thus, the number of arrangements not having the desired property is at least $(n-1)W(n)$. Hence, $n! \geq (n-1)W(n) + W(n) = nW(n)$. Therefore,

$$\frac{W(n)}{n!} \;\leq\; \frac{1}{n}\,, \quad n \;\geq\; 3\,,$$

giving the result. Therefore, if the dwarves just arrange themselves randomly, then the probability that the arrangement has the desired property tends to zero as n tends to infinity.

We now attempt to find a non-recursive formula for $W(n)$ by means of a generating function for $t(n)$. From (1), replacing $i-1$ by i, we have

$$W(n) \;=\; \frac{1}{4}\sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-1-i)!} W(i)W(n-1-i)\,, \quad n \;\geq\; 2\,.$$

Hence,

$$nt(n) \;=\; \frac{1}{4}\sum_{i=0}^{n-1} t(i)t(n-1-i)\,, \quad n \geq 2\,, \tag{2}$$

with $t(0) = 2$ and $t(1) = 2$. Consider the generating function $h(x)$:

$$h(x) \;=\; t(0) + t(1)x + t(2)x^2 + \cdots + t(n)x^n + \cdots\,.$$

In $\frac{1}{4}[h(x)]^2$, the first term is 1 and the coefficient of x^{n-1}, for $n \geq 2$, is $\frac{1}{4}\sum\limits_{i=0}^{n-1} t(i)t(n-1-i) = nt(n)$, using (2). Also,

$$h'(x) = t(1) + 2t(2)x + \cdots + nt(n)x^{n-1} + \cdots .$$

Therefore, $h'(x) = \frac{1}{4}[h(x)]^2 + 1$. Letting $h(x) = y$, we have the differential equation $\frac{dy}{dx} = \frac{1}{4}y^2 + 1$, which has solution $\arctan\left(\frac{y}{2}\right) = \frac{x}{2} + c$. When $x = 0$, we get $y = h(0) = t(0) = 2$, and also $c = \arctan(1) = \frac{\pi}{4}$. Thus, $y = h(x) = 2\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$.

By considering the Taylor expansion of $h(x)$, we find that $t(n)$, the coefficient of x^n, is given by $t(n) = \frac{2}{n!}D^n \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\Big|_{x=0}$. Thus,

$$W(n) \;=\; 2D^n \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\Bigg|_{x=0} \;=\; \frac{1}{2^{n-1}}D^n \tan x\Bigg|_{x=\frac{\pi}{4}} .$$

The table below verifies this result for small values of n.

n	$D^n \tan(x)$	$D^n \tan(x)$ at $x = \frac{\pi}{4}$	$\frac{1}{2^{n-1}}D^n \tan(x)$ at $x = \frac{\pi}{4}$
0	$\tan x$	1	2
1	$1 + \tan^2 x$	2	2
2	$2\tan x(1 + \tan^2 x)$	4	2
3	$2(3\tan^2 x + 1)(1 + \tan^2 x)$	16	4
4	$8\tan x(3\tan^2 x + 2)(1 + \tan^2 x)$	80	10
5	$8(1 + \tan^2 x)$ $\times(15\tan^4 x + 15\tan^2 x + 2)$	512	32
6	$16\tan x(1 + \tan^2 x)$ $\times(45\tan^4 x + 60\tan^2 x + 17)$	3904	122

Our conclusion is that the number of ways, $w(n)$, of arranging n objects of different magnitudes in order so that the differences in magnitude alternate between positive and negative is given by

$$w(n) \;=\; \frac{1}{2^{n-1}}D^n \tan x\Bigg|_{x=\frac{\pi}{4}} ,$$

$n \geq 2$, with $w(1) = 1$.

Zhe Li is a 20-year-old student from China at Atlantic College who investigated this problem for his extended essay in Maths in the International Baccalaureate. Paul Belcher is Head of Maths at Atlantic College and was supervisor for the extended essay.

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Paul Ottaway

This month I would like to spend some time examining some difficult easy problems. That may seem rather contradictory. The three problems I have selected are taken from past grade 7 and 8 mathematics contests. This is what will make them easier than some of the problems I have posed in the past. On the other hand, they are the last questions from their respective contests and will therefore require a little more insight and creativity on the part of the reader. I should also note that, when attempting these problems, we must put on our "grade 8 hats" and forget all the mathematics we have come to learn since that time. I find it remarkable how much mathematics I take for granted. It is a worthwhile exercise to return to the basics.

One of my favourite aspects of grade 7 and 8 problems is that they can be used as a teaching tool. After years of doing mathematics contests, the tricks and techniques seem commonplace. I have to remember, however, that at some point even I saw them for the first time.

The following problem has a solution that is related to a particular set of numbers that a grade 7 student may not be familiar with. Naturally, this pattern is rather intriguing and can be used as a platform for many more problems in the future.

Gauss, Grade 7, $\#25$, 2001

A triangle can be formed having side lengths 4, 5, and 8. It is impossible, however, to construct a triangle having side lengths 4, 5, and 9. Ron has 8 sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. The shortest possible length of the longest of the eight sticks is

(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

Solution: Since we want the shortest possible length for the longest stick, it makes sense to start with the smallest sticks, each having length 1. Although a grade 8 student may not recognize it as such, the triangle inequality is involved in the question. It is clear that if sticks of length a and b are in our set, there cannot be any other stick of length less than $a+b$, since these three would then form a triangle. Using this information, we systematically add more sticks, taking the smallest possible length at each step. This gives us sticks with lengths 1, 1, 2, 3, 5, 8, 13, and 21. Is it much of a coincidence that these numbers form the Fibonacci sequence? Although this is not a formal proof that 21 is the shortest possible, a simple proof by contradiction could be used to show it is indeed a lower bound. This is left as an exercise.

I have found that the most prevalent formulas when solving problems in a contest setting are the formulas for the sums of arithmetic and geometric sequences. These cropped up in at least one question, it seems to me, from every contest that I can remember writing. Even though Gauss is said to have found one of these formulas at a ridiculously young age, most of our grade 7 and 8 students do not know them. How would we then approach the following problem?

Gauss, Grade 8, $\#21$, 1999:

The sum of seven consecutive integers is always:

(A) odd (B) a multiple of 7 (C) even
(D) a multiple of 4 (E) a multiple of 3

Solution 1: If we jump right to the formula for the sum of an arithmetic sequence, we can see that the sum starting at a is $7(a-1)+28$ which is always a multiple of 7. Easy enough, right? What if we did not know the formula for the sum of an arithmetic sequence?

Solution 2: It is easy for a grade 8 student to check the first couple of values obtained by computing these sums. For instance, $1+2+3+4+5+6+7=28$ and $2+3+4+5+6+7+8=35$. Now, since we know that one of the 5 possible answers must be correct we can now eliminate ones that are not satisfied by the values we have computed. Looking at 28, we can eliminate (A) and (E). Then, looking at 35 we can eliminate (C) and (D). That leaves only one possible answer, namely (B).

To quote Sir Arthur Conan Doyle, "*When you have eliminated the impossible, that which remains, however improbable, must be the truth.*"

This technique, although not appropriate when trying to solve problems in general, can be very useful and time-saving when attacking multiple choice questions. In my experience, this particular problem takes an average high school student longer than a grade 7 student. First, they remember that a formula exists; then, they spend quite a bit of time trying to remember or re-derive it before they can solve the problem!

For my last problem this month, I have chosen one of my favourites. It exemplifies the idea of using a problem as a teaching tool, as well as appealing to the more limited mathematical knowledge of a grade 7 student.

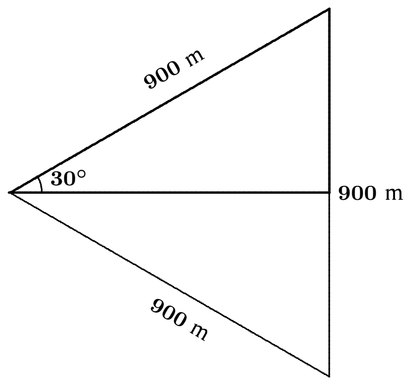
Variant on a Gauss, Grade 7**, problem**:

A particular ski lift takes passengers from the base of a mountain to its peak at a rate of 1.5 m/s. The lift travels at a 30 degree incline and it takes 10 minutes to reach the top. What is the height of the mountain?

Solution 1: We begin with a simple calculation: Ten minutes is the same as 600 seconds; hence, we have travelled a total of $(1.5)(600)=900$ metres. Fitting this to a diagram, we get a $30°$-$60°$-$90°$ triangle with 900 m along its hypotenuse. Some simple trigonometry will tell us that the height of the mountain is 450 m.

But wait! How many grade 7 students do you know who have a working understanding of trigonometry? It certainly is not common. How do you think a grade 7 student can solve this problem? The following solution is my best guess.

Solution 2: First, we recognize that a $30°$-$60°$-$90°$ triangle is really half of an equilateral triangle (we reflect along the side opposite the 60-degree angle).



Each side of this triangle is exactly 900 m. Now we see that the two $30°$-$60°$-$90°$ triangles are identical (congruent). Therefore, 900 m is twice the length that we are looking for. Thus, the height of the mountain is 450 m.

This, in my opinion, is one of the best forms of problem solving. It is not simply that we can calculate what the answer should be, but that we can calculate it without some of the "higher" mathematics that many of us take for granted. In some ways, knowing too much can slow us down and prevent us from being creative when the need arises.

I am always impressed when a student can solve the problem I have just shown here. If you remember back to when you first learned the $30°$-$60°$-$90°$ triangle and the 1-$\sqrt{3}$-2 side lengths, it was probably proven to you by taking an equilateral triangle, splitting it in two, and applying the Pythagorean theorem. By solving this problem, a grade 7 student has demonstrated that she can independently develop the same result years before it is taught in her curriculum. In my mind, this is problem-solving at its best.