SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2718. [2002 : 112] *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $A_k \in M_m(\mathbb{R})$ tels que $A_iA_j = O_m$, $i,\, j \in \{1,\,2,\,\ldots,\,n\}$, avec $i < j$ et $x_k \in \mathbb{R}^*$, $(k = 1,\,2,\,\ldots,\,n)$. Montrer que

$$\det\left(I_m + \sum_{k=1}^{n}\left(x_kA_k + x_k^2A_k^2\right)\right) \;\geq\; 0\,.$$

Solution par Michel Bataille, Rouen, France.

Pour chaque $p \in \{1,\,2,\,\ldots,\,n\}$ posons

$$B_p \;=\; I_m + \sum_{k=1}^{p}(x_kA_k + x_k^2A_k^2) \qquad \text{et} \qquad C_p \;=\; I_m + x_pA_p + x_p^2A_p^2\,.$$

On désire prouver : $\det(B_n) \geq 0$.

En utilisant $A_1A_2 = 0_m$, on voit aussitôt que $C_1C_2 = B_2$. Puis, en utilisant $A_1A_3 = A_1A_2 = 0_m$, il suit que $C_1C_2C_3 = B_2C_3 = B_3$. Par une récurrence immédiate, on arrive à $B_n = C_1C_2\cdots C_n$, et donc

$$\det(B_n) \;=\; \det(C_1)\det(C_2)\cdots\det(C_n)\,.$$

Il suffit donc de prouver $\det(C_p) \geq 0$, $(p = 1,\,2,\,\ldots,\,n)$. Maintenant,

$$C_p \;=\; (I_m - \omega x_pA_p)(I_m - \overline{\omega}x_pA_p) \;=\; (I_m - \omega x_pA_p)\overline{(I_m - \omega x_pA_p)}\,,$$

où $\omega = \exp(2\pi i/3)$. (La barre indiquant la conjugaison complexe. Il s'ensuit que

$$\begin{aligned}\det(C_p) \;&=\; \det(I_m - \omega x_pA_p)\det\overline{(I_m - \omega x_pA_p)}\\ &=\; \det(I_m - \omega x_pA_p)\overline{\det(I_m - \omega x_pA_p)}\\ &=\; |\det(I_m - \omega x_pA_p)|^2 \;\geq\; 0\,.\end{aligned}$$

Also solved by TIM D. AUSTIN, student, Colchester Royal Grammar School, Colchester, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

2724★. [2002 : 174] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle and h_a, h_b, h_c, respectively, the corresponding altitudes. Prove that the maximum range of validity of the inequality

$$\left(\frac{h_a^t+h_b^t+h_c^t}{3}\right)^{1/t} \;\le\; \frac{\sqrt{3}}{2}\left(\frac{a^t+b^t+c^t}{3}\right)^{1/t},$$

where $t \neq 0$, is $\dfrac{-\ln 4}{\ln 4 - \ln 3} \;<\; t \;<\; \dfrac{\ln 4}{\ln 4 - \ln 3}$.

Partial solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.

We show that the inequality is valid for a narrower range of values, namely $0 < t \le \dfrac{\ln 9 - \ln 4}{\ln 4 - \ln 3}$.

If we denote the area of the triangle by F and the circumradius by R, then letting $h_a = 2F/a$, $a = 2R^2 \sin A \sin B \sin C$, etc., the given inequality can be rewritten as

$$\sum_{\text{cyclic}} \sin^t A \;\ge\; \left(\frac{2}{\sqrt{3}}\right)^t \sum_{\text{cyclic}} \sin^t B \sin^t C\,. \tag{1}$$

The inequality

$$\sum_{\text{cyclic}} \sin^t A \;\le\; 3\left(\frac{\sqrt{3}}{2}\right)^t \tag{2}$$

is a known inequality ([1]) having maximum range $0 \le t < \dfrac{\ln 9 - \ln 4}{\ln 4 - \ln 3}$. Since $3(yz+zx+xy) \le (x+y+z)^2$, we have

$$3\left(\sum_{\text{cyclic}} \sin^t B \sin^t C\right) \;\le\; \left(\sum_{\text{cyclic}} \sin^t A\right)^2$$

$$\frac{3\left(\sum_{\text{cyclic}} \sin^t B \sin^t C\right)}{\sum_{\text{cyclic}} \sin^t A} \;\le\; \sum_{\text{cyclic}} \sin^t A \;\le\; 3\left(\frac{\sqrt{3}}{2}\right)^t,$$

from which (1) follows. This proves (1) for $0 < t < \dfrac{\ln 9 - \ln 4}{\ln 4 - \ln 3}$, which proves the proposed inequality for this range of t.

Incidentally, the value $\dfrac{\ln 9 - \ln 4}{\ln 4 - \ln 3}$ is obtained from (2) by equality for the degenerate triangle with angles of 90°, 90°, and 0°. The proposed upper bound on t is obtained from (1) by equality for the same degenerate triangle.

References.

[1] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969.

[*Editor's Note:* For negative values of t, when we replace t by $-t$ in the inequality (1), we obtain

$$\left(\frac{2}{\sqrt{3}}\right)^{-t}\left(\sin^{-t}B\sin^{-t}C+\sin^{-t}C\sin^{-t}A+\sin^{-t}A\sin^{-t}B\right) \leq \sin^{-t}A+\sin^{-t}B+\sin^{-t}C\,.$$

Then, multiplying through by $\left(\frac{2}{\sqrt{3}}\right)^{t}\sin^t A\sin^t B\sin^t C$, we get the opposite inequality to (1). Thus, the proposed range of validity for the inequality in this problem must be amended to

$$0 < t < \frac{\ln 4}{\ln 4-\ln 3}\,.$$

We invite readers to find a solution that considers this entire range.]

2725. [2002 : 175] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $k \geq 1$, let $S_k(n) = \sum\limits_{j=1}^{n}(2j-1)^k$ be the sum of the k^{th} powers of the first n odd numbers.

1. Show that the sequence $\{S_3(n), n \geq 1\}$ contains *infinitely many squares*.
2. ⋆ Prove that this sequence contains only *finitely many squares* of other exponents k.

[*Editor's comments:* Clearly, for Part 2 of the problem to be non-trivial, we must state that $k > 1$ (since $S_1(n) = n^2$, which is a square for all n). This was actually stipulated by the proposer in his original submission, but was inadvertently omitted. Apparently, all the solvers realized this omission and interpreted the question correctly. Also, ".. OF other exponents" was clearly a typo for ".. FOR other exponents".]

Solution to Part 1 *by the Austrian IMO Team 2002* (*slightly modified by the editor*).

Note first that

$$\begin{aligned} S_3(n) &= \sum_{j=1}^{2n} j^3 - \sum_{j=1}^{n}(2j)^3 \\ &= \left(\frac{2n(2n+1)}{2}\right)^2 - 8\left(\frac{n(n+1)}{2}\right)^2 \;=\; n^2(2n^2-1)\,. \end{aligned}$$

Hence, $S_3(n)$ is a square if and only if $2n^2 - 1 = m^2$ for some integer m. The Diophantine equation $m^2 - 2n^2 = -1$ is a Pell-type equation which is well known to have infinitely many solutions, since 2 is not a perfect square. Indeed, $m = n = 1$ is clearly a solution. If we define m_k and n_k for $k \in \mathbb{N}$ by

$$m_1 = n_1 = 1\,, \qquad m_k + \sqrt{2}n_k \;=\; (m_1 + \sqrt{2}n_1)^{2k-1}\,,$$

then $m_k - \sqrt{2}n_k = (m_1 - \sqrt{2}n_1)^{2k-1}$, and hence,

$$\begin{aligned} m_k^2 - 2n_k^2 &= \Big((m_1 + \sqrt{2}n_1)(m_1 - \sqrt{2}n_1)\Big)^{2k-1} \\ &= (m_1^2 - 2n_1^2)^{2k-1} \;=\; -1\,. \end{aligned}$$

Therefore, every pair (m_k, n_k) is a solution. It follows that the sequence $\{S_3(n)\}_{n=1}^{\infty}$ contains infinitely many squares.

Also solved (Part 1 only) by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

All the submitted solutions are similar to the one featured above. Both Klamkin and Zhou simply quoted known results about the existence of infinitely many solutions to the Pell equation involved. In fact, from known theory, the solutions obtained above actually yield all the n values for which $S_3(n)$ is a square. This was explicitly pointed out only by Guersenzvaig. No solution to Part 2 was received, so it remains open. Klamkin commented that this may be a very difficult problem in general, but the case $k = 2$ might be solvable, since the corresponding Diophantine equation is $n(2n-1)(2n+1) = 3m^2$.

2726. [2002 : 175] *Proposed by Armend Shabini, University of Prishtina, Prishtina, Kosovo, Serbia.*

Given the finite sequence of real numbers, $\{a_k\}$, $1 \le k \le 2n$, where the terms satisfy

$$a_{2k} - a_{2k-1} \;=\; d\,,\; 1 \le k \le n\,, \qquad \text{and} \qquad \frac{a_{2k+1}}{a_{2k}} \;=\; q\,,\; 1 \le k \le n-1\,,$$

prove that, when $q \neq 1$,

(a) $\displaystyle\sum_{k=1}^{2n} a_k \;=\; \frac{2qa_{2n} - 2a_1 - nd(1+q)}{q-1}$, and

(b) $\displaystyle a_{2n} \;=\; a_1\,q^{n-1} + d\left(\frac{1-q^n}{1-q}\right)$.

[*Editor's Note:* The formula in (b) was misprinted originally but has been corrected above. The editors were at fault here, not the proposer. All solvers corrected the error.]

Solution by Joe Howard, Portales, NM, USA.

$$\begin{aligned}
\sum_{k=1}^{2n} a_k &= a_1 + (1+q)a_2 + \cdots + (1+q)a_{2n} - qa_{2n} \\
&= (1+q)(a_2 + a_4 + \cdots + a_{2n}) - qa_{2n} + a_1 \\
&= (1+q)(a_1 + a_3 + \cdots + a_{2n-1} + nd) - qa_{2n} + a_1 \\
&= (a_3 + a_5 + \cdots + a_{2n-1}) + q(a_1 + a_3 + \cdots + a_{2n-1}) \\
&\quad -qa_{2n} + 2a_1 + nd(1+q) \\
&= q(a_2 + a_4 + \cdots + a_{2n}) + q(a_1 + a_3 + \cdots + a_{2n-1}) \\
&\quad -2qa_{2n} + 2a_1 + nd(1+q) \\
&= q\sum_{k=1}^{2n} a_k - 2qa_{2n} + 2a_1 + nd(1+q)\,,
\end{aligned}$$

from which (a) follows.

An easy induction argument shows that, for $k = 1, 2, \ldots, n$,

$$a_{2k} = a_1 q^{k-1} + d(1 + q + q^2 + \cdots + q^{k-1})\,.$$

By setting $k = n$ and summing the geometric progression, we obtain (b).

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

2727. [2002 : 176] *Proposed by Armend Shabini, University of Prishtina, Prishtina, Kosovo, Serbia.*

Given the finite sequence of real numbers, $\{a_k\}$, $1 \le k \le n$, where the terms satisfy

$$a_k - a_{k-1} \;=\; a_{k-1} - a_{k-2} + d\,, \quad k > 2\,,\; d \in \mathbb{R}\,,$$

find a closed form expression for $\sum\limits_{k=1}^{n} a_k$.

Use this to find the value of $\sum\limits_{k=0}^{n-1} \binom{2k+2}{2k}$.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Solving the non-homogeneous recurrence relation, we see that $a_k = \frac{1}{2}dk^2 + Bk + C$, for constants B and C depending on a_1 and a_2. To verify that this is a solution, observe:

$$a_k - 2a_{k-1} + a_{k-2} \;=\; \frac{1}{2}d\left(k^2 - 2(k-1)^2 + (k-2)^2\right) \;=\; d$$

(the linear terms clearly cancel). Solving for B and C gives

$$B \;=\; a_2 - a_1 - \frac{3}{2}d \quad \text{and} \quad C \;=\; -a_2 + 2a_1 + d\,.$$

Hence,

$$\sum_{k=1}^{n} a_k \;=\; \frac{1}{2}d\left[\frac{1}{6}n(n+1)(2n+1)\right] + B\left[\frac{1}{2}n(n+1)\right] + Cn\,,$$

using standard summation results.

We see that if $a_k = \binom{2k}{2}$, then $a_k - a_{k-1} = 4k - 3$. Thus, $d = 4$. Checking the first two terms, we have $B = -1$, $C = 0$. Therefore,

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{2k+2}{2k} &= \sum_{k=1}^{n} a_k \;=\; 2\left[\frac{1}{6}n(n+1)(2n+1)\right] - \left[\frac{1}{2}n(n+1)\right] \\ &= \frac{1}{6}n(n+1)(4n-1)\,.\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Bergen, Norway; VEDULA N. MURTY, Dover, PA, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Editor's note: Some solutions used the fact that the third differences of a_k are zero; therefore, an expression for a_k can be obtained using Newton-Gregory forward differences.

2728. [2002 : 177] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

The distance between two well-known points in $\triangle ABC$ is

$$\frac{bc}{a+b+c}\sqrt{2(\cos A + 1)}\,.$$

What are the points?

A combination of almost identical solutions by Nikolaos Dergiades, Thessaloniki, Greece and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let I be the incentre of $\triangle ABC$, and let r be the inradius. Let $[ABC]$ denote the area of $\triangle ABC$, and let s be its semiperimeter. Then

$$\begin{aligned}AI &= \frac{r}{\sin(A/2)} = \frac{[ABC]}{s\,\sin(A/2)} = \frac{bc\,\sin(A)}{(a+b+c)\sin(A/2)} \\ &= \frac{bc}{a+b+c}\left(2\cos(A/2)\right) = \frac{bc}{a+b+c}\sqrt{2(\cos A + 1)}\,.\end{aligned}$$

The answer is: the points are the vertex A and the incentre I.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; TITU ZVONARU and BOGDAN IONIȚĂ, Bucharest, Romania; and the proposer.

2729. [2002 : 177] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let $Z(n)$ denote the number of trailing zeroes of $n!$, where $n \in \mathbb{N}$.

(a) Prove that $\dfrac{Z(n)}{n} < \dfrac{1}{4}$.

(b)⋆ Prove or disprove that $\lim\limits_{n\to\infty} \dfrac{Z(n)}{n} = \dfrac{1}{4}$.

Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.

(a) Clearly, $\mathbb{Z}(n) = E_5(n)$, where $E_5(n)$ denotes the largest integer d such that $5^d \mid n!$. By the well-known Legendre formulas [Ed: see, for example, Calvin T. Long, *Elementary Introduction to Number Theory*, 3rd edition, pp 64–67, Theorems 2.29 and 2.30], we have

$$E_5(n) = \sum_{k\geq 1} \left\lfloor \frac{n}{5^k} \right\rfloor = \frac{n - S_5(n)}{5-1} = \frac{n - S_5(n)}{4},$$

where $S_5(n)$ is the sum of the digits in the base-5 representation of n. Since $S_5(n) \geq 1$, it follows immediately that $\dfrac{\mathbb{Z}(n)}{n} < \dfrac{1}{4}$.

(b) Let $n = (n_r \cdots n_1 n_0)_5$ be the base-5 representation of n, where $n_i \in \{0, 1, 2, 3, 4\}$ for all $i = 0, 1, 2, \ldots, r$. From $5^r \leq n < 5^{r+1}$ we get $r = \lfloor \log_5 n \rfloor$. Hence, $S_5(n) \leq 4(r+1) = 4(\lfloor \log_5 n \rfloor + 1)$, and therefore, $\dfrac{S_5(n)}{4n} \leq \dfrac{\lfloor \log_5 n \rfloor + 1}{n}$. Since $0 \leq \dfrac{S_5(n)}{4n}$ and $\lim\limits_{n\to\infty} \dfrac{\lfloor \log_5 n \rfloor + 1}{n} = 0$, we have $\lim\limits_{n\to\infty} \dfrac{S_5(n)}{4n} = 0$ by the Squeeze Theorem. It follows that

$$\lim_{n\to\infty} \frac{\mathbb{Z}(n)}{n} = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{S_5(n)}{4n} \right) = \frac{1}{4}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; TIM D. AUSTIN, Student, Colchester Royal Grammar School, Colchester, UK; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PIERRE BORNSZTEIN, Pontoise, France; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, WA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, *Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; OLOV WILANDER, student, Christ's College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA. Part* (a) *only was also solved by CHARLES ASHBACHER, Hiawatha, IA, USA; The AUSTRIAN IMO TEAM, 2002; NIKOLAOS DERGIADES, Thessaloniki, Greece; and the proposer. There were two incorrect solutions to part* (b).

Both Loeffler and Wilander studied the general problem of expanding $n!$ in base b. Both of them remarked that if $b = \prod_i p_i^{\alpha_i}$, then

$$\lim_{n\to\infty} \frac{\mathbb{Z}_b(n)}{n} \;=\; \min_i \left(\frac{1}{\alpha_i(p_i-1)}\right) ,$$

where $\mathbb{Z}_b(n)$ denotes the number of trailing zeroes of $n!$ in base b.

2730. [2002 : 177] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ and $\mathrm{GM}(x_1, x_2, \ldots, x_n)$ denote the arithmetic mean and the geometric mean of the real numbers x_1, x_2, $\ldots$, x_n, respectively.

Given positive real numbers a_1, a_2, $\ldots$, a_n, b_1, b_2, $\ldots$, b_n, prove that

(a) $\mathrm{GM}(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$
$\quad\geq\; \mathrm{GM}(a_1, a_2, \ldots, a_n) + \mathrm{GM}(b_1, b_2, \ldots, b_n)$.

For each real number $t \geq 0$, define $f(t) = \mathrm{GM}(t+b_1, t+b_2, \ldots, t+b_n) - t$.

(b) Prove that $f(t)$ is a monotonic increasing function of t, and that

$$\lim_{t\to\infty} f(t) \;=\; \mathrm{AM}(b_1, b_2, \ldots, b_n) .$$

[*Editor's note:* Several solvers pointed out that part (a) is ***CRUX with MAYHEM*** problem **2176** [1996 : 275; 1997 : 444].]

Solution to Part (b) *by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Let $G(x) = \mathrm{GM}(1+b_1x, 1+b_2x, \ldots, 1+b_nx)$. Then $f(t) = \dfrac{G(x)-1}{x}$ where $x = \dfrac{1}{t}$. Thus, $\lim\limits_{t\to\infty} f(t) = \lim\limits_{x\to 0} \dfrac{G(x)-1}{x}$. By l'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to 0} \frac{G(x)-1}{x} \;&=\; \lim_{x\to 0} G'(x) \;=\; \lim_{x\to 0} \frac{G(x)}{n} \sum_{k=1}^{n} \frac{b_k}{1+b_kx} \\ &=\; \mathrm{AM}(b_1, b_2, \ldots, b_n) .\end{aligned}$$

Also solved by TIM AUSTIN, Colchester Royal Grammar School, Colchester, UK; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; JOSÉ LUIS DÍAZ-BARRERO and JUAN JOSÉ EGOZCUE, Universitat Politècnica de Catalunya, Barcelona, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; M. PERISASTRY, Maharaja's College, Vizianagaram, India (part (a)); *JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de*

Valladolid, Valladolid, Spain; VEDULA N. MURTY, Dover, PA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

According to Howard, Part (a) *also appears in* Equations and Inequalities, *by Herman, Kucera, and Simsa, Springer-Verlag, p. 158, and in* Aspects of Calculus, *by Klambauer, p. 203.*

2731. [2002 : 178] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let C be a conic with foci F_1, F_2, and directrices D_1, D_2, respectively.

Given any point M on the conic, draw the line passing through M, perpendicular to the directrices, intersecting D_1, D_2, at M_1, M_2, respectively. Let R be the point of intersection of the lines M_1F_1 and M_2F_2. Prove that

(a) $\dfrac{\overline{F_1R}}{\overline{M_1R}}$ is independent of the choice of M;

(b) the normal to the conic at M passes through R.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) Since M_1M_2 is parallel to F_1F_2 it represents the (fixed) distance between the directrices. Moreover, by similar triangles, $\dfrac{F_1R}{M_1R} = \dfrac{F_1F_2}{M_1M_2}$. Thus, the ratio is independent of the choice of M.

(b) Suppose that MR intersects F_1F_2 at S. Then

$$\frac{F_1S}{M_1M} = \frac{RS}{RM} = \frac{F_2S}{M_2M}\,.$$

By the definition of focus and directrix, we also have $\dfrac{F_1M}{M_1M} = e = \dfrac{F_2M}{M_2M}$, where e is the eccentricity. Hence, $\dfrac{F_1S}{F_1M} = \dfrac{F_2S}{F_2M}$. In the case of an ellipse, this condition means that RM is the angle bisector of $\angle F_1MF_2$, and is thus normal to the conic at M, as desired. For the hyperbola case, we draw through F_1 the line parallel to F_2M, intersecting RS at T. Then $\dfrac{F_1S}{F_1T} = \dfrac{F_2S}{F_2M}$; hence, $F_1T = F_1M$. Consequently, $\angle F_1MS = \angle F_2MR$, and we conclude again that RM is the normal at M.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Loeffler notes that M does not need to lie on the conic for Part (a) *of the problem. Furthermore, Bataille shows that the ratio, F_1R/M_1R, in Part* (a) *is equal to e^2 as follows: In standard notation, $c = F_1F_2/2$, a is half the distance between the intersection points of F_1F_2 with the conic, a/e is the distance from the centre of the conic to a directrix, and $c/a = e$. Then*

$$\frac{F_1R}{M_1R} = \frac{F_1F_2}{M_1M_2} = \frac{c}{(a/e)} = e^2\,.$$

2733★. [2002 : 179] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

It is a known result that if O is the circumcentre of $\triangle A_1A_2A_3$, and if O_1, O_2, O_3, are the circumcentres of $\triangle OA_2A_3$, $\triangle OA_3A_1$, $\triangle OA_1A_2$, respectively, then the lines A_1O_1, A_2O_2 and A_3O_3 are concurrent.

Does the corresponding result hold for simplexes? That is, if O is the circumcentre of a simplex $A_0A_1\ldots A_n$ and O_k is the circumcentre of the simplex determined by O and the face opposite A_k, are the lines O_kA_k, $k=0, 1, \ldots, n$, concurrent?

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Professor Klamkin's result does not hold in 3 dimensions. If we consider a Cartesian coordinate system and consider the tetrahedron $ABCD$ where

$$A_1 = (0,0,0)\,,\quad A_2 = (1,0,0)\,,\quad A_3 = (0,1,0)\,,\quad A_4 = (0,0,2)\,,$$

then it is readily verified that

$$O = \left(\frac{1}{2},\frac{1}{2},1\right)\,,\quad O_1 = \left(-1,-1,\frac{1}{4}\right)\,,\quad O_2 = \left(-1,\frac{1}{2},1\right)\,,$$
$$O_3 = \left(\frac{1}{2},-1,1\right)\,,\quad O_4 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{4}\right)\,.$$

Then A_1, O_1, A_2, and O_2 are not coplanar, since

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & \frac{1}{4} & 1 \\ -1 & \frac{1}{2} & 1 & 1 \end{vmatrix} = \frac{9}{8} \neq 0\,.$$

Therefore, the lines A_1O_1 and A_2O_2 are not concurrent.

No other solutions were submitted. Is there some other way to generalize to 3 dimensions?

2734. [2002 : 179] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Prove that

$$(bc)^{2n+3} + (ca)^{2n+3} + (ab)^{2n+3} \geq (abc)^{n+2}\left(a^n + b^n + c^n\right)\,,$$

where a, b, c, are non-negative reals, and n is a non-negative integer.

I. *Solution by Michel Bataille, Rouen, France.*

The inequality obviously holds if a, b, or c is 0. Thus, we may assume that a, b, $c > 0$. Dividing by $(abc)^{2n+3}$ and setting $x=\frac{1}{a}$, $y=\frac{1}{b}$, $z=\frac{1}{c}$, the given inequality becomes

$$x^{2n+3} + y^{2n+3} + z^{2n+3} \geq xy^{n+1}z^{n+1} + yz^{n+1}x^{n+1} + zx^{n+1}y^{n+1}\,, \quad (1)$$

for x, y, $z > 0$. Without loss of generality, we suppose that $x \geq y \geq z$. Now,

$$x^{2n+3} + y^{2n+3} + z^{2n+3} \;\geq\; x^{n+1}y^{n+2} + y^{n+1}z^{n+2} + z^{n+1}x^{n+2}\,,$$

since $(x^{n+2} - y^{n+2})(x^{n+1} - z^{n+1}) + (y^{n+2} - z^{n+2})(y^{n+1} - z^{n+1}) \geq 0$, and

$$\begin{aligned} & x^{n+1}y^{n+2} + y^{n+1}z^{n+2} + z^{n+1}x^{n+2} \\ & \qquad \geq \; xy^{n+1}z^{n+1} + yz^{n+1}x^{n+1} + zx^{n+1}y^{n+1}\,, \end{aligned}$$

since $z^{n+1}(x - y)(x^{n+1} - y^{n+1}) + y^{n+1}(y - z)(x^{n+1} - z^{n+1}) \geq 0$. Then (1) readily follows.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

The inequality is clearly true if $abc = 0$. Therefore, we may suppose $0 < a \leq b \leq c$. Then $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$ and $\frac{bc}{a} \geq \frac{ca}{b} \geq \frac{ab}{c}$. Hence, by the Rearrangement Inequality [*Ed.* See, for example, Dragos Hrimiuc, *π in the Sky*, Pacific Institute for the Mathematical Sciences, Dec 2000, pp. 21–23], we have

$$\begin{aligned} & \frac{1}{a}\left(\frac{bc}{a}\right)^{n+1} + \frac{1}{b}\left(\frac{ca}{b}\right)^{n+1} + \frac{1}{c}\left(\frac{ab}{c}\right)^{n+1} \\ & \quad \geq \; \frac{1}{2}\left[\frac{1}{c}\left(\frac{bc}{a}\right)^{n+1} + \frac{1}{a}\left(\frac{ca}{b}\right)^{n+1} + \frac{1}{b}\left(\frac{ab}{c}\right)^{n+1}\right. \\ & \qquad\qquad \left. + \frac{1}{b}\left(\frac{bc}{a}\right)^{n+1} + \frac{1}{c}\left(\frac{ca}{b}\right)^{n+1} + \frac{1}{a}\left(\frac{ab}{c}\right)^{n+1}\right] \\ & \quad = \; \frac{1}{2}\left[\frac{1}{c}\left(\left(\frac{bc}{a}\right)^{n+1} + \left(\frac{ca}{b}\right)^{n+1}\right) + \frac{1}{a}\left(\left(\frac{ca}{b}\right)^{n+1} + \left(\frac{ab}{c}\right)^{n+1}\right)\right. \\ & \qquad\qquad \left. + \frac{1}{b}\left(\left(\frac{ab}{c}\right)^{n+1} + \left(\frac{bc}{a}\right)^{n+1}\right)\right] \\ & \quad = \; \frac{1}{2}\left[c^n\left(\left(\frac{b}{a}\right)^{n+1} + \left(\frac{a}{b}\right)^{n+1}\right) + a^n\left(\left(\frac{c}{b}\right)^{n+1} + \left(\frac{b}{c}\right)^{n+1}\right)\right. \\ & \qquad\qquad \left. + b^n\left(\left(\frac{a}{c}\right)^{n+1} + \left(\frac{c}{a}\right)^{n+1}\right)\right] \\ & \quad \geq \; c^n + a^n + b^n\,, \end{aligned}$$

since $x + \frac{1}{x} \geq 2$ for all $x > 0$. Multiplying by $(abc)^{n+2}$ yields the desired inequality.

III. *Essentially the same solution by Pierre Bornsztein, Pontoise, France; Zeljko Hanjš, University of Zagreb, Zagreb, Croatia; Joe Howard, Portales, NM, USA; and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

The desired inequality may be rewritten as:

$$\sum_{\text{cyclic}} a^{2n+3}b^{2n+3} \;\geq\; \sum_{\text{cyclic}} a^{2n+2}b^{2n+2}c^{n+2} \,.$$

Since the vector $(2n+3, 2n+3, 0)$ majorizes the vector $(2n+2, n+2, n+2)$, this inequality is a direct consequence of the Majorization Inequality, also known as Muirhead's Theorem. (See, for example, G. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. Cambridge University Press.) Note that equality occurs if and only if $a = b = c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; THE AUSTRIAN IMO TEAM 2002; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; ZENOFON PAPANICOLAOU, Athens, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Using the Rearrangement Inequality, Janous obtained the more general result that if real numbers α and β satisfy $\alpha \geq \beta > 0$ and $\alpha \leq 2\beta$, then

$$(bc)^{\alpha} + (ca)^{\alpha} + (ab)^{\alpha} \geq (abc)^{\beta}(a^{2\alpha-3\beta} + b^{2\alpha-3\beta} + c^{2\alpha-3\beta})$$

for all positive reals a, b, and c. Of course, this also follows immediately from the Majorization Inequality, since clearly the vector $(2\alpha - 2\beta, \beta, \beta)$ is majorized by the vector $(\alpha, \alpha, 0)$.

2735★. [2002 : 179] *Proposed by Richard I. Hess, Rancho Palos Verdes, CA, USA.*

Given three Pythagorean triangles with the same hypotenuse, is it possible that the area of one triangle is equal to the sum of the areas of the other two triangles?

Editor's Note: No solutions have been submitted for this problem.

2736. [2002 : 180] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $ABCD$ be a convex quadrilateral. From points A and B, draw lines parallel to sides BC and AD, respectively, giving points G and F on CD, respectively.

Let P and Q be the points of the intersection of the diagonals of the trapezoids $ABFD$ and $ABCG$, respectively.

Prove that $PQ \parallel CD$.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Consider the following locus problem: points R and S are fixed, and T varies on a fixed line l, where R and S are on the same side of l. Point U lies on l such that $RSTU$ is a convex trapezoid. Point X is the intersection of the diagonals of this trapezoid. Show that the locus of X is a line parallel to l.

This is not difficult.Let R', S', and X' be the feet of the perpendiculars from R, S, and X, respectively, to l. Then

$$XX' = \frac{UX}{US} \cdot SS' = \frac{XT}{RT} \cdot RR' .$$

However, $\triangle RUX$ is similar to $\triangle TSX$, so that

$$\frac{UX}{XS} = \frac{RX}{XT} ,$$

and therefore,

$$\frac{UX}{UX + XS} = \frac{RX}{RX + XT} \qquad \text{or} \qquad \frac{UX}{US} = \frac{RX}{RT} .$$

Hence,

$$\frac{XX'}{RR'} + \frac{XX'}{SS'} = \frac{UX}{US} + \frac{XT}{RT} = \frac{RX}{RT} + \frac{XT}{RT} = 1 ,$$

implying that XX' is constant as T varies (since RR' and SS' are clearly constant). Hence, the locus of X is a line parallel to l, as claimed.

This immediately implies the result in the problem. Let $R = A$, $S = B$, and let l be the line CD. Let C and F be two possible positions of the variable point T. Then U is successively G and D, and X is first Q, then P. Hence, both P and Q lie on the locus defined above, which we showed to be a line parallel to l. Thus, PQ is parallel to CD, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. Zhou, Zvonaru and Ioniţă have noted that, if AD is parallel to BC, then $P = Q$ and PQ is not well-defined.

2737. [2002 : 180] *Proposed by Lyubomir Lyubenov, teacher, and Ivan Slavov, student, Foreign Language High School "Romain Rolland", Stara Zagora, Bulgaria.*

Find all solutions of the equation

$$x^n - 2nx^{n-1} + 2n(n-1)x^{n-2} + ax^{n-3} + bx^{n-4} + \cdots + c = 0 ,$$

given that there are n real roots.

Solution by Mihály Bencze, Brasov, Romania and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $x_1, x_2, \ldots, x_n$ be the real roots of the equation. Then

$$\sum_{i=1}^{n} x_i = 2n \quad \text{and} \quad \sum_{1\le i<j\le n} x_i x_j = 2n(n-1),$$

so that

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2\sum_{1\le i<j\le n} x_i x_j = 4n^2 - 4n(n-1) = 4n.$$

Therefore,

$$\sum_{i=1}^{n}(x_i-2)^2 = \sum_{i=1}^{n} x_i^2 - 4\sum_{i=1}^{n} x_i + 4n = 4n - 8n + 4n = 0,$$

whence $x_1 = x_2 = \cdots = x_n = 2$.

Also solved by TIM AUSTIN, student, Colchester Royal Grammar School, Colchester, UK; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposers. There was one incomplete solution submitted.

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