PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 4Z9. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 November 2003**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX.) *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

2826. *Proposed by Bernardo Recamán Santos, Bogota, Colombia.*

Show that, for every sufficiently large integer n, it is possible to split the integers 1, 2, $\dots$, n into two disjoint subsets such that the sum of the elements in one set equals the product of the elements in the other.

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Montrer que pour tout entier n suffisamment grand, il est possible de séparer les entiers $1, 2, \dots, n$ en deux sous-ensembles disjoints de telle sorte que la somme des éléments du premier soit égale au produit des éléments du second.

2827. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a non-negative integer. Determine

$$\sum_{k=0}^{\infty} \frac{\tanh\left(2^k\right)}{2 + 2\sinh^2\left(2^k\right)}\,.$$

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Soit n un entier non négatif. Calculer

$$\sum_{k=0}^{\infty} \frac{\tanh\left(2^k\right)}{2 + 2\sinh^2\left(2^k\right)}\,.$$

2828. *Proposed by Achilleas Pavlos Porfyriadis, Student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece (adapted by the Editors).*

Suppose that f satisfies the functional equation

$$f(x) + 2f\left(\frac{x+2000}{x-1}\right) \;=\; 4011 - x\,.$$

Find the value of $f(2002)$.

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On suppose que f satisfait l'équation fonctionnelle

$$f(x) + 2f\left(\frac{x+2000}{x-1}\right) \;=\; 4011 - x\,.$$

Trouver la valeur de $f(2002)$.

2829. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Given $\triangle ABC$ with sides a, b, c, prove that

$$\frac{2\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

. . . .

Montrer que, dans un triangle ABC de côtés a, b, c,

$$\frac{2\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

2830. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that $\Gamma(O, R)$ is the circumcircle of $\triangle ABC$. Suppose that side AB is fixed and that C varies on Γ (always on the same side of AB).

Suppose that I_a, I_b, I_c, are the centres of the excircles of $\triangle ABC$ opposite A, B, C, respectively. If Ω is the centre of the circumcircle of $\triangle I_a I_b I_c$, determine the locus of Ω as C varies.

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Soit $\Gamma(O, R)$ le cercle circonscrit du triangle ABC. Le côté AB étant fixé, on fait varier C sur Γ (mais sans le faire passer de l'autre côté de AB).

Soit I_a, I_b, et I_c les centres des cercles exinscrits du triangle ABC opposés aux sommets A, B, et C, respectivement. Ω désignant le centre du cercle circonscrit au triangle $I_a I_b I_c$, déterminer le lieu de Ω lorsque C varie.

2831. *Proposed by Achilleas Pavlos Porfyriadis, Student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece.*

For a convex polygon, prove that it is impossible for two sides without a common vertex to be longer than the longest diagonal.

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Montrer que, dans un polygone convexe, il est impossible que deux côtés non adjacents soient plus longs que la plus longue diagonale.

2832★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n be a positive integer, and let

$$a(n) = \left| \sum_{j=0}^{3n} (-2)^j \left(\binom{6n+2-j}{j+1} + \binom{6n+1-j}{j} \right) \right| .$$

Prove that

(a) $a(n) = 3$ if and only if $n = 1$, and

(b) the sequence $\{a(n)\}_{n=1}^{\infty}$ is strictly increasing.

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Soit n un entier positif, et soit

$$a(n) = \left| \sum_{j=0}^{3n} (-2)^j \left(\binom{6n+2-j}{j+1} + \binom{6n+1-j}{j} \right) \right| .$$

Montrer que

(a) $a(n) = 3$ si et seulement si $n = 1$, et

(b) la suite $\{a(n)\}_{n=1}^{\infty}$ est strictement croissante.

2833★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a be a positive real number, and let $n \geq 2$ be an integer. For each $k = 1, 2, \ldots, n$, let x_k be a non-negative real number, λ_k be a positive real number, and let $y_k = \lambda_k x_k + \frac{x_{k+1}}{\lambda_{k+1}}$. Here and elsewhere, indices greater than n are to be reduced modulo n.

(a) If $a > 1$, prove that

$$n+\sum_{k=1}^{n} a^{y_k} \geq 2\sum_{k=1}^{n} a^{x_k} \quad \text{and} \quad 3n+\sum_{k=1}^{n} a^{y_k+y_{k+1}} \geq \sum_{k=1}^{n} (1+a^{x_k})^2 .$$

(b) If $0 < a < 1$, prove that the opposite inequalities hold.

[The proposer has proofs for the cases $n = 3$ and $n = 4$.]

. . . .

Soit a un nombre réel positif, et soit $n \geq 2$ un entier. Pour chaque $k = 1, 2, \ldots, n$, soit x_k un nombre réel non négatif et λ_k un nombre réel positif, et soit $y_k = \lambda_k x_k + \frac{x_{k+1}}{\lambda_{k+1}}$.Ici et dans ce qui suit, on convient que les indices plus grands que n sont réduits modulo n.

(a) Si $a > 1$, montrer que

$$n+\sum_{k=1}^{n} a^{y_k} \geq 2\sum_{k=1}^{n} a^{x_k} \quad \text{et} \quad 3n+\sum_{k=1}^{n} a^{y_k+y_{k+1}} \geq \sum_{k=1}^{n} (1+a^{x_k})^2 .$$

(b) Si $0 < a < 1$, montrer que les inégalités sont inversées.

[Le proposeur a une preuve pour les cas $n = 3$ et $n = 4$.]

2834. *Proposed by Michel Bataille, Rouen, France.*

Let $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for integers $n > 2$. Then define

$$g_n = f_{n+6} + 3f_{n+2} + 3f_{n-2} + f_{n-6}$$

for integers $n > 6$. Find $\gcd\{g_{f_{6666}}, g_{f_{666}}\}$.

. . . .

Soit $f_1 = f_2 = 1$ et $f_n = f_{n-1} + f_{n-2}$ où n est un entier, $n > 2$. On définit

$$g_n = f_{n+6} + 3f_{n+2} + 3f_{n-2} + f_{n-6}$$

avec n entier et $n > 6$. Trouver le plus grand commun diviseur de $g_{f_{6666}}$ et $g_{f_{666}}$.

2835. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
For non-negative real numbers x and y, not both equal to 0, prove that

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{5}{8}.$$

....

Si x et y sont deux nombres réels non négatifs non tous nuls, montrer que

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{5}{8}.$$

2836. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
Suppose that $\triangle ABC$ is equilateral and that P is an interior point. The lines AP, BP, CP intersect the opposite sides at D, E, F, respectively. Suppose that $PD=PE=PF$. Determine the locus of P.

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Soit ABC un triangle équilatéral et P un point intérieur. Les droites AP, BP, CP coupent respectivement les côtés opposés en D, E, F. Si l'on suppose que $PD=PE=PF$, déterminer le lieu de P.

2837. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that Γ is a circle and that I, J, and K are three distinct points in the plane of Γ, but not on Γ. Let A be any point on Γ. Points B, C, D, E, F, and G on Γ are defined by the conditions that chords AB and DE intersect at I, chords BC and EF intersect at J, and chords CD and FG intersect at K. (A tangent is to be regarded as a chord with its point of contact defined to be a pair of coincident points.)

Is it possible to select the positions of I, J, and K so that G coincides with A for all points A lying on Γ? (Justification required!)

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Soit I, J, et K trois points dans le plan d'un cercle Γ et A un point quelconque sur Γ. On désigne par B, C, D, E, F, et G les points sur Γ tels que les cordes AB et DE, BC et EF, CD et FG se coupent respectivment en I, J, et K. (Une tangente est considérée comme une corde engendrée par deux points qui coïncident.)

Est-il possible de choisir les points I, J, et K de telle sorte que G coïncide avec A pour tous les points A situés sur Γ ? (On demande une justification !)