THE OLYMPIAD CORNER

No. 229

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We begin this issue with the problems of the 2000 Hungarian Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to IMO 2000 in Korea, for collecting them.

2000 HUNGARIAN MATHEMATICAL OLYMPIAD

1. Consider the number of positive even divisors for each of the first n positive integers, and form the sum of these numbers. Form a similar sum of the numbers of positive odd divisors of the first n positive integers. Prove that the two sums differ by at most n.

2. Construct the point P inside a given triangle such that the feet of the perpendiculars from P to the sides of the triangle determine a triangle whose centroid is P.

3. Let k be a positive integer, and suppose that more than 2^k distinct integers are given. Prove that $k+2$ of these numbers can be chosen so that, for some positive integer m, the sums of the chosen numbers taken m at a time are all distinct.

Next we give the problems of the 2000 Iranian Mathematical Olympiad. Yet again we thank Andy Liu, Canadian Team Leader to IMO 2000 in Korea, for obtaining them for our use.

2000 IRANIAN MATHEMATICAL OLYMPIAD

1. In a tennis tournament, there are n participants A_1, A_2, $\ldots$, A_n. Any two of them play at most once against each other, and the winner of each match receives 1 point. The number of matches that have been played is $k \leq n(n-1)/2$. Prove that the non-negative integers d_1, d_2, $\ldots$, d_n are the scores obtained by A_1, A_2, $\ldots$, A_n, respectively, if and only if $\sum_{i=1}^{n} d_i = k$ and, for every subset $X \subseteq \{1, 2, \ldots, n\}$, the number of matches played among A_j with $j \in X$ is not greater than $\sum_{j\in X} d_j$.

2. Triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed outside triangle $A_1A_2A_3$, with $O_2A_3 = O_2A_1$ and $O_3A_1 = O_3A_2$. A point O_1 is outside $A_1A_2A_3$ such that $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$ and $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$, and T is the foot of the perpendicular from O_1 to A_2A_3. Prove that

(a) A_1O_1 is perpendicular to O_2O_3, and

(b) $\dfrac{A_1O_1}{O_2O_3} = 2\dfrac{O_1T}{A_2A_3}$.

3. A circle Γ with radius R and centre W, and a line d are drawn in a plane, such that the distance of W from d is greater than R. Let M and N be two variable points on the line d such that the circle with diameter MN is tangent to the circle Γ. Prove that there exists a point P in the plane such that $\angle MPN$ is constant.

4. Let n be a positive integer, and let S be a set containing ordered n-tuples of non-negative integers such that if $(a_1, a_2, \ldots, a_n) \in S$, then every $(b_1, b_2, \ldots, b_n)$ for which $b_i \leq a_i$, $1 \leq i \leq n$, is also in S. If $h_m(S)$ is the number of elements of S the sum of whose components is equal to m, prove that h_m is a polynomial in m for all sufficiently large m.

5. Suppose a, b, and c are real numbers such that for any positive real numbers $x_1, x_2, \ldots, x_n$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^a \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^b \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)^c \geq 1 .$$

Prove that the vector (a, b, c) can be represented as a non-negative linear combination of the vectors $(-2, 1, 0)$ and $(1, -2, 1)$.

6. Prove that for every positive integer n, there exists a polynomial $p(x)$ with integer coefficients such that $p(1)$, $p(2)$, $\ldots$, $p(n)$ are distinct powers of 2.

Before turning to solutions from our readers to problems from the 2001 numbers of the *Corner*, we revisit problem 1 of the XXXIII Spanish Mathematical Olympiad 1996–97, First Round [2000 : 196–197]. Jean-Claude Andrieux offers the following geometric companion to the algebraic solution in [2002 : 294–295].

1. Show that any complex number $z \neq 0$ can be expressed as a sum of two complex numbers such that their difference and their quotient are purely imaginary (that is, with real part zero).

Additional observations by Jean-Claude Andrieux, Beaune, France.

Il me semble qu'une solution totalement géométrique pourrait éclairer la résolution algébrique.

On cherche deux complexes α et β tels que:

$$\begin{cases} \text{(i)} & z = \alpha + \beta\,, \\ \text{(ii)} & \beta - \alpha \in i\mathbb{R}\,, \\ \text{(iii)} & \dfrac{\beta}{\alpha} \in i\mathbb{R}\,. \end{cases}$$

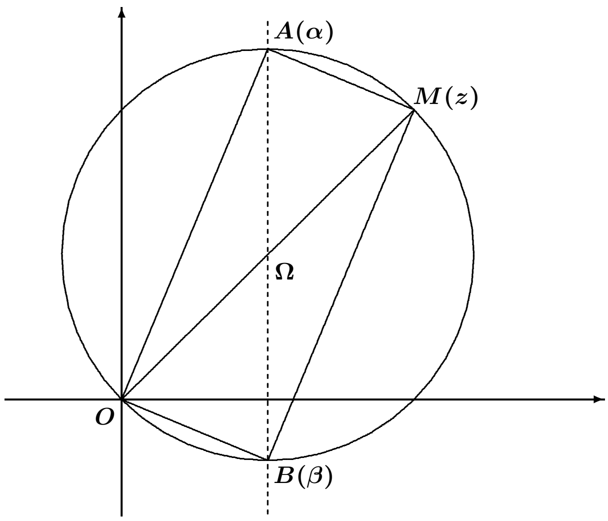
Le plan complexe étant rapporté à un repère orthonormal $(0, \vec{u}, \vec{v})$, notons M, A et B les points d'affixes respectives z, α et β.

(i) Traduit le fait que $OAMB$ est un parallélogramme.

(ii) Traduit le fait que $(\overrightarrow{OA}, \overrightarrow{OB}) = \frac{\pi}{2} \bmod \pi$. Donc $OAMB$ est un rectangle.

(iii) Traduit le fait que $\overrightarrow{AB}$ est colinéaire à $\vec{v}$. Donc $[AB]$ est le diamètre de direction $(0, \vec{v})$ du cercle circonscrit à $OAMB$.

On en déduit alors la construction des points A et B, uniques à l'ordre près.



Now we turn to solutions to the problems of the Hungary-Israel Bi-National Mathematical Competition 1997 [2001 : 8–9].

1. Is there an integer N such that

$$(\sqrt{1997} - \sqrt{1996})^{1998} = \sqrt{N} - \sqrt{N-1}\,?$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, *Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. Here is Wang's solution, with historical comments.*

One such integer is

$$N = \left(\frac{(\sqrt{1997}+\sqrt{1996})^{1998}+(\sqrt{1997}-\sqrt{1996})^{1998}}{2}\right)^2 .$$

In the following, we shall establish a more general result.

Theorem. Let a, $b \in \mathbb{R}$, with $0 \le b \le a$. Then for all $n \in \mathbb{N}$, we have $(a-b)^n = \sqrt{k^2} - \sqrt{k^2-(a^2-b^2)^n}$, where $k = \frac{1}{2}\big((a+b)^n + (a-b)^n\big)$.

Proof. Since

$$k^2 - \left(\frac{(a+b)^n-(a-b)^n}{2}\right)^2 = (a^2-b^2)^n ,$$

we get

$$\sqrt{k^2} - \sqrt{k^2-(a^2-b^2)^n} = k - \frac{(a+b)^n-(a-b)^n}{2} = (a-b)^n .$$

Corollary 1. Let d, m, $n \in \mathbb{N}$, with $d \le m$. Then

$$(\sqrt{m}-\sqrt{m-d})^n = \sqrt{k^2} - \sqrt{k^2-d^n} ,$$

where $k = \frac{1}{2}\big((\sqrt{m}+\sqrt{m-d})^n + (\sqrt{m}-\sqrt{m-d})^n\big)$.

Proof. In the theorem, let $a = \sqrt{m}$ and $b = \sqrt{m-d}$.

Corollary 2. Let m, $n \in \mathbb{N}$. Then

$$(\sqrt{m}-\sqrt{m-1})^n = \sqrt{k^2} - \sqrt{k^2-1} ,$$

where $k = \frac{1}{2}\big((\sqrt{m}+\sqrt{m-1})^n + (\sqrt{m}-\sqrt{m-1})^n\big)$.

The given problem is the special case of Corollary 2 when $m = 1997$, $n = 1998$, and $N = k^2$. Note that $k \in \mathbb{N}$, since n is even and

$$k = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \left(\sqrt{m}\right)^{n-2i} \left(\sqrt{m-1}\right)^{2i} .$$

Comments. Here is a brief history of this problem, as I know it. Over fifty years ago, the following problem appeared in the *American Mathematical Monthly* E950 [1951, 566]: Show that every positive integral power of $\sqrt{2}$ is of the form $\sqrt{m}-\sqrt{m-1}$. According to C.W. Trigg in the article "The Monthly Problems Departments, 1894–1954" (*Monthly*, Vol 64, No 7, 1957, Part II, *The Otto Dunkel Memorial Problem Book*, pp. 3–8), this was the second most popular problem proposed during that 50-year span, in terms of the number of people who submitted solutions. Browsing through the list of 65 solvers' names, I recognize such well-known mathematicians as Frank Harary, Leo Moser, and Albert Wilansky, to name just a few.

The published solution by S.T. Thompson dealt with the more general situation discussed in the solution above. The solution given here is basically the same as his, with minor changes. The original *Monthly* problem was used as the second question on the 1994 Canadian Mathematical Olympiad.

2. Find all real numbers α with the following property: for any positive integer n there exists an integer m such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{3n}.$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Pontoise, France. We present the solution by Bornsztein.

We will prove that the real numbers α with the specified property are the integers.

Note that if m, n are integers with $n > 0$, the condition $\left|\alpha - \frac{m}{n}\right| < \frac{1}{3n}$ may be rewritten as $|n\alpha - m| < \frac{1}{3}$, which is the same as $d(n\alpha, m) < \frac{1}{3}$, where d denotes the usual distance. Thus, the property described in the problem is equivalent to the following property P:

$$d(n\alpha, \mathbb{Z}) < \frac{1}{3}, \qquad \text{for any positive integer } n.$$

Case 1. Suppose α is irrational. Then, from a well-known theorem of Kronecker (see [1]), the set $\{n\alpha - \lfloor n\alpha \rfloor \mid n \in \mathbb{N}^*\}$ is dense in $[0,1]$. Hence, there exists $n \in \mathbb{N}^*$ such that $d\left(n\alpha - \lfloor n\alpha \rfloor, \frac{1}{2}\right) < 0.1$. Let $a = \lfloor n\alpha \rfloor$. Then $a + 0.4 < n\alpha < a + 0.6$ which leads to

$$d(n\alpha, \mathbb{Z}) = \min\{d(n\alpha, a), d(n\alpha, a+1)) > 0.4 > \frac{1}{3}.$$

Thus, α does not have the property P.

Case 2. Suppose α is rational. Let $\alpha = \frac{a}{b}$ where a, b are relatively prime integers and $b > 1$.

- If $b = 2k$ is even, then a is odd, and we have $k\alpha = a/2$. It follows that $d(k\alpha, \mathbb{Z}) = \frac{1}{2} > \frac{1}{3}$. Thus, α does not have the property P.
- If $b = 2k+1$ is odd and $k > 1$, we note that since a, b are relatively prime, there exist integers u, v with $u > 0$ such that $au + bv = 1$. Then

$$u\alpha = \frac{ua}{b} = \frac{1-bv}{b} = -v + \frac{1}{b} = -v + \frac{1}{2k+1},$$

and hence,

$$ku\alpha \;=\; -kv + \frac{k}{2k+1}\,.$$

Then $d(ku\alpha,\mathbb{Z}) = \frac{k}{2k+1}$ or $d(ku\alpha,\mathbb{Z}) = 1 - \frac{k}{2k+1} = \frac{k+1}{2k+1}$. In either case $d(ku\alpha,\mathbb{Z}) > \frac{1}{3}$ (noting that $k > 1$). Thus, α does not have the property P.

- If $b = 3$, then $\alpha = p + \frac{c}{3}$ where p is an integer and $c \in \{1, 2\}$. Then $d(\alpha,\mathbb{Z}) = \frac{1}{3}$, and α does not have the property P.

We have shown that if $\alpha \notin \mathbb{Z}$, then α does not have the property P. Moreover, if $\alpha \in \mathbb{Z}$, then obviously α has the property P (choose $m = n\alpha$). Thus, α has the property P if and only if α is an integer, as claimed.

Reference:

[1] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers*, Oxford.

3. ABC is an acute-angled triangle whose circumcentre is O. The intersection points of the diameters of the circumcircle, passing through A, B, C, with the opposite sides are A_1, B_1, C_1, respectively. The circumradius of the triangle ABC is of length $2p$, where p is a prime. The lengths OA_1, OB_1, OC_1 are integers. What are the lengths of the sides of the triangle?

Solved by Mohammed Aassila, Strasbourg, France; Geoffrey A. Kandall, Hamden, CT, USA; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Kandall's solution.

The circumcentre O lies in the interior of ABC, and $OA = OB = OC = 2p$. Let $OA_1 = r$, $OB_1 = s$, $OC_1 = t$. Thus, r, s, t are positive integers.

Assertion I. $r = s = t = p$.
Proof. We have

$$\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1} \;=\; \frac{[OBC]}{[ABC]} + \frac{[OAC]}{[ABC]} + \frac{[OAB]}{[ABC]} \;=\; \frac{[ABC]}{[ABC]}\,;$$

that is,

$$\frac{r}{r+2p} + \frac{s}{s+2p} + \frac{t}{t+2p} \;=\; 1\,.$$

This is equivalent to

$$r(s+2p)(t+2p) + s(r+2p)(t+2p) + t(r+2p)(s+2p) \\ \;=\; (r+2p)(s+2p)(t+2p)\,. \qquad (1)$$

It follows that $2rst \equiv 0 \pmod{2p}$; hence, $p \mid rst$. Witout loss of generality, suppose $p \mid r$. Since $0 < r < 2p$, we have $r = p$. Replacing r by p in (1), we obtain

$$3s(t+2p) + 3t(s+2p) \;=\; 2(s+2p)(t+2p)\,, \qquad (2)$$

which implies $4st \equiv 0 \pmod{2p}$; that is, $p \mid 2st$. Thus, either $p = 2$ or $p \mid st$.

(i) Suppose $p = 2$. Equation (2) reduces to $st + s + t = 8$; that is, $(s+1)(t+1) = 9$. Therefore, $s = t = 2$. Also $r = p = 2$.

(ii) Suppose $p \mid st$. Without loss of generality, $p \mid s$. Since $0 < s < 2p$, we have $s = p$. It then follows easily from equation (2) that $t = p$.

Assertion I has now been proved.

Assertion II. $AB = AC = BC = 2p\sqrt{3}$.

Proof. We have three pairs of congruent triangles: $\triangle OAB_1 \cong \triangle OBA_1$, $\triangle OBC_1 \cong \triangle OCB_1$, $\triangle OCA_1 \cong \triangle OAC_1$. Let

$$\begin{aligned} X &= [OAB_1] = [OBA_1]\,, \\ Y &= [OBC_1] = [OCB_1]\,, \\ Z &= [OAC_1] = [OCA_1]\,. \end{aligned}$$

Since $AO : OA_1 = 2 : 1$, we have $Y + Z = 2X$ and $X + Y = 2Z$, from which it follows easily that $X = Z$. Consequently, $BA_1 = CA_1$; hence, $OA_1 \perp BC$ and $BA_1 = CA_1 = p\sqrt{3}$. Therefore, $BC = 2p\sqrt{3}$. Similarly, $AB = AC = 2p\sqrt{3}$.

4. What is the number of distinct sequences of length 1997 that can be formed by using the letters A, B, C, where each letter appears an odd number of times?

Solved by Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We use Klamkin's write-up.

We will find the number of distinct sequences of length $2n + 1$. We need only set $n = 998$ to solve the given problem.

The number we seek is

$$S = \sum \frac{(2n+1)!}{r!s!t!}\,,$$

where the summation is over all odd non-negative integers r, s, t such that $r+s+t = 2n+1$. To find this sum explicitly, we start with the multinomial expansion

$$(x+y+z)^{2n+1} = \sum \frac{x^r y^s z^t (2n+1)!}{r!s!t!}\,,$$

where the summation is over *all* non-negative integers r, s, t such that $r + s + t = 2n + 1$. It then follows that the latter sum where r, s, t are all odd is given by

$$\begin{aligned} &\tfrac{1}{8}\big[(x+y+z)^{2n+1} \\ &\quad -(-x+y+z)^{2n+1} - (x-y+z)^{2n+1} - (x+y-z)^{2n+1} \\ &\quad +(-x-y+z)^{2n+1} + (-x+y-z)^{2n+1} + (x-y-z)^{2n+1} \\ &\quad -(-x-y-z)^{2n+1}\big]\,. \end{aligned}$$

Now, setting $x = y = z = 1$, it follows that

$$S = \frac{2(3)^{2n+1} - 6}{8} = \frac{3(3^{2n} - 1)}{4}.$$

In a similar fashion, by *adding* all the latter eight trinomials but this time each to the power $2n$, we find that the number of distinct sequences of length $2n$ that can be formed by using the letters A, B, C, where each letter appears an even number of times, is $[2(3)^{2n} + 6]/8$.

5. The three squares ACC_1A'', ABB_1A', $BCDE$ are constructed on the sides of a given triangle ABC, outwards. The center of the square $BCDE$ is P. Prove that the three lines $A'C$, $A''B$ and PA pass through one point.

Solved by Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We use Kandall's solution.

We may assume $\angle A (= \angle BAC) < 90^\circ$. Say $A'C$ meets AB at R, AP meets BC at S, and $A''B$ meets AC at T. Draw $A'B$, $A''C$, BP, and CP.

$$\begin{aligned}
\frac{AR}{RB} &= \frac{[A'AC]}{[A'BC]} = \frac{A'A \cdot AC \cdot \sin(A + 90^\circ)}{A'B \cdot BC \cdot \sin(B + 45^\circ)} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{AC}{BC} \cdot \frac{\sin(A + 90^\circ)}{\sin(B + 45^\circ)}, \\
\frac{BS}{SC} &= \frac{[ABP]}{[ACP]} = \frac{AB \cdot BP \cdot \sin(B + 45^\circ)}{AC \cdot CP \cdot \sin(C + 45^\circ)} = \frac{AB}{AC} \cdot \frac{\sin(B + 45^\circ)}{\sin(C + 45^\circ)}, \\
\frac{CT}{TA} &= \frac{[A''CB]}{[A''AB]} = \frac{A''C \cdot BC \cdot \sin(C + 45^\circ)}{A''A \cdot AB \cdot \sin(A + 90^\circ)} \\
&= \frac{\sqrt{2}}{1} \cdot \frac{BC}{AB} \cdot \frac{\sin(C + 45^\circ)}{\sin(A + 90^\circ)}.
\end{aligned}$$

It follows easily that $\frac{AR}{RB} \cdot \frac{BS}{SC} \cdot \frac{CT}{TA} = 1$. Hence, $A'C$, $A''B$, and PA are concurrent.

6. Can a closed disk be decomposed into a union of two congruent parts having no common points?

Solved by Pierre Bornsztein, Pontoise, France; and Robert Bilinski, Outremont, QC. We give the comment by Murray S. Klamkin, University of Alberta, Edmonton, AB.

This problem was given as B-6 in the 25^{th} William Lowell Putnam Mathematical Competition, December 1964, and is given with solution in [1].

Reference:

[1] A.M. Gleason, R.E. Greenwood, L.M. Kelly, *The William Lowell Putnam Mathematical Competition, Problems and Solutions:* 1938–1964, Math. Assoc. of America, 1980, p. 599.

We turn next to solutions to some of the problems of the 36^{th} Armenian National Olympiad in Mathematics [2001 : 9–10].

1. Let

$$p(x) \;=\; (x-a_1)^{n_1}(x-a_2)^{n_2}(x-a_3)^{n_3}$$

be a polynomial, such that

$$p(x)-1 \;=\; (x-b_1)^{k_1}(x-b_2)^{k_2}(x-b_3)^{k_3}\,,$$

where the numbers a_1, a_2, a_3, as well as b_1, b_2, b_3, are distinct, and n_1, n_2, n_3, k_1, k_2, k_3 are natural numbers. Prove that the degree of the polynomial $p(x)$ does not exceed 5.

Solution by Michel Bataille, Rouen, France.

First, note that since $p(a_i)=0$ and $p(b_j)=1$, we cannot have $a_i=b_j$ $(i,\,j=1,\,2,\,3)$. Thus, a_1, a_2, a_3, b_1, b_2, b_3 are six distinct numbers.

Now, if $(x-c)^m$ divides the polynomial $q(x)$, then $(x-c)^{m-1}$ divides its derivative $q'(x)$. Noticing that $\big(p(x)-1\big)'=p'(x)$, we see that the polynomials $(x-a_i)^{n_i-1}$, $(x-b_j)^{k_j-1}$ divide $p'(x)$ (for $i,\,j=1,\,2,\,3$). Since a_1, a_2, a_3, b_1, b_2, b_3 are six distinct numbers, the product

$$(x-a_1)^{n_1-1}(x-a_2)^{n_2-1}(x-a_3)^{n_3-1}(x-b_1)^{k_1-1}(x-b_2)^{k_2-1}(x-b_3)^{k_3-1}$$

divides $p'(x)$ as well. We deduce that

$$\begin{aligned}\deg p'(x) &\geq (n_1-1)+(n_2-1)+(n_3-1)\\ &\qquad\qquad +(k_1-1)+(k_2-1)+(k_3-1)\\ &= 2\deg p(x)-6\,,\end{aligned}$$

since $n_1+n_2+n_3=k_1+k_2+k_3=\deg p(x)$. Also, $\deg p'(x)=\deg p(x)-1$. The desired result $\deg p(x)\leq 5$ follows at once.

2. Suppose a and b are natural numbers, such that $(a+b)$ is an odd number. Prove that for any division of the set of natural numbers into two groups, there will be two numbers from the same group, the difference of which is either a or b.

Solved by Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bornsztein's solution.

Since $a+b$ is odd, we deduce that a and b have opposite parity. With no loss of generality, we may suppose that a is even and b is odd. For a contradiction, suppose that $\mathbb{N}=A\cup B$ with $A\cap B=\emptyset$, and for any x, $y\in A$ (respectively, B), $x-y\neq a$ and $x-y\neq b$.

With no loss of generality, we may suppose that $1\in A$. Then, $1+a$ and $1+b$ are in B. Thus, $1+2a$ and $1+2b$ are in A. By induction, we easily prove that, for each non-negative integer n, both $1+2na$ and $1+2nb$ are in A, while $1+(2n+1)a$ and $1+(2n+1)b$ are in B. In particular,

$1 + ab \in A$, since a is even, and $1 + ab \in B$, since b is odd. This contradicts the hypothesis that $A \cap B = \emptyset$.

The conclusion follows.

3. Prove that, for any points A, B, C, D, E, F, the following inequality holds:

$$AD^2 + BE^2 + CF^2 \;\leq\; 2(AB^2 + BC^2 + CD^2 + DE^2 + EF^2 + FA^2)\,.$$

Solved by Michel Bataille, Rouen, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Klamkin's write-up.

Let $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{D}$, $\overrightarrow{E}$, $\overrightarrow{F}$ denote the respective vectors AB, BC, CD, DE, EF, FA. Then the inequality can be rewritten as

$$\begin{aligned}\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}\right)^2 + \left(\overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}\right)^2 + \left(\overrightarrow{C} + \overrightarrow{D} + \overrightarrow{E}\right)^2 \\ \leq\; 2\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2 + \overrightarrow{D}^2 + \overrightarrow{E}^2 + \overrightarrow{F}^2\right).\end{aligned}$$

Replacing $\overrightarrow{F}$ by $-\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D} + \overrightarrow{E}\right)$, the inequality can be rewritten in terms of a sum of squares:

$$\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{D} + \overrightarrow{E}\right)^2 + \left(\overrightarrow{A} + \overrightarrow{C} + \overrightarrow{E}\right)^2 + \left(\overrightarrow{A} + \overrightarrow{D}\right)^2 + \left(\overrightarrow{B} + \overrightarrow{E}\right)^2 \;\geq\; 0\,.$$

Thus, the inequality holds.

There is equality if and only if $\overrightarrow{A} + \overrightarrow{C} + \overrightarrow{E} = \overrightarrow{A} + \overrightarrow{D} = \overrightarrow{B} + \overrightarrow{E} = \overrightarrow{0}$. This requires that $ABCDEF$ be a planar centro-symmetric hexagon whose sides are parallel to the three main diagonals. Here, any side length is half the length of its parallel main diagonal.

4. It is known that the function $f(x)$ is defined on the set of natural numbers, taking values from the natural numbers, and that it satisfies the following conditions:

(a) $f(xy) = f(x) + f(y) - 1$ for any x, $y \in \mathbb{N}$,

(b) the equality $f(x) = 1$ is true for finitely many numbers,

(c) $f(30) = 4$.

Find $f(14400)$.

Solved by Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We present the solution by Andrieux.

Prenons $x = y = 1$, on obtient $f(1) = f(1) + f(1) - 1$, donc $f(1) = 1$. Soit n un entier différent de 1. Une récurrence immédiate montre que, pour tout entier k :

$$f(n^k) \;=\; kf(n) - (k-1)\,.$$

Si $f(n) = 1$, alors pour tout entier k on a $f(n^k) = 1$. Les entiers n^k avec $k \in \mathbb{N}$ étant tous distincts, l'équation $f(x) = 1$ possède alors une infinité de solutions ce qui contredit (b). Donc, pour $n \geq 2$, on a $f(n) \geq 2$.

Soient alors 3 entiers n, p, et q, on a

$$\begin{aligned} f(npq) &= f((np)q) \;=\; f(np) + f(q) - 1 \\ &= f(n) + f(p) - 1 + f(q) - 1 \;=\; f(n) + f(p) + f(q) - 2\,. \end{aligned}$$

Puisque $f(30) = 4$, on a

$$\begin{aligned} f(2 \times 3 \times 5) &= 4\,, \\ f(2) + f(3) + f(5) - 2 &= 4\,, \\ f(2) + f(3) + f(5) &= 6\,. \end{aligned}$$

De $f(2) \geq 2$, $f(3) \geq 2$ et $f(5) \geq 2$, on en déduit

$$f(2) \;=\; f(3) \;=\; f(5) \;=\; 2\,.$$

Décomposons alors 14400 en produit de facteurs premiers,

$$14400 \;=\; 2^6 \times 3^2 \times 5^2\,,$$

d'où

$$\begin{aligned} f(14400) &= f(2^6 \times 3^2 \times 5^2) \\ &= f(2^6) + f(3^2) + f(5^2) - 2 \\ &= 6f(2) - 5 + 2f(3) - 1 + 2f(5) - 1 - 2 \\ &= 12 - 6 + 4 - 1 + 4 - 1 - 2 \\ &= 11\,. \end{aligned}$$

Finalement, $f(14400) = 11$.

Next we look at solutions to some of the problems of the Croatian National Mathematical Competition, Novi Vinodolski, IV$^{\text{th}}$ Class, 1997 [2001 : 89].

1. Find the last four digits of the number 3^{1000} and the number 3^{1997}.

Solved by Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Tsaoussoglou.

(a) $3^{1000} = (3^4)^{250} = 81^{250}$, showing that the last digit is 1. Now

$$\begin{aligned} 3^{1000} &= \left(3^2\right)^{500} \;=\; (10-1)^{500} \\ &= 10^{500} - \frac{500}{1}10^{499} + \frac{499 \cdot 500}{1 \cdot 2}10^{498} - \cdots \\ &\qquad - \frac{498 \cdot 499 \cdot 500}{1 \cdot 2 \cdot 3}10^3 + \frac{499 \cdot 500}{2}10^2 - \frac{500}{1}10 + 1\,. \end{aligned}$$

For the last 4 digits, we need only the last 3 terms, which simplify to

$$\begin{aligned} 499 \cdot 25 \cdot 10^3 - 5 \cdot 10^3 + 1 &= 5(499 \cdot 5 - 1) \cdot 10^3 + 1 \\ &= 1247 \cdot 10^4 + 1 \\ &= 12470001\,. \end{aligned}$$

Therefore, the last four digits are 0001.

(b) $3^{1997} = 3 \cdot 3^{1996} = 3(3^4)^{499}$, showing that the last digit is 3. Now

$$\begin{aligned} 3^{1996} &= \left(3^2\right)^{998} = (10-1)^{998} \\ &= 10^{998} - \frac{998}{1}10^{997} + \frac{997 \cdot 998}{1 \cdot 2}10^{996} - \cdots \\ &\qquad - \frac{996 \cdot 997 \cdot 998}{1 \cdot 2 \cdot 3}10^3 + \frac{997 \cdot 998}{1 \cdot 2}10^2 - \frac{998}{1}10 + 1 \\ &= \ldots 4321\,. \end{aligned}$$

Then

$$3^{1997} = 3(\ldots 4321) = \ldots 2963\,.$$

Thus, the last four digits of 3^{1997} are 2963.

We also give Klamkin's solution.

By Fermat's general theorem, $3^{\varphi(10000)} \equiv 1 \pmod{10000}$. Here, $\varphi(10000) = 4000$. Now $3^{4000} - 1 = (3^{1000} - 1)(3^{1000} + 1)(3^{2000} + 1)$. Since $3^{4n} \equiv 1 \pmod{10}$, it follows that $3^{1000} \equiv 1 \pmod{10000}$. Hence, the last four digits of the number 3^{1000} are 0001.

Let $3^{1997} \equiv a + 10b + 100c + 1000d \pmod{10000}$. Then

$$3^{2000} \equiv 1 \equiv 27a + 270b + 2700c + 27000d \pmod{10000}\,.$$

Hence, $a = 3$ and then $-8 \equiv 27b + 270c + 2700 \pmod{1000}$; hence, $b = 6$ and then $-17 \equiv 27c + 270d \pmod{100}$; hence, $c = 9$ and then $-26 \equiv 27d \pmod{10}$. Finally, $d = 2$. Thus, the last four digits of 3^{1997} must be 2963.

2. A circle k and the point K are on the same plane. For every two distinct points P and Q on k, the circle k' contains the points P, Q, and K. Let M be the intersection of the tangent to the circle k' at the point K and the line PQ. Find the locus of the points M when P and Q move over all points on k.

Solution by Michel Bataille, Rouen, France.

First two easy particular cases:

If K lies on k, the locus of M is clearly the tangent to k at K.

If K is the centre O of k, the tangent to k' at K is always parallel to PQ. Thus, the locus of M is empty.

Now to the general case where $K \notin k$ and $K \neq 0$ (Figure 1). Since the line PQ is the radical axis of k and k', the point M has the same power with respect to k and k'. The relation $MO^2 - R^2 = MK^2$ (in which R denotes the radius of k) follows immediately and shows that M belongs to the line L whose points N are characterized by $NO^2 - NK^2 = R^2$.

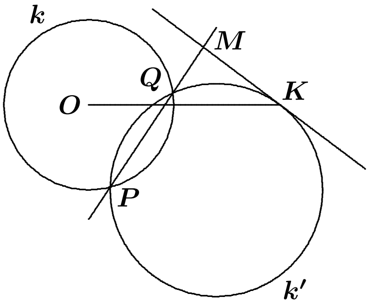


Figure 1

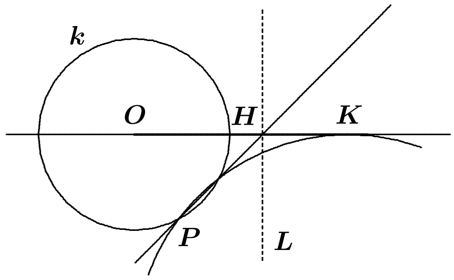


Figure 2

Conversely, let N be any point on L. Then

$$NO^2 - R^2 = NK^2 . \qquad (1)$$

Choose any point P on k, not on NK and not a point of tangency of k with a circle tangent to NK at K [this leaves infinitely many choices!]. Then there exists a unique circle Γ through P and tangent to NK at K: its centre is the point of intersection of the perpendicular bisector of KP and the perpendicular to NK at K. The circle Γ cuts k again at Q distinct from P (by the choice of P), and the line PQ is the radical axis of Γ $(= k')$ and k. Thus, PQ passes through any point that has the same power with respect to the two circles. In particular, PQ passes through N, in view of (1). Thus, $N = M$ is a point of the locus. In conclusion, the locus we seek is the line L.

Note. This line L is the perpendicular to OK at the point H defined by $IH = \frac{1}{2}R^2/OK$, where I is the mid-point of OK. Observe that K is not on L (since $K \notin k$) and that L is exterior to k (since $NO^2 > R^2$ for all N on L). The line L can easily be constructed by remarking that H is also the intersection of OK with the line through the common points of k and a circle tangent to OK at K (see Figure 2).

3. A function f is defined on the set of positive numbers, which has the following properties:

$$f(1) = 1 , \quad f(2) = 2 ,$$

$$f(n+2) = f\big(n+2-f(n+1)\big) + f\big(n+1-f(n)\big) , \quad (n \geq 1) .$$

(a) Show that $f(n+1) - f(n) \in \{0, 1\}$ for every $n \geq 1$.

(b) If $f(n)$ is odd, show that $f(n+1) = f(n) + 1$.

(c) For every natural number k determine all numbers n for which

$$f(n) \;=\; 2^{k-1} + 1\,.$$

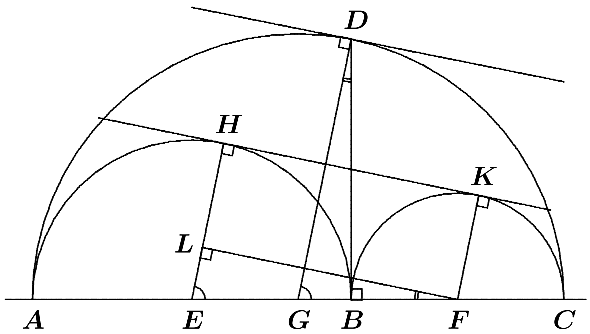
Comment by Michel Bataille, Rouen, France.

This problem is very similar to problem No 5, 1990 Canadian Mathematical Olympiad. A reference is *Mathematics Magazine* Vol 63, No 3, June 1990, p. 204.

Next we turn to solutions to some of the problems of the Croation National Mathematical Competition, Additional Competition for Selection of the 38^{th} IMO Team, 1997 [2001 : 90].

1. Three points A, B, C, are given on the same line, such that B is between A and C. Over the segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, as diameters, semicircles are constructed on the same side of the line. The perpendicular from B to AC intersects the largest circle at point D. Prove that the common tangent of the two smaller semicircles, different from BD, is parallel to the tangent on the largest semicircle through the point D.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.



Let the semicircles with diameters AB, BC, and AC be $\Gamma_1(E, R_1)$, $\Gamma_2(F, R_2)$, and $\Gamma_3(G, R_1 + R_2)$, respectively. Assume $R_1 > R_2$. The common tangent to Γ_1 and Γ_2 touches Γ_1 at H and Γ_2 at K. Let $L \in EH$ such that $FL \| KH$. Thus, $GD = EF = R_1 + R_2$, $GB = EL = R_1 - R_2$, and $\angle ELF = \angle GBD = 90^\circ$. We see that $\triangle ELF \cong \triangle GBD$, and therefore, $\angle FEL = \angle BGD$. This implies that EH is parallel to GD. We conclude that the tangent to Γ_3 at D is parallel to KH.

2. Let a, b, c, d be real numbers such that at least one is different from zero. Prove that all roots of the polynomial

$$P(x) \;=\; x^6 + ax^3 + bx^2 + cx + d$$

cannot be real.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Heinz-Jürgen Seiffert, Berlin, Germany. We give Seiffert's write-up.

More generally, let $a_0, a_1, \ldots, a_{n-3}$, $n \geq 3$, be real numbers such that at least one is different from zero. Then, the polynomial

$$P(x) \;=\; x^n + \sum_{j=0}^{n-3} a_j x^j \tag{1}$$

cannot have only real roots. (The proposal's result is the particular case $n = 6$.)

Proof. Assume, by way of contradiction, that $P(x)$ has the (not necessarily distinct) real roots x_1, x_2, ..., x_n. Since $P(x)$ has leading coefficient 1,

$$P(x) \;=\; \prod_{k=1}^{n} (x - x_k)\,. \tag{2}$$

Comparing the coefficients of x^{n-1} and x^{n-2} gives

$$\sum_{k=1}^{n} x_k \;=\; 0 \quad \text{and} \quad \sum_{1\le j<k\le n} x_j x_k \;=\; 0\,.$$

Hence,

$$\sum_{k=1}^{n} x_k^2 \;=\; \left(\sum_{k=1}^{n} x_k\right)^2 - \; 2 \sum_{1\le j<k\le n} x_j x_k \;=\; 0\,,$$

which implies $x_1 = x_2 = \cdots = x_n = 0$, because $x_1, x_2, \ldots, x_n$ are all real. Thus, by (2), $P(x) = x^n$, and then, by (1), $a_0 = a_1 = \cdots = a_{n-3} = 0$, a contradiction.

Next we look at reader solutions to problems of the 1997 St. Petersburg City Mathematical Olympiad, Selection Round – 10^{th} Grade [2001 : 91].

1. Positive integers x, y, z satisfy the equation $2x^x + y^y = 3z^z$. Prove that they are equal.

Solution by Pierre Bornsztein, Pontoise, France.

Let x, y, z be positive integers, such that $2x^x + y^y = 3z^z$.

Suppose that $y > z$. Then, since they are integers, we have $y \geq z + 1$. Using the Binomial Theorem,

$$3z^z \;=\; 2x^x + y^y \;>\; (z+1)^{z+1} \;\geq\; z^{z+1} + (z+1)z^z \;\geq\; 3z^z\,.$$

This is a contradiction. Thus, $y \leq z$.

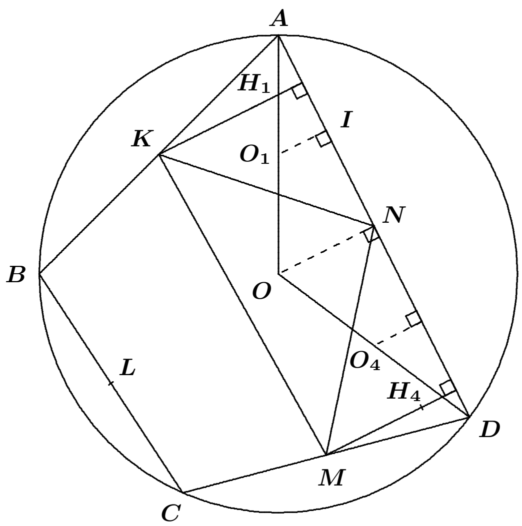
If $y < z$, then $2x^x = 3z^z - y^y > 2z^z$, and so $x > z$. As above, we then have $x \geq z+1$ and

$$3z^z \;=\; 2x^x + y^y \;>\; 2(z+1)^{z+1} \;\geq\; 2z^{z+1} + (z+1)z^z \;>\; 3z^z\,.$$

This is a contradiction. Thus, $y = z$. Then $x^x = z^z$, which gives $x = z$, and we are done.

3. K, L, M, N are the mid-points of sides AB, BC, CD, DA, respectively, of an inscribed quadrangle $ABCD$. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are vertices of a parallelogram.

Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's solution.



Let O be the centre of the circle $(ABCD)$, and let H_1, H_2, H_3, H_4 be the orthocentres of triangles AKN, BKL, CLM, DMN, respectively. Note that $\triangle AKN$ is the homothetic image of $\triangle ABD$ under the homothety with centre A and factor $\frac{1}{2}$. It follows that the circumcentre O_1 of $\triangle AKN$ is the mid-point of AO. Denoting by I the mid-point of AN, we thus have $\overrightarrow{O_1I} = \frac{1}{2}\overrightarrow{ON}$. Using the well-known relation $\overrightarrow{KH_1} = 2\overrightarrow{O_1I}$, we deduce that $\overrightarrow{KH_1} = \overrightarrow{ON}$.

Similarly, $\overrightarrow{MH_4} = \overrightarrow{ON}$. It then follows that $\overrightarrow{KH_1} = \overrightarrow{MH_4}$. Hence, $\overrightarrow{H_1H_4} = \overrightarrow{KM}$. In the same way, $\overrightarrow{H_2H_3} = \overrightarrow{KM}$. Thus, $\overrightarrow{H_1H_4} = \overrightarrow{H_2H_3}$, which means that $H_1H_2H_3H_4$ is a parallelogram.

Now we look at solutions from our readers to problems of the 1997 St. Petersburg City Mathematical Olympiad, Selection Round – 11^{th} Grade [2001 : 91–92].

1. Can a 75×75 table be partitioned into dominoes (that is, 1×2 rectangles) and crosses (that is, five-square figures consisting of a square and its four horizontal and vertical neighbours)?

Solved by Robert Bilinski, Outremont, QC; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the solution by Klamkin.

Our solution is indirect in that we assume it can be done and obtain a contradiction. Let the table be coloured black and white in the manner of a chess board, with the corner squares being black. Then the total number of black squares is one greater than the total number of white squares. Let x and y denote the number of dominoes and crosses, respectively, that cover the table. Then, $2x + 5y = 75^2$, and therefore, $x = 5n$. It follows that $2n + y = 75 \times 15 = 1125$. Now y must be odd, say $y = 2m + 1$.

While each domino covers one black and one white square, a cross can cover one white and four black squares, or one black and four white. Let p denote the number of crosses used each of which cover one white and four black. The remaining $2m+1-p$ crosses each cover one black and four white. The total number of black squares covered by the crosses is $4p+(2m+1-p)$, while the total number of white squares covered is $p+4(2m+1-p)$. Their difference is $6p - 6m - 3$. This difference must equal 1. Thus, $3p - 3m = 2$, which is impossible. Hence, the table cannot be partitioned in the given manner.

2. Prove that for $x \geq 2$, $y \geq 2$, $z \geq 2$

$$(y^3 + x)(z^3 + y)(x^3 + z) \;\geq\; 125xyz\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Klamkin's generalization, adapted by the editors.

We show that

$$(x_1^{nr} + ax_2^{ns})(x_2^{nr} + ax_3^{ns})\cdots(x_n^{nr} + ax_1^{ns}) \;\geq\; (k^{n(r-s)} + a)^n P^{ns}\,, \quad (1)$$

where $P = x_1 x_2 \cdots x_n$, $r \geq s \geq 0$, $a \geq 0$, and $x_i \geq k \geq 0$.

Since $k^n \leq P$, we have $k^{n(r-s)} \leq P^{r-s}$, and hence,

$$\begin{aligned}(k^{n(r-s)} + a)P^s &\leq P^r + aP^s \\ &\leq (x_1^{nr} + ax_2^{ns})^{\frac{1}{n}}(x_2^{nr} + ax_3^{ns})^{\frac{1}{n}}\cdots(x_n^{nr} + ax_1^{ns})^{\frac{1}{n}}\,,\end{aligned}$$

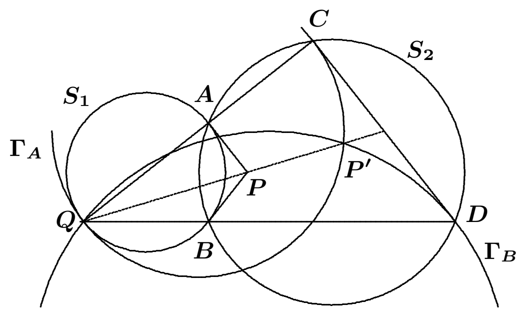
where the last step follows by Hölder's Inequality. Now raise both sides to the power n to obtain (1). There is equality if and only if $x_i = k$ for all i.

The given inequality corresponds to the special case $n = 3$, $r = 1$, $s = \frac{1}{3}$, $a = 1$ and $k = 2$.

Other extensions can be obtained by replacing each of the factors on the left side of (1) by a sum of more terms, and using Hölder's Inequality. For example, the first factor can be replaced by $x_1^{nr} + ax_2^{ns} + bx_3^{nt}$, with the other $n - 1$ factors given cyclically.

3. Circles S_1 and S_2 intersect at points A and B. A point Q is chosen on S_1. The lines QA and QB meet S_2 at points C and D; the tangents to S_1 at A and B meet at point P. The point Q lies outside S_2, the points C and D lie outside S_1. Prove that the line QP goes through the mid-point of CD.

Solved by Michel Bataille, Rouen, France; and Babis Stergiou, Lycio Psachnon Evias, Greece. We give Bataille's solution.



Since $QA \cdot QC = QB \cdot QD$, points A, C as well as B, D are inverse points through an inversion with centre Q. This inversion transforms the line CD into S_1 and the lines AP, BP into circles Γ_A, Γ_B passing through Q and tangent to CD at C, D, respectively. Let P' be the inverse of P. We see that Γ_A, Γ_B intersect at Q and P' and that CD is a common exterior tangent (at C, D) to Γ_A, Γ_B. As is well-known, QP' meets CD at its mid-point. Since Q, P, P' are collinear, the desired result is obtained.

That completes the *Corner* for this issue. Send me your nice solutions, generalizations, and also Olympiad contests!