# SKOLIAD No. 69

## Shawn Godin

Solutions may be sent to Shawn Godin, 2191 Saturn Cres., Orleans, ON, K4A 3T6, or emailed to

mayhem-editors@cms.math.ca.

We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 October 2003*. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The items in this issue come from the 2001 Invitational Mathematics Challenge, a Canadian Mathematics Competition run by the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. This competition is written by the top 200 students who write the Cayley (Grade 10) and Fermat (Grade 11) contests (by invitation). Thanks go to Peter Crippin at CEMC for allowing us to use the contest material.

## 2001 Invitational Mathematics Challenge
## Défi invitation de mathématiques 2001

**(Grade 10 / 10$^{e}$ année - Sec. IV au Québec)**

**TIME ALLOWED:** 2 hours.

Calculators are permitted. It is expected that all calculations and answers will be expressed as exact numbers such as $4\pi$, $2+\sqrt{7}$, etc. Marks are awarded for completeness, clarity and style of presentation. A correct solution, poorly presented, will not earn full marks.

**1**. Thirty years ago, the ages of Xavier, Yolanda, and Zoë were in the ratio $1 : 2 : 5$. Today, the ratio of Xavier's age to Yolanda's age is $6 : 7$. What is Zoë's present age?

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Il y a trente ans, l'âge de Xavier, l'âge de Yolande, et l'âge de Zoë étaient dans un rapport de $1 : 2 : 5$. Aujourd'hui, l'âge de Xavier et celui de Yolande sont dans un rapport de $6 : 7$. Quel est l'âge actuel de Zoë ?

**2**. (a) Determine the number of integers between $100$ and $999$, inclusive, that contain exactly two digits that are the same.

(b) Determine the probability that a positive integer less than $1000$ contains exactly two digits that are the same.

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(a) Déterminer le nombre d'entiers, de $100$ à $999$, qui ont exactement deux chiffres identiques.

(b) Déterminer la probabilité pour qu'un entier strictement positif, inférieur à $1000$, ait exactement deux chiffres identiques.

**3**. Solve the system of equations:

$$\begin{aligned} x+y+z &= 2 \\ x^2-y^2-z^2 &= 2 \\ x-3y^2+z &= 0\,. \end{aligned}$$

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Résoudre le système d'équations :

$$\begin{aligned} x+y+z &= 2 \\ x^2-y^2-z^2 &= 2 \\ x-3y^2+z &= 0. \end{aligned}$$

**4**. A flat mirror is perpendicular to the $xy$–plane and stands on the line $y = x + 4$. A laser beam from the origin strikes the mirror at $P(-1, 3)$ and is reflected to the point $Q$ on the $x$–axis. Determine the coordinates of the point $Q$.

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Un miroir plat est placé sur la droite d'équation $y = x + 4$, tout en étant perpendiculaire au plan formé par les axes des $x$ et des $y$. Un rayon laser part de l'origine, atteint le miroir au point $P(-1, 3)$, puis est réfléchi de manière à atteindre le point $Q$ sur l'axe des $x$. Déterminer les coordonnées du point $Q$.

**5**. Determine all pairs of non-negative integers $(m, n)$ which are solutions to the equation $3\,(2^m) + 1 = n^2$.

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Déterminer toutes les paires $(m, n)$ d'entiers non négatifs qui vérifient l'équation $3\,(2^m) + 1 = n^2$.

**(11$^{e}$ année / Grade 11 - Sec.V au Québec)**

**DURÉE :** $2$ heures.

L'usage de la calculatrice est permis. Les réponses et les calculs doivent être exprimés à l'aide de nombres exacts, tels que $4\pi$, $2+\sqrt{7}$, etc. Dans l'évaluation, on tiendra compte de la qualité, de la clarté et de la précision de la présentation. Une solution correcte, mais mal présentée, ne recevra pas le nombre maximum de points.

**1**. Un cultivateur a six contenants pouvant contenir respectivement $15$, $16$, $18$, $19$, $20$, et $31$ litres. Un des contenants est rempli de créme, tandis que les cinq autres sont remplis de lait blanc ou de lait au chocolat. Il y a deux fois plus de lait blanc que de lait au chocolat.

(a) Quel est le volume du contenant qui est rempli de crème ?

(b) La crème se vend $3$ \$ le litre, le lait au chocolat se vend $2$ \$ le litre et le lait blanc se vent $1$ \$ le litre. Quelle est la valeur totale de ce qui est dans les six contenants ?

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Farmer Haas has six containers with capacities of $15$, $16$, $18$, $19$, $20$, and $31$ litres. One of these containers is filled with cream and the other five are filled with either white milk or chocolate milk. Farmer Haas has twice as much white milk as chocolate milk.

(a) What is the volume of the container that is filled with cream?

(b) The price of the cream is \$$3$ per litre, the price of the chocolate milk is \$$2$ per litre, and the price of the white milk is \$$1$ per litre. What is the total value of the contents of the six containers?

**2**. Voir question # $3$ du concours de 10$^{e}$.

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See question # $3$ from the grade 10 contest.

**3**. Les points $X$ et $Y$ sont situés sur les côtés respectifs $BC$ et $CD$ du carré $ABCD$. Les segments $XY$, $AX$, et $AY$ ont une longueur respective de $3$, $4$, et $5$ unités. Déterminer la longueur d'un côté du carré $ABCD$.

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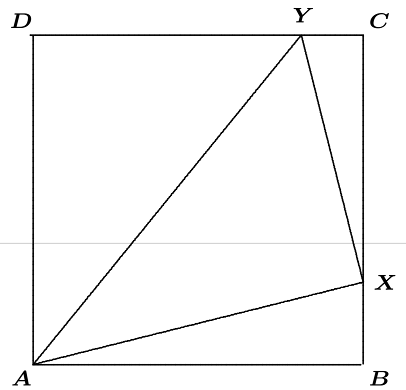
Points $X$ and $Y$ are on sides $BC$ and $CD$ of square $ABCD$, as shown below. The lengths of $XY$, $AX$, and $AY$ are $3$, $4$, and $5$, respectively. Determine the side length of square $ABCD$.




**4**. Un miroir plat est placé le long d'une droite $L$, perpendiculairement au plan formé par les axes des $x$ et des $y$. Un rayon laser part de l'origine, atteint le miroir au point $P(-1,5)$, puis est réfléchi pour atteindre le point $Q(24,0)$. Déterminer l'équation de la droite $L$.

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A flat mirror is perpendicular to the $xy$–plane and stands along a line $L$. A laser beam from the origin strikes the mirror at $P(-1,5)$ and is reflected to the point $Q(24,0)$. Determine the equation of the line $L$.

**5**. Soit $f(n) = n^4 + 2n^3 - n^2 + 2n + 1$.

(a) Démontrer que $f(n)$ peut être exprimé comme produit de deux polynômes du second degré ayant chacun des coefficients entiers.

(b) Déterminer tous les entiers $n$ pour lesquels $|f(n)|$ est un nombre premier.

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Let $f(n) = n^4 + 2n^3 - n^2 + 2n + 1$.

(a) Show that $f(n)$ can be written as the product of two quadratic polynomials with integer coefficients.

(b) Determine all integers $n$ for which $|f(n)|$ is a prime number.

Next we give solutions to the 1991 Canadian Mathematical Society Prize Exam given in the October 2002 issue [2002 : 391].

**1**. Show that $1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \frac{5}{81} - \cdots \frac{100}{3^{99}} = \frac{9}{16}\left[1 - \frac{403}{3^{101}}\right]$.

*Solution by Jefferson Lin, student, Brooklyn, NY, USA.*

Let $S = 1 - \frac{2}{3} + \frac{3}{9} - \cdots \frac{100}{3^{99}}$. Then $\frac{S}{3} = \frac{1}{3} - \frac{2}{9} + \cdots + \frac{99}{3^{99}} - \frac{100}{3^{100}}$.

Adding these two we get

$$S+\frac{S}{3} \;=\; \left(1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\cdots-\frac{1}{3^{99}}\right)-\frac{100}{3^{100}}\,.$$

But the expression in the brackets is just a geometric series. Thus, we get

$$\frac{4}{3}S \;=\; \frac{1-\left(-\frac{1}{3}\right)^{100}}{1-\left(-\frac{1}{3}\right)}-\frac{100}{3^{100}}\,.$$

Multiplying both sides of this equation by $\frac{4}{3}$ yields

$$\frac{16}{9}S \;=\; 1-\frac{1}{3^{100}}-\frac{400}{3^{101}} \;=\; \frac{3^{101}-403}{3^{101}}\,.$$

Thus, $S=\frac{9}{16}\left[1-\frac{403}{3^{101}}\right]$.

*Also solved by Siwen Sun, grade 11 student, Collège Saint-Louis, Lachine, QC.*

**2**. Solve for all real $x$: $\sqrt{x^2-x+2}+\sqrt{x^2-x-2}=1$.

*Solution by Geneviève Lalonde, Massey, ON.*

Completing the square yields

$$x^2-x+2 \;=\; x^2-x+\tfrac{1}{4}-\tfrac{1}{4}+2 \;=\; \left(x-\tfrac{1}{2}\right)^2+\tfrac{7}{4}\,.$$

We notice that $\sqrt{x^2-x+2}\geq\sqrt{\frac{7}{4}}>1$ for all $x\in\mathbb{R}$. Therefore, the original equation has no solutions.

*Three incorrect solutions were received. All incorrect solutions solved by squaring to eliminate the radicals, but failed to check for extraneous roots.*

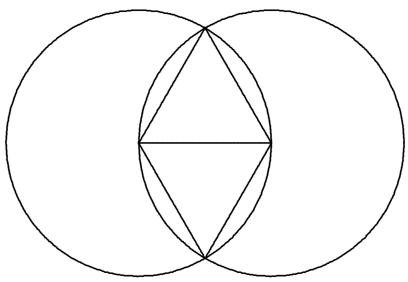
**3**. Two circles of equal radius pass through each other's centres. What are

(a) the perimeter and

(b) the area of the whole region enclosed by the circles?

*Solution to part (a) by Siwen Sun, grade 11 student, Collège Saint-Louis, Lachine, QC.*

Because both circles have the same radius and pass through each other's centres, they form 2 equilateral triangles with the 2 centres and the 2 points of intersection as vertices.



Therefore, the angle of the part inside the $2$ circles is $2 \times 60^\circ = 120^\circ$. Thus,

$$P \;=\; 4\pi r - 2 \times \frac{2\pi r}{3} \;=\; \frac{8\pi r}{3}\,.$$

*Also solved by Jefferson Lin, student, Brooklyn, NY, USA.*

*Solution to part* (b) *by Geneviève Lalonde, Massey, ON.*

The area is made up of $\frac{2}{3}$ of each circle, plus two equilateral triangles. Thus,

$$\begin{aligned} A &= 2 \times \frac{2}{3}\pi r^2 + 2 \times \frac{\sqrt{3}r^2}{4} \\ &= \left(\frac{4}{3}\pi + \frac{\sqrt{3}}{2}\right) r^2\,. \end{aligned}$$

*Two incorrect solutions were also received.*

**4**. When the polynomial $x^4 + ax^3 - 7x^2 + bx - 49$ is divided by $(x - 3)$ the remainder is $53$, and by $(x + 2)$ the remainder is $-87$. Find $a$ and $b$.

*Solution by Alfian, grade* $11$ *student, SMU Methodist* $1$*, Palearbang, Indonesia.*

Let $f(x) = x^4 + ax^3 - 7x^2 + bx - 49$. The remainder theorem tells us that the remainder when $f(x)$ is divided by $(x - 3)$ is $f(3)$. Thus, $f(3) = 53$ which, after substitution, yields

$$27a + 3b \;=\; 84\,. \tag{1}$$

Similarly, we must have $f(-2) = -87$ which, after substitution, yields

$$4a + b \;=\; 13\,. \tag{2}$$

Multiplying (2) by $3$ and subtracting from (1) gives $a = 3$. Then substitution gives $b = 1$.

*Also solved by Jefferson Lin, student, Brooklyn, NY, USA.*

**5**. From the letters of the word "antenna", we want to make all possible four-letter "words" (they may be nonsensical, for example, "aann"). How many can we make?

*Solution by Geneviève Lalonde, Massey, ON.*

We must examine a number of cases.

Case 1: The word has $3$ n's. Then we must choose the $4^{\text{th}}$ letter from $\{a, e, t\}$. Thus, the total number of such words is $\binom{3}{1} \times 4 = 12$.

Case 2: The word has $2$ n's and *not* $2$ a's. Thus, we must choose two letters from $\{a, e, t\}$. The total number of such words is $\binom{3}{2} \times \frac{4!}{2!} = 36$.

Case 3: The word has $2$ n's and $2$ a's. The total number of such words is $\frac{4!}{2!2!} = 6$.

Case 4: The word has $2$ a's and *not* $2$ n's. Thus, we must choose two letters from $\{e, n, t\}$. The total number of such words is $\binom{3}{2} \times \frac{4!}{2!} = 36$.

Case 5: The word is made up of $4$ distinct letters. The total number of such words is $4! = 24$.

The total number of four-letter words is $12 + 36 + 6 + 36 + 24 = 118$.

*Two incorrect solutions were received.*

**6**. If $a$ and $b$ are positive integers larger than 2, prove that $(2^a + 1)$ cannot be divisible by $(2^b - 1)$.

*Solution by Jefferson Lin, student, Brooklyn, NY, USA.*

Assume that we have $2^b - 1 \mid 2^a + 1$ with $a > b$, and assume that $a$ is the smallest value for which this is true. Then there is an integer $k$ such that $k(2^b - 1) = 2^a + 1$. Thus,

$$\begin{aligned} k \cdot 2^b - k &= 2^a + 1\,, \\ k \cdot 2^b - 2^a &= k + 1\,, \\ 2^b(k - 2^{a-b}) &= k + 1\,. \end{aligned}$$

Therefore, if we let $n = k - 2^{a-b}$, we get

$$\begin{aligned} k + 1 &= n \cdot 2^b\,, \\ k &= n \cdot 2^b - 1\,, \\ (n \cdot 2^b - 1)(2^b - 1) &= 2^a + 1\,. \end{aligned}$$

Simplifying yields $n(2^b - 1) = 2^{a-b} + 1$. Thus, $2^b - 1 \mid 2^{a-b} + 1$. But $2^{a-b} + 1 < 2^a + 1$, and $2^a + 1$ was assumed to be the smallest value with this property. Therefore, we have a contradiction, and $(2^a + 1)$ cannot be divisible by $(2^b - 1)$ for positive integers $a, b > 2$.

*One incorrect solution was received.*

That brings us to the end of another issue of Skoliad. This issue's winner of a copy of **MATHEMATICAL MAYHEM VOL. 6** is (drum roll ...) Jefferson Lin! Please continue sending in contest problems and solutions.