SKOLIAD No. 68

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We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by 1 September 2003. A copy of MATHEMATICAL MAYHEM Vol. 2 will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The items in this issue come from the Mandelbrot competition. This competition has four rounds that occur during the school year. Each round has two competitions: an individual test, and a team test written a couple of days later. A school's score is made up of the top four individual scores plus the score of the team (comprised of students selected by the school). Each test is 40 minutes in duration, and no aids of any type are allowed. My thanks go to Sam Vandervelde at Greater Testing Concepts for forwarding the material to me. For more information about the contest you can visit the website

www.mandelbrot.org

The Mandelbrot Competition
Division B Round One Individual Test
November 1997

1. (*) What angle less than $180^\circ$ is formed by the hands of a clock at 2:30 pm? (Express the answer in degrees.) (1 point)

2. (*) If $x = \sqrt{\frac{6}{7}}$, then evaluate \( (x + \frac{1}{x})^2 \). (1 point)

3. (*) How many possible values are there for the sum $a + b + c$ if $a$, $b$, and $c$ are positive integers and $abc = 50$? (2 points)
4. List the numbers $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ in order from greatest to least. (2 points)

5. Compute $3^A$ where $A = \frac{(\log 1 - \log 4)(\log 9 - \log 2)}{(\log 1 - \log 9)(\log 8 - \log 4)}$. All logarithms are base three. (2 points)

6. Joe and Andy are playing a simple game on a circular board with $n$ spaces. First Joe advances five spaces from the starting space, then Andy advances seven, then Joe advances five, then Andy advances seven, and so on. The first player to land back on the original space wins. If $n$ is a random two-digit number, what is the probability that Joe wins? (3 points)

7. In the diagram, $M$ is the mid-point of $AB$ and $Y$ is the mid-point of $AC$. Hence, $Q$ is a trisection point of $CM$; we call the other trisection point $P$ and extend $BP$ to meet $AC$ at $X$. Evaluate $(CX + AY)/XY$. (3 points)

![The Mandelbrot Competition Diagram](image)

The Mandelbrot Competition
Division B Round One Team Test
November 1997

**Facts:** The Weighted Power Mean Inequality for two positive variables states that if $x_1$, $x_2$, $w_1$ and $w_2$ are positive real numbers, and $m$ and $n$ are non-zero integers with $m > n$, then

$$
\left( \frac{w_1 x_1^m + w_2 x_2^m}{w_1 + w_2} \right)^\frac{1}{m} \geq \left( \frac{w_1 x_1^n + w_2 x_2^n}{w_1 + w_2} \right)^\frac{1}{n},
$$

which is quite a mouthful. The positive variables are $x_1$ and $x_2$. The two sides are equal if and only if $x_1 = x_2$. The numbers $w_1$ and $w_2$ "weight" the variables in different proportions. Try $w_1 = w_2 = 1$ to see the standard Power Mean Inequality; then compare with $w_1 = 1$ and $w_2 = 2$, which emphasizes $x_2$. Finally, the non-zero integers $m$ and $n$ vary the powers. For example, use $m = 1$ and $n = -1$ to obtain the Arithmetic Mean–Harmonic Mean Inequality.
**Setup:** On the planet Garth a certain laser has the curious property that when reflected off a special mirror it always continues in a direction perpendicular to the original path. Some popular Garthian children's games are based on this phenomenon. The ones described below involve a player situated in the corner of a rectangular mirrored hallway of width one plog (about 7.3 metres), as shown below. The player directs the laser beam an angle of $\theta$ away from the left wall, hitting the far wall a distance $d$ from the end of the hall on the first bounce.

![Diagram of laser path](image)

**Problems:** *(Please, no calculus-based solutions.)*

**Part i:** (4 points) Show that the laser's path length up to the second bounce is $\frac{1}{\cos \theta} + \frac{1}{\sin \theta}$.

**Part ii:** (4 points) The object of one of the simpler games is to have the shortest path length after two bounces. Use the standard power mean inequality with $m = 2$, $n = -1$, $x_1 = \cos \theta$, and $x_2 = \sin \theta$ to prove that the shortest possible path length is $2\sqrt{2}$. Invoke the equality condition to show that we need $\tan \theta = 1$ (that is, $d = 1$) to obtain this minimum.

**Part iii:** (4 points) Show that the total path length after three bounces (a more challenging version) is $2 \left( \frac{1}{\cos \theta} \right) + \frac{1}{\sin \theta}$. To minimize this, we employ a clever strategy. Begin by finding numbers $w_1$ and $\lambda_1$ such that $w_1 \lambda_1^2 = 1$ and $w_1 / \lambda_1 = 2$.

**Part iv:** (4 points) Now apply the weighted power mean inequality with $m = 2$ and $n = -1$ as before, using $w_1$ from **Part iii**, $w_2 = 1$, $x_1 = \lambda_1 \cos \theta$, and $x_2 = \sin \theta$, to prove that the minimum path length is $(1 + 2^{2/3})^{3/2}$. What value of $d$ should the player aim for?

**Part v:** (5 points) Use this technique to find the shortest path length after five bounces.
Correction: Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON, points out that our featured solution to problem 8 of the Eighteenth W.J. Blundon Contest [2002 : 529] contains an extraneous root. The value $6 - 2\sqrt{5}$ does not satisfy the original equation. The editor apologizes for letting this go unnoticed.

Next we move on to the solutions to the 2000 Maritime Mathematics Competition presented in the September 2002 issue.

Concours de Mathématiques des Maritimes 2000
2000 Maritime Mathematics Competition

1. At a meeting, one mathematician remarked to another, “There are nine fewer of us here than twice the product of the two digits of our total number.” How many mathematicians were at the meeting?

Solution by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

Let $AB$ be the two-digit number. Then we have

\[
\begin{align*}
2 \cdot A \cdot B - 9 &= 10 \cdot A + B \\
2 \cdot A \cdot B - 10 \cdot A &= B + 9 \\
A &= \frac{B + 9}{2B - 10}
\end{align*}
\]

Note that $B$ must be odd, and the only value of $B$ that gives a positive integer for $A$ is $B = 7$, which gives $A = 4$. Thus, there were 47 mathematicians at the meeting.

Also solved by Robert Bilinski, Outremont, QC.

2. Suppose that two circles with radii $r$ and $R$ intersect in a single point and that the straight line $L$ is tangent to both circles at $t$ and $T$ respectively, as in the diagram below. Determine the distance between the points $t$ and $T$.

\[\begin{array}{c}
L \\
t \\
T
\end{array}\]
Solution by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

Note that $AB = R + r$ and $AC = R - r$. Thus, by the Pythagorean Theorem

$$BC^2 = AB^2 - AC^2 = (R + r)^2 - (R - r)^2 = 4Rr$$

Thus, $Tt = BC = 2\sqrt{Tr}$.

Also solved by Robert Bilinski, Outremont, QC.

3. Trouver la somme de tous les nombres à quatre chiffres dont les chiffres sont choisis, sans répétition, parmi 1, 2, 3, 4, 5. (Il y en a 120.)

Solution by Robert Bilinski, Outremont, QC.

Parmi les 120 nombres que l'on peut écrire, on en a 24 qui commencent par chaque unité. On peut dire la même chose des autres positions puisque la recherche de nombres est exhaustive et équiprobable. Donc le total de chaque position (chaque colonne dans l'addition) est $24 \times (1 + 2 + 3 + 4 + 5) = 360$. Ainsi, en additionnant les 120 nombres trouvés, on obtient $360 \times 1111 = 399960$.

Also solved by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

4. A cubic box with edges 1 metre long is placed against a vertical wall. A ladder $\sqrt{15}$ metres long is placed so that it touches the wall as well as the free horizontal edge of the box. Find at what height the ladder touches the wall.

Official Solution

Let $x$ be the height at which the ladder touches the wall. Let $y$ be the distance between the foot of the ladder and the wall. By the Pythagorean Theorem, $x^2 + y^2 = (\sqrt{15})^2 = 15$, and, using similar triangles,

$$\frac{x - 1}{1} = \frac{1}{y - 1} \implies (x - 1)(y - 1) = 1 \implies xy = x + y.$$
Now
\[ 15 = x^2 + y^2 = (x^2 + 2xy + y^2) - 2xy = (x + y)^2 - 2(x + y). \]
Thus, letting \( z = x + y \), we have
\[ z^2 - 2z - 15 = 0 \implies (z - 5)(z + 3) = 0 \]
\[ \implies z = 5 \text{ or } z = -3. \]

As \( x \) and \( y \) are both positive, \( z = -3 \) is inadmissible. Thus, \( 5 = z = x + y \), whence \( y = 5 - x \). Substituting into \( xy = x + y \), we obtain
\[ x(5 - x) = 5 \implies x^2 - 5x + 5 = 0. \]
By the Quadratic Formula,
\[ x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \frac{5 \pm \sqrt{5}}{2}. \]
Therefore, there are two solutions. The ladder touches the wall at a height of either \((5 + \sqrt{5})/2\) metres or \((5 - \sqrt{5})/2\) metres.

5. Une pelouse circulaire de 12 mètres de diamètre est traversée d’une allée de gravier de 3 mètres de large dont un des bords passe par le centre de la pelouse. Trouver l’aire du reste de la pelouse.

Solution by Robert Bilinski, Outremont, QC.
L’aire de la pelouse est donc composée d’un demi-cercle complet, d’aire \( \pi r^2 / 2 = 18\pi \text{ m}^2 \), et d’une partie de cercle qui correspond à un secteur angulaire amputé d’un triangle isocèle.

Mettons le cercle de telle sorte que l’allée soit verticale, un de ces côtés est un diamètre, l’autre est une corde verticale. Faisons le rapprochement entre le cercle et un cercle trigonométrique. On remarque que la corde verticale coupe le rayon horizontal en son milieu, donc le cosinus des 2 sommets de la corde verticale sur le cercle est \( \frac{1}{2} \). Ainsi l’angle entre les deux sommets est \( 2\pi / 3 \). Le secteur angulaire défini par ces deux sommets et le centre du cercle a une aire de \( 12\pi \text{ m}^2 \).

Mais la partie triangulaire de ce secteur est dans l’allée, le reste étant de la pelouse. Cette partie triangulaire est isocèle de côtés égaux 6 m (le rayon) et d’hauteur principale 3 m (le demi-rayon horizontal). De plus, on sait que le troisième côté égale \( 6\sqrt{3} \), car la corde verticale correspond au double du sinus de dans un cercle de rayon 6 m. Donc l’aire du triangle est \( 9\sqrt{3} \text{ m}^2 \).

Ainsi, la seconde partie de la pelouse a une aire de \((12\pi - 9\sqrt{3}) \text{ m}^2 \). Au total, la pelouse aura une aire de \((30\pi - 9\sqrt{3}) \text{ m}^2 \).

Also solved by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.
6. Consider decompositions of an $8 \times 8$ chessboard into $p$ non-overlapping rectangles subject to the following two conditions.

- Each rectangle has the same number of white squares and black squares.
- No two rectangles have the same number of squares.

Find the maximum value of $p$ for which such a decomposition is possible. For this maximum value of $p$, determine all corresponding decompositions of the chessboard into $p$ rectangles.

**Official Solution.**

Consider a decomposition of the chessboard into $p$ non-overlapping rectangles subject to the two given conditions. Let $a_1, a_2, \ldots, a_p$ be the number of squares in the rectangles in the decomposition.

Because of the second condition, the $a_i$'s are all distinct so we may assume, without loss of generality, that $a_1 < a_2 < \cdots < a_p$. Furthermore, each $a_i$ is even, by the first condition.

We claim that $p \leq 7$. To show this, suppose to the contrary that $p \geq 8$. Then the number of squares covered by the rectangles in the decomposition is

$$a_1 + a_2 + \cdots + a_p \geq 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 72,$$

which is impossible since the chessboard has only 64 squares.

For $p = 7$, we obtain the following five sequences as the only possibilities for $(a_1, a_2, \ldots, a_7)$:

- $(2, 4, 6, 8, 10, 12, 22)$,
- $(2, 4, 6, 8, 10, 14, 20)$,
- $(2, 4, 6, 8, 10, 16, 18)$,
- $(2, 4, 6, 8, 12, 14, 18)$,
- $(2, 4, 6, 10, 12, 14, 16)$.

To establish that the maximum value of $p$ is indeed 7, it remains to show that there is an actual decomposition of the chessboard into 7 rectangles. Now, no rectangle may have 22 squares since it is impossible to find a rectangle contained in the chessboard having dimensions $1 \times 22$ or $2 \times 11$. There is, therefore, no decomposition of the board corresponding to the first sequence. Each of the other 4 sequences, however, does have a corresponding decomposition, as the diagram on the next page illustrates.

Therefore, the maximum value of $p$ is 7, and there are 4 decompositions of the chessboard into 7 non-overlapping rectangles subject to the given conditions.

*Also solved by Robert Bilinski, Ouremont, QC, and Alexandre Ortan, student, École Joseph-Francois-Perrault, Montréal, QC.*
That brings us to the end of another number of Skoliad. This issue’s winner of a copy of MATHEMATICAL MAYHEM Volume 5 is Alexandre Ortan. Congratulations Alexandre! Please continue to send me your solutions and contests.