PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 5N3. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (*) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator’s permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard 8½ ×11” or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than 1 September 2003. They may also be sent by email to crux-editors@cms.math.ca. (It would be appreciated if email proposals and solutions were written in LaTeX.) Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

2801. Proposed by Heinz-Jürgen Seiffert. Berlin, Germany. Suppose that \( \triangle ABC \) is not obtuse. Denote (as usual) the sides by \( a \), \( b \), and \( c \) and the circumradius by \( R \). Prove that

\[
\left( \frac{2A}{\pi} \right)^{\frac{1}{a}} \left( \frac{2B}{\pi} \right)^{\frac{1}{b}} \left( \frac{2C}{\pi} \right)^{\frac{1}{c}} \leq \left( \frac{2}{3} \right)^{\frac{\sqrt{\pi}}{R}} .
\]

When does equality hold?

Supposons que le triangle \( \triangle ABC \) n’ait pas d’angle obtus et soit \( a \), \( b \), et \( c \) ses côtés et \( R \) le rayon du cercle circonscrit. Montrer que

\[
\left( \frac{2A}{\pi} \right)^{\frac{1}{a}} \left( \frac{2B}{\pi} \right)^{\frac{1}{b}} \left( \frac{2C}{\pi} \right)^{\frac{1}{c}} \leq \left( \frac{2}{3} \right)^{\frac{\sqrt{\pi}}{R}} .
\]

Quand l’égalité a-t-elle lieu?
2802. Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.

Four positive integers, a, b, c, d, are said to have property $\mathcal{PS}$ if all of $bc + cd + db, ac + cd + da, ab + bd + da$, and $ab + bc + ca$ are Perfect Squares.

Suppose that the positive integers $m, p, q, r$ satisfy $p \leq q \leq r$ and $pq + qr + rp = m^2$. Let $s = p + q + r + 2m$.

Prove that $p, q, r,$ and $s$ have property $\mathcal{PS}$.

On dit que quatre entiers positifs, $a, b, c, d,$ possèdent la propriété $\mathcal{CP}$ si tous les nombres $bc + cd + db, ac + cd + da, ab + bd + da,$ et $ab + bc + ca$ sont des Carrés $\mathcal{P}$arfaits.

Supposons que les entiers positifs $m, p, q, r$ satisfassent $p \leq q \leq r$ et $pq + qr + rp = m^2$. Soit $s = p + q + r + 2m$.

Montrer que $p, q, r, e s$ possèdent la propriété $\mathcal{CP}$.

2803. Proposed by I.C. Draghicescu, Bucharest, Romania.

Suppose that $x_1, x_2, \ldots, x_n$ ($n > 2$) are real numbers such that the sum of any $n - 1$ of them is greater than the remaining number. Let $s = \sum_{k=1}^{n} x_k$.

Prove that

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \geq \frac{s}{n - 2}.$$ 

Soit $x_1, x_2, \ldots, x_n$ ($n > 2$) des nombres réels tels que la somme de $n - 1$ d’entre eux est plus grande que le nombre restant. On pose $s = \sum_{k=1}^{n} x_k$.

Montrer que

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \geq \frac{s}{n - 2}.$$ 

2804. Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.

Given three non-concentric circles $\Gamma_j (M_j, R_j)$, let $\mu_j$ denote the power of a point $P$ with respect to $\Gamma_j$.

Determine the locus of $P$ if $2\mu_2 = \mu_1 + \mu_3$.

On donne trois cercles non concentriques $\Gamma_j (M_j, R_j)$ et soit $\mu_j$ la puissance d’un point $P$ par rapport à $\Gamma_j$.

Déterminer le lieu des points $P$ tels que $2\mu_2 = \mu_1 + \mu_3$. 
2805. Proposed by Mihály Benze, Brasov, Romania.

Let \( k \) be a fixed positive integer. For all positive integers \( n \), prove that there exist positive integers \( a_1, a_2, \ldots, a_n \) such that \((n, a_n) = 1\) and

\[
\sum_{j=1}^{n} \frac{j^k}{a_j} = 1. 
\]

2806. Proposed by Mihály Benze, Brasov, Romania.

Suppose that \( x, y, z > 0, \alpha \in \mathbb{R} \) and \( x^\alpha + y^\alpha + z^\alpha = 1 \). Prove that

(a) \( x^2 + y^2 + z^2 \geq x^{\alpha+2} + y^{\alpha+2} + z^{\alpha+2} + 2x^2y^2z^2(x^{\alpha-2} + y^{\alpha-2} + z^{\alpha-2}) \),

(b) \( \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq x^{\alpha-2} + y^{\alpha-2} + z^{\alpha-2} + \frac{2(x^{\alpha+1} + y^{\alpha+1} + z^{\alpha+1})}{xyz} \).

2807. Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.

In \( \triangle ABC \), denote its area by \([ABC]\) (and its semi-perimeter by \( s \)). Show that

\[
\min \left\{ \frac{2s^4 - (a^4 + b^4 + c^4)}{[ABC]^2} \right\} = 38.
\]

Soit \([ABC]\) l'aire d'un triangle \( ABC \), et \( s \) son demi-périmètre. Montrer que

\[
\min \left\{ \frac{2s^4 - (a^4 + b^4 + c^4)}{[ABC]^2} \right\} = 38.
\]
2808. Proposed by Aram Tangboondouangijit, student. University of Maryland, College Park, Maryland, USA.

In \( \triangle ABC \), we have \( b < c \) and \( a \left( 3b^2 + c^2 - a^2 \right) = 2b (c^2 - b^2) \).

Determine the ratio \( a : b : c \).

Dans le triangle \( ABC \), on a \( b < c \) et \( a \left( 3b^2 + c^2 - a^2 \right) = 2b (c^2 - b^2) \).

Démontrer les rapports \( a : b : c \).

2809. Proposed by Mihály Benze, Brasov, Romania.

Suppose that \( k \geq 2 \) is a fixed integer. For each non-negative integer \( n \), let \( x_n \) denote the left most digit of \( n^k \).

Prove that the number \( 0.x_0x_1x_2 \ldots x_n \ldots \) is irrational.

Soit \( k \geq 2 \) un entier donné. Pour tout entier non négatif \( n \), désignons par \( x_n \) le premier chiffre du nombre \( n^k \).

Montrer que le nombre \( 0, x_0x_1x_2 \ldots x_n \ldots \) est irrationnel.

2810. Proposed by I.C. Draghicescu, Bucharest, Romania.

Suppose that \( a, b \) and \( x_1, x_2, \ldots, x_n (n \geq 2) \) are positive real numbers.

Let \( s = \sum_{k=1}^{n} x_k \). Prove that

\[
\prod_{k=1}^{n} \left( a + \frac{b}{x_k} \right) \geq \left( a + \frac{nb}{s} \right)^n.
\]

Supposons que \( a, b \) et \( x_1, x_2, \ldots, x_n (n \geq 2) \) soient des nombres réels positifs et posons \( s = \sum_{k=1}^{n} x_k \). Montrer que

\[
\prod_{k=1}^{n} \left( a + \frac{b}{x_k} \right) \geq \left( a + \frac{nb}{s} \right)^n.
\]

2811. Proposed by Mihály Benze, Brasov, Romania.

Determine all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) which satisfy, for all real \( x \),

\[
f \left( x^3 + x \right) \leq x \leq f^3(x) + f(x).
\]

Déterminer toutes les fonctions \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfaisant, pour tous les \( x \) réels,

\[
f \left( x^3 + x \right) \leq x \leq f^3(x) + f(x).
\]
2812. Proposed by Mihály Benze, Brasov, Romania.
Determine all injective functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$(2a + b)f(ax + b) \geq a f^2 \left( \frac{1}{x} \right) + b f \left( \frac{1}{x} \right) + a$$

for all positive real $x$, where $a, b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ and $2a + b > 0$.

Déterminer toutes les fonctions injectives $f : \mathbb{R} \to \mathbb{R}$ satisfaisant

$$(2a + b)f(ax + b) \geq a f^2 \left( \frac{1}{x} \right) + b f \left( \frac{1}{x} \right) + a$$

pour tous les $x$ réels positifs, où $a, b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ et $2a + b > 0$.

2813. Proposed by Barry R. Monson, University of New Brunswick, Fredericton, NB and J. Chris Fisher, University of Regina, Regina, SK.

Suppose that $M$ is the mid-point of side $AB$ of the square $ABCD$. Let $P$ and $Q$ be the points of intersection of the line $MD$ with the circle, centre $M$, radius $MA (= MB)$, where $P$ is inside the square $ABCD$ and $Q$ is outside.

Prove that rectangle $APBQ$ is a golden rectangle; that is,

$$PB : PA = (\sqrt{5} + 1) : 2.$$