A Revival of an Old Construction Problem

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The purpose of this note is to revive an old triangle construction problem and to provide an apparently missing elementary justification for part of it.

In 1988, I. Sakmar proposed the following problem [1]:

Inverting a Transformation by Equilateral Triangles


(a) Show that any triangle $PQR$ which can be obtained in this way arises from a unique triangle $ABC$, and give a construction for recovering triangle $ABC$ from triangle $PQR$.

(b) Show that not every triangle $PQR$ can be so obtained.

The solution followed three years later [2]. The first part of the solution is really delightful in its simplicity and beauty. It goes as follows: Let $U$, $V$, $W$ be the new vertices of equilateral triangles constructed inwardly on the edges of the triangle $PQR$. If $P$, $Q$, $R$ are vertices of equilateral triangles constructed outwardly on the sides of a given triangle $ABC$, then $A$, $B$, $C$ are necessarily the mid-points of the sides of the triangle $UVW$ (Figure 1).

![Figure 1](image)

Later, from other sources [3, 4], the author has learned that this problem has had a long history. It is attributed to E. Lemoine who proposed it as early as 1868 in Nouvelles Annales de Mathématique (Question 864), and the first solution by L. Kiepert followed a year later (his solution was similar to that of Mauldon but proceeded by constructing outward equilateral triangles). The problem reappeared thereafter in various sources ([4] and a list of references cited therein). When justifying the construction,
various methods were used, from analytical to purely elementary. For instance, Kiepert exploits the properties of the Fermat point of the triangle. The most elegant argument, as pointed out by J.E. Wetzel [3], was given in 1956 by H.G. Steiner, who used rotations: If $X^\alpha$ designates the rotation about a point $X$ through an angle $\alpha$ (a positive angle is measured counterclockwise), then the sequence of rotations $Q^{60}P^{60}R^{60}$ is a half turn that fixes point $A$, and takes $W$ into $V$. Therefore, $A$ is a mid-point of $VW$. The same argument holds for vertices $B$ and $C$.

And what about the second part of the problem? In fact, it is the discussion of the first part, and without it no solution to a construction problem may be considered complete. In [2], the existence condition was stated and proved analytically. In a much earlier note [5] a geometric condition was stated, without proof (see Figure 2):

Consider any side of the given triangle $PQR$, say $QR$, and the corresponding outward equilateral triangle $QRD$. Extend the sides $DQ$ and $DR$ beyond $Q$ and $R$ to points $Q'$ and $R'$ so that $DQ = QQ'$ and $DR = RR'$. Let $\Gamma$ be the circle tangent to $DQ$ at $Q'$ and $DR$ at $R'$. The solution exists if vertex $P$ lies inside $\Gamma$.

J. Wetzel in his elaboration of this and similar constructions [3] gave the same condition and provided proof using complex coordinates. In this note, the purely elementary, geometric argument is presented.

We begin with the claim that when triangle $PQR$ is constructed from triangle $ABC$, then $PQR$ and $ABC$ have the same orientation. To prove this, assume that $C$ is the largest angle in $ABC$. Because angles $RAB$ and $RBA$ are $60^\circ$ (Figure 1), both angles $RCA$ and $RCB$ are less than $120^\circ$, so that $P$ and $Q$ are separated by the line $CR$. Also, because angles $CBP$ and $QAC$ are $60^\circ$, angles $ABP$ and $BAQ$ are less than $180^\circ$, so that pairs $P$, $R$ and $Q$, $R$ are separated by the line $AB$, and the claim follows. It then follows immediately that if after constructing $ABC$ as the mid-point triangle of $UVW$, its orientation is opposite to $PQR$, then the vertices $P$, 

Figure 2
$Q$, $R$ cannot be obtained in the prescribed way from any triangle. Thus, we need to show that $PQR$ has the same orientation as the mid-point triangle of $UVW$ exactly when $P$ lies inside the circle $\Gamma$.

No generality is lost in assuming triangle $PQR$ counterclockwise oriented. Construct equilateral triangles $P'R'V'$ and $Q'PW'$ oriented the same as $PQR$ (Figure 2). Since $\overrightarrow{DQ} = \overrightarrow{QQ'}$ and $\overrightarrow{DR} = \overrightarrow{RR'}$, each of $Q$, $Q'$, $U$ and $R$, $R'$, $U$ form equilateral triangles. Also, because chord $Q'R'$ subtends an arc of 120° on $\Gamma$, one sees that, for $P$ inside $\Gamma$, angle $Q'PR'$ varies from 60° to 240°. Then, angle $V'PW'$ is always between 0° and 180°, meaning the triangle $PV'W'$ is counterclockwise oriented. Similarly, when $P$ is outside $\Gamma$, the angle $V'PW'$ is between 180° and 360°, which corresponds to the clockwise oriented triangle $PV'W'$. Consider then the transformation $R^{-60}P^{-60}$; that is, the translation carrying point $U$ into $V$ and $V'$ into $P$. Thus, the segment $PV'$ is carried into segment $UV$ by a half turn. Analogously, the translation $Q^{-60}P^{60}$ carries $U$ into $W$ and $W'$ into $P$, so that the segment $PW'$ is carried into $UW$ by a half turn. Therefore, triangles $UVW$ and $PV'W'$ are interchanged by a half turn and hence have the same orientation. Of course, the mid-point triangle of $UVW$ will have the same orientation as $UVW$. This concludes the proof.

It is interesting to note that $P$ lying on the circle $\Gamma$ corresponds to angle $V'PW' = 0°$ or 180°; that is, points $A$, $B$, $C$ are collinear. Specifically, when $P$ lies on a small arc $Q'R'$, point $A$ lies outside segment $BC$; when $P$ lies on a big arc, $A$ lies on the segment $BC$.

References


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