MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!). The electronic address is mayhem-editors@cms.math.ca

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Mayhem Problems

Envoyez vos propositions et solutions à MATHEMATICAL MAYHEM, Faculté de mathématiques, Université de Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, ou par courriel à mayhem-editors@cms.math.ca

N’oubliez pas d’inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d’étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le 1er avril 2003. Les solutions reçues après cette date ne seront prises en compte que s’il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l’anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l’anglais.

Pour être admissibles au DÉFI MAYHEM de ce mois-ci, les solutions doivent avoir été postées avant le 1er février 2003, cachet de la poste faisant foi.

M57. Proposé par J. Walter Lynch, Athens, GA, USA.

Quatre points sont également espacés autour d’un cercle ayant un rayon r. Le cercle est ensuite divisé par 4 arcs égaux. Renversez les arcs en laissant le point du bout en place. Trouvez l’aire de la figure ainsi obtenue.

Four points are equally spaced around a circle with radius r. This divides the circle into 4 equal arcs. Flip over each arc, leaving the endpoints in place. Find the area enclosed by the figure thus obtained.
M58. Proposé par l’équipe de Mayhem.
Trouvez tous les entiers positifs $x$ et $y$ qui satisfont l’équation $x^y = y^x$.

Find all positive integers $x$ and $y$ which satisfy the equation $x^y = y^x$.

M59. Proposé par Izidor Hafner, Tržaška 25, Ljubljana, Slovenia.
Le diagramme ci-dessous représente le développement d’un polyèdre sur un plan. Les faces du solide sont divisées en polygones plus petits. Le problème consiste à colorer les polygones (ou à les numéroter) de telle sorte que chaque face du solide original soit d’une couleur différente.

The diagram above represents the net of a polyhedron in which the faces of the solid are divided into smaller polygons. The task is to colour the polygons (or number them), so that each face of the original solid is a different colour.

M60. Proposé par Mihăi Bencze, Brăsov, Romania

Déterminez tous les entiers positifs dont $\left\lfloor \sum_{k=1}^{n} \sqrt{k} \right\rfloor = n$, et que $[x]$ est le plus grand entier plus petit ou égal à $x$.

Determine all positive integers for which $\left\lfloor \sum_{k=1}^{n} \sqrt{k} \right\rfloor = n$, where $[x]$ is the greatest integer less than or equal to $x$. 
M61. Proposé par l'équipe de Mayhem.
On vous donne 54 poids qui pèsent $1^2$, $2^2$, $3^2$, $\ldots$, $54^2$. Regroupez ceux-ci en trois groupes de poids égales.

You are given 54 weights which weigh $1^2$, $2^2$, $3^2$, $\ldots$, $54^2$. Group these into three sets of equal weight.

Disons que $ABCD$ est un trapèze dont les côtés $AB$ et $CD$ sont parallèles et que les diagonales $AC$ et $BC$ se croisent au point $P$. Supposons que $AB = 50$, $CD = 160$, et l'aire du triangle $PAD$ est 2000. Trouvez l'aire du trapèze.

Let $ABCD$ be a trapezoid where sides $AB$ and $CD$ are parallel and the diagonals $AC$ and $BC$ intersect at point $P$. Suppose $AB = 50$, $CD = 160$, and the area of triangle $PAD$ is 2000. Determine the area of the trapezoid.

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Mayhem Problem Solutions

M7. *From the Australian Mathematics Trust problems set.*

A circle of radius 6 has an isosceles triangle $PQR$ inscribed in it, where $PQ = PR$. A second circle touches the first circle and the mid-point of the base $QR$ of the triangle as shown. The side $PQ$ has length $4\sqrt{5}$. Find the radius of the smaller circle.

\[ PQ^2 - QX^2 = (12 - 2r)^2 \\
80 - QX^2 = 144 - 48r + 4r^2 \\
-QX^2 = 64 - 48r + 4r^2. \quad (1) \]
Similarly,

\[
OQ^2 - QX^2 = (6 - 2r)^2 \\
36 - QX^2 = 36 - 24r + 4r^2.
\]

Substituting (1):

\[
36 + 64 - 48r + 4r^2 = 36 - 24r + 4r^2 \\
64 = 24r.
\]

Therefore, the radius of the smaller circle is \( \frac{8}{3} \).

**M8. Proposed by the Mayhem staff.**

Find all right-angled triangles with integer sides if one of the sides is 2001 units long.

*Solution by Paul Jefferys, Berkhamsted Collegiate School, UK.*

We consider the formula for Pythagorean triplets: \((x^2 - y^2)k, (2xy)k, \text{ and } (x^2 + y^2)k\), where \(x, y, \text{ and } k\) are positive integers and \(x > y\).

Now the side length given by \((2xy)k\) is even, so that we need only consider the cases in which \((x^2 - y^2)k = 2001\) and \((x^2 + y^2)k = 2001\).

**Case 1:** \((x^2 - y^2)k = 2001\). Then \(k\) is either 1, 3, 23, 29, 69, 87, 667, or 2001, since \(k\) divides 2001.

If \(k = 2001\), then \(x^2 - y^2 = 1\), so that \((x - y)(x + y) = 1\) forcing \(y = 0\), a contradiction since \(y\) must be a positive integer.

If \(k = 667\), then \(x^2 - y^2 = 3\) and we must have \(x + y = 3\) and \(x - y = 1\), giving the triplet (2001, 2668, 3335).

If \(k = 23\), then \(x^2 - y^2 = 23\) and we must have \(x + y = 23\) and \(x - y = 1\), giving the triplet (2001, 22968, 23055).

If \(k = 29\), then \(x^2 - y^2 = 69 = 3 \times 23\). Then either \(x + y = 69\) and \(x - y = 1\), giving \(x = 35\) and \(y = 34\), or \(x + y = 23\) and \(x - y = 3\), giving \(x = 13\) and \(y = 10\). This gives the pair of triplets (2001, 69020, 69049) and (2001, 7540, 7801).

If \(k = 1\), then \(x^2 - y^2 = 2001 = 3 \times 23 \times 29\). Then there are four ways to factor 2001 as the product of two factors: \(2001 = 2001 \times 1 = 667 \times 3 = 87 \times 23 = 69 \times 29\). These four factorizations yield the four triplets (2001, 2002000, 2002001), (2001, 222440, 222449), (2001, 3520, 4049), and (2001, 1960, 2801).
Case 2: \((x^2 + y^2)k = 2001\). Again, \(k\) is either 1, 3, 23, 29, 69, 87, 667, or 2001. But \(x^2 + y^2\) must leave a remainder of 1 upon division by 4, and 2001 also leaves a remainder of 1 upon division by 4, so that \(k\) itself must leave a remainder of 1 upon division by 4. Thus, the possibilities for \(k\) are 2001, 69, 29, and 1.

If \(k = 2001\), then \(x^2 + y^2 = 1\), but this is impossible when \(x\) and \(y\) are positive integers.

If \(k = 69\), then \(x^2 + y^2 = 29\), and checking gives the only solution \(x = 5\) and \(y = 2\), giving the triplet \((1449, 1380, 2001)\).

If \(k = 29\), then \(x^2 + y^2 = 69\), and checking shows there are no integer solutions.

If \(k = 1\), then \(x^2 + y^2 = 2001\), and checking shows there are no integer solutions.

**M9** Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.

Find integers \(a, b,\) and \(c\) (not all equal) with \(a + b + c = 2001\), such that \(a, b,\) and \(c\) form an arithmetic sequence (in that order) and \(a + b, b + c,\) and \(c + a\) form a geometric sequence (in that order).

Solutions by Mihály Benze, Brasov, Romania and Paul Jefferys, Berkhamsted Collegiate School, UK. One incorrect solution was received. We give the solution of Benze.

Since \(a, b,\) and \(c\) form an arithmetic series, let \(a = b - r\) and \(c = b + r\) for some integer \(r\).

Then

\[
\begin{align*}
a + b + c &= 2001 \\
b - r + b + b + r &= 2001 \\
3b &= 2001 \\
b &= 667.
\end{align*}
\]

Then, since \(a + b, b + c,\) and \(c + a\) form a geometric sequence we have

\[
\begin{align*}
(a + b)(c + a) &= (b + c)^2 \\
(2b - r)(2b) &= (2b + r)^2 \\
4b^2 - 2br &= 4b^2 + 4br + r^2 \\
6br + r^2 &= 0 \\
r(6b + r) &= 0.
\end{align*}
\]

Either \(r = 0\) (inadmissible since \(a, b,\) and \(c\) are not all equal), or \(r = -6b = -4002\).

Then we get \(a = 4669, b = 667,\) and \(c = -3335\).
(a) Factor fully $2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$.
(b) Find the geometric interpretation of the above expression if $a$, $b$, and $c$ are sides of a non-degenerate triangle.
1. Solution to (a) by Mihály Bencze, Brasov, Romania.

\[
2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4
= 4a^2b^2 - 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4
= 4a^2b^2 - (2a^2b^2 - 2b^2c^2 - 2c^2a^2 + a^4 + b^4 + c^4)
= (2ab - (a^2 + b^2 - c^2))(2ab + (a^2 + b^2 - c^2))
= (c^2 - (a^2 - 2ab + b^2))(a^2 + 2ab + b^2) - c^2)
= (c - (a - b))(c + (a - b))(a + b) - c)((a + b) + c)
= (a + b - c)(b + c - a)(c + a - b)(a + b + c).
\]

II. Solution to (b) by proposer.
If $a$, $b$, $c$ are the sides of a scalene triangle, then if we let $a + b + c = 2s$, we get

\[
(b + c - a) = 2(s - a)
(c + a - b) = 2(s - b)
(a + b - c) = 2(s - c),
\]

so that $(a + b - c)(b + c - a)(c + a - b)(a + b + c) = 16(s - a)(s - b)(s - c)$. But from Heron’s formula, we know $A = \sqrt{s(s - a)(s - b)(s - c)}$. Thus, $2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = 16A^2$ where $a$, $b$, $c$ are the sides of a scalene triangle with area $A$.

[Ed. This is still valid without the restriction "scalene".]

M11. Proposed by the Mayhem staff.
Two sequences $a_1, a_2, \ldots, a_{2001}$ and $b_1, b_2, \ldots, b_{2001}$ are formed by the following rules:

- $a_1 = 5$ and $a_2 = 3$,
- $b_1 = 9$ and $b_2 = 7$,
- $\frac{a_n}{b_n} = \frac{a_{n-1} + a_{n-2}}{b_{n-1} + b_{n-2}}$ for $n > 2$ such that each $\frac{a_n}{b_n}$ is in lowest terms.

What is the smallest fraction of the form $\frac{a_n}{b_n}$?
Solutions by Mihály Bencze, Brasov, Romania, and Geneviève Lalonde, Massey, ON. One incorrect solution was received. We give the solution of Lalonde.

First note that if \( \frac{a}{b} < \frac{c}{d} \), where both fractions are in lowest terms (and \( a, b, c, d \) are positive integers), then \( \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \).

**Proof:** Since \( \frac{a}{b} < \frac{c}{d} \) then \( ad < cb \), giving

\[
\frac{a + c}{b + d} = \frac{b(a + c)}{b(b + d)} = \frac{ab + cb}{b(b + d)} > \frac{ab + ad}{b(b + d)} = \frac{a(b + d)}{b(b + d)} = \frac{a}{b}.
\]

We can similarly prove the other inequality. Thus, if we let \( c_n = \frac{a_n}{b_n} \), we see that every time we create a new term, it is in between the two previous terms. Therefore, we get

\[ c_2 < c_4 < c_6 < c_8 < \cdots < c_{2n} < \cdots < c_{2n-1} < c_7 < c_5 < c_3 < c_1, \]

so that the smallest fraction is \( c_2 = \frac{3}{7} \).

**M12.** Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.

Determine all ordered pairs \((x, y)\) with \( \gcd(x, y) = 1 \), and \( x < y \) such that \( 2000 \left( \frac{x}{y} + \frac{y}{x} \right) \) is an odd integer.

**Solution by Paul Jefferys, Berkhamsted Collegiate School, UK.**

\[
2000 \left( \frac{x}{y} + \frac{y}{x} \right) = 2000 \left( \frac{x^2 + y^2}{xy} \right) .
\]

Since \( \gcd(x, y) = 1 \), \( xy \) shares no common factors with \( x^2 + y^2 \). Thus, if \( 2000 \left( \frac{x^2 + y^2}{xy} \right) \) is to be an integer, \( xy \) must divide 2000. Further, 16 must divide \( xy \), since the expression is to be odd. Then, since \( \gcd(x, y) = 1 \), we have either 16 divides \( x \) or 16 divides \( y \).

**Case 1:** 16 divides \( x \).

If \( x > 16 \), then since \( x \) divides 2000, \( x \) is at least \( 16 \times 5 = 80 \), in which case \( x \) would be greater than \( y \) since \( xy \) must divide 2000. Thus, \( x = 16 \). Then \( y > x \) and \( y \) divides 2000. But \( y \) is also odd, so that the possibilities for \( y \) are 25 and 125.

**Case 2:** 16 divides \( y \).

If 5 divides \( x \), then \( y \) divides 2000 but 5 does not divide \( y \). Thus, \( y = 16 \) and \( x = 5 \) since \( x < y \).
If 5 divides \( y \), then \( x = 1 \) since \( x \) divides 2000 but neither 2 nor 5 divide \( x \). Then the possibilities for \( y \) are 80, 400, and 2000.

If 5 divides neither \( x \) nor \( y \), then \( x = 1 \) and \( y = 16 \).

Therefore, in the end, we have the pairs (16, 25), (16, 125), (5, 16), (1, 16), (1, 80), (1, 400), and (1, 2000).

**Advanced Solutions**

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**A264. Proposed by Mohammed Aassila, Strasbourg, France.**

Prove that for any integer \( a \) and natural number \( m \),

\[
a^m \equiv a^{m-\phi(m)} \pmod{m}.
\]

(This is a generalization of Euler’s theorem.)

**Solution by Kenneth Williams, Carleton University, Ottawa.**

For \( \mathbb{N} = \{1, 2, 3, \ldots, \} \) and \( a, m \in \mathbb{N} \), we are required to solve

\[
a^m \equiv a^{m-\phi(m)} \pmod{m}.
\]

If \( a = 1 \) or \( m = 1 \), the result is trivial. We may suppose \( a > 1 \) and \( m > 1 \).

Let

\[
m_1 = \prod_{\substack{p \mid m \\mid a}} p^{\nu_p(m)}, \quad m_2 = \prod_{\substack{p \mid m \\mid a}} p^{\nu_p(m)},
\]

where \( p^{\nu_p(m)} \mid m \), and \( p \) is prime.

Clearly, \( m_1 \in \mathbb{N}, m_2 \in \mathbb{N}, m_1m_2 = m, (m_1, m_2) = 1 \), and \( (m_2, a) = 1 \).

Since \( (m_2, a) = 1 \), we have \( a^{\phi(m_2)} \equiv 1 \pmod{m_2} \).

Since \( (m_1, m_2) = 1 \), we have \( \phi(m) = \phi(m_1m_2) = \phi(m_1)\phi(m_2) \), so that

\[
a^{\phi(m_1m_2)} = a^{\phi(m_1)\phi(m_2)} = \left(a^{\phi(m_2)}\right)^{\phi(m_1)} \equiv 1 \pmod{m_2}.
\]

Hence, \( m_2 \mid a^{\phi(m)} - 1 \).

Next, let \( p \) be a prime dividing \( m_1 \), so that

\[
\nu_p(m) \geq 1, \quad \nu_p(a) \geq 1, \quad p^{\nu_p(m)} \mid m_1.
\]
Then,

\[ \nu_p(m) \leq 2^{\nu_p(m)-1} \leq p^{\nu_p(m)-1} \]
\[ \leq (p - 1)p^{\nu_p(m)-1} \]
\[ \leq (p - 1)p^{\nu_p(m)-1}\phi\left(\frac{m}{p^{\nu_p(m)}}\right) \]
\[ = p^{\nu_p(m)}\phi\left(\frac{m}{p^{\nu_p(m)}}\right) - p^{\nu_p(m)-1}\phi\left(\frac{m}{p^{\nu_p(m)}}\right) \]
\[ \leq p^{\nu_p(m)}\left(\frac{m}{p^{\nu_p(m)}}\right) - \phi\left(p^{\nu_p(m)}\left(\frac{m}{p^{\nu_p(m)}}\right)\right) \]
\[ = m - \phi(m) \leq \nu_p(a)(m - \phi(m)) \]

so that

\[ m_1 = \prod_{p \mid m} p^{\nu_p(m)} \left| \prod_{p \mid m} p^{(m - \phi(m))\nu_p(a)} \right| \prod_{p \mid a} p^{(m - \phi(m))\nu_p(a)} \]

\[ m_1 \mid a^{m - \phi(m)} \]

Hence,

\[ m_1 m_2 \mid a^{m - \phi(m)} \left(a^{\phi(m)} - 1\right) \]

so that

\[ m \mid a^{m} - a^{m - \phi(m)} \]

Also solved by MICHIEL BATAILLE, Rouen, France.