SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


The sides \(a, b\) and \(c\) of a non-degenerate triangle \(ABC\) satisfy the relations \(b^2 = ca + a^2\) and \(c^2 = ab + b^2\). Find the measures of the angles of triangle \(ABC\).

[Editor's note: A solution to this problem by Václav Konečný appeared in CRUX [2001 : 416]. The solution relied heavily on trigonometry. The following solution was submitted later and is featured here since it is quite elegant and does not use trigonometry.]

Solution by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

In [1] the following problem and solution appears (loosely translated from the Spanish by the editors):

349. Show that if among the sides \(a, b,\) and \(c\) of a triangle the relation \(a^2 = b^2 + bc\) holds, then the angles \(A\) and \(B\) opposite the sides \(a, b\) respectively, satisfy the equation \(\angle A = 2\angle B\).

Proof: Let \(\triangle ABC\) be given (as in the diagram above). Let \(D\) be on the side \(AC\) produced in the direction of \(A\) such that \(AD = c\). From the equation \(a^2 = b^2 + bc\), it follows that

\[
\frac{a}{b} = \frac{b + c}{a}.
\]

This implies that \(\triangle CAB\) and \(\triangle CBD\) are similar and \(\angle A = \angle CBD\). Furthermore, \(\angle B = \angle BDA = \angle DBA\).

Consequently, \(\angle A = \angle B + \angle DBA = 2\angle B\).
A solution to #2568 now proceeds as follows:
\[
\begin{align*}
    b^2 &= ca + a^2 \quad \Rightarrow \quad \angle B = 2\angle A \\
    c^2 &= ab + b^2 \quad \Rightarrow \quad \angle C = 2\angle B = 4\angle A .
\end{align*}
\]
Since \( \pi - \angle A = \angle B + \angle C = 6\angle A \), we conclude that
\[
\angle A = \frac{\pi}{7} , \quad \angle B = \frac{2\pi}{7} , \quad \angle C = \frac{4\pi}{7} .
\]


Suppose that \( a, b \) and \( c \) are positive real numbers. Prove that
\[
\frac{2 \left( a^3 + b^3 + c^3 \right)}{abc} + \frac{9(a + b + c)^2}{a^2 + b^2 + c^2} \geq 33 .
\]

Editor's comment.
Šefket Arslanagić was the first to note the (obvious) error in solution II [2002 : 279]. The inequality \( \frac{1}{3} (a + b + c)^2 \geq a^2 + b^2 + c^2 \) is incorrect, which makes the entire solution II incorrect. The correct inequality is \( \frac{1}{3} (a + b + c)^2 \leq a^2 + b^2 + c^2 \) (AM-QM). The editors apologize for the error and promise to be constantly on high alert.

2656★. [2001 : 336] Proposed by Vedula N. Murty, Dover, PA, USA.

For positive real numbers \( a, b \) and \( c \), show that
\[
\frac{(1 - b)(1 - bc)}{b(1 + a)} + \frac{(1 - c)(1 - ca)}{c(1 + b)} + \frac{(1 - a)(1 - ab)}{a(1 + c)} \geq 0 .
\]

Editor's comment.

It turns out that this inequality is false as stated. Several solvers sent us their counterexample. Perhaps readers might try restricting the reals \( a, b \) and \( c \) to the interval \([0, 1]\).


Prove that
\[
\sum_{n=0}^{2k-1} \tan \left( \frac{(4n - 1)\pi + (-1)^n 4\theta}{8k} \right) = \frac{2k}{1 + (-1)^{k+1}\sqrt{2} \sin \theta} .
\]
Solution by Stanley Rabinowitz, Westford, MA, USA.

We start with the formulas for $\sin(nA)$ and $\cos(nA)$ which can be found in any book on advanced trigonometry, such as [1], page 52:

\[
\cos(nA) = (\cos^n A) \left[ \binom{n}{0} t - \binom{n}{2} t^2 + \binom{n}{4} t^4 - \cdots \right],
\]
\[
\sin(nA) = (\cos^n A) \left[ \binom{n}{1} t - \binom{n}{3} t^3 + \binom{n}{5} t^5 - \cdots \right],
\]

where $t = \tan A$. [Ed: These are immediate consequences of the binomial theorem and De Moivre’s formula.] Dividing, we get

\[
\cot(nA) = \frac{\binom{n}{0} t - \binom{n}{2} t^2 + \binom{n}{4} t^4 - \cdots}{\binom{n}{1} t - \binom{n}{3} t^3 + \binom{n}{5} t^5 - \cdots} = \frac{x^n - \binom{n}{2} x^{n-2} + \binom{n}{4} x^{n-4} - \cdots}{nx^{n-1} - \binom{n}{3} x^{n-3} + \binom{n}{5} x^{n-5} - \cdots},
\]

where $x = \frac{1}{t} = \cot A$. [Ed: Assuming $\sin A \neq 0$ and $\cos A \neq 0$.] Note that the equation

\[
x^n - \binom{n}{2} x^{n-2} + \binom{n}{4} x^{n-4} - \cdots = \left( \cot(nA) \right) \left[ nx^{n-1} - \binom{n}{3} x^{n-3} + \binom{n}{5} x^{n-5} - \cdots \right], \tag{1}
\]

is an $n^{th}$ degree polynomial equation in $x$. Since

\[
\cot \left( n \left( A + \frac{j\pi}{n} \right) \right) = \cot (nA + j\pi) = \cot(nA)
\]

for $j = 0, 1, 2, \ldots, n - 1$, and since $\cot(A + \frac{j\pi}{n})$ are all distinct, they are exactly the $n$ roots of the equation in (1). Hence, we have

\[
\sum_{j=0}^{n-1} \cot \left( A + \frac{j\pi}{n} \right) = n \cot(nA). \tag{2}
\]

Let $S$ denote the summation on the left side of the given identity. Setting $n = 2r$ for the even-indexed terms, $n = 2r + 1$ for the odd-indexed terms, and separating these terms, we have

\[
S = \sum_{r=0}^{k-1} \tan \left( \frac{(8r-1)\pi + 4\theta}{8k} \right) + \sum_{r=0}^{k-1} \tan \left( \frac{(8r+3)\pi - 4\theta}{8k} \right)
\]
\[
= \sum_{r=0}^{k-1} \tan \left( \frac{r\pi}{k} + \frac{4\theta - \pi}{8k} \right) + \sum_{r=0}^{k-1} \tan \left( \frac{r\pi}{k} + \frac{3\pi - 4\theta}{8k} \right).
\]
Since \( \tan x = -\cot(x + \frac{\pi}{2}) \), we have by (2) that

\[
S = - \sum_{r=0}^{k-1} \cot \left( \frac{r\pi}{k} + \frac{4\theta - \pi}{8k} + \frac{\pi}{2} \right) - \sum_{r=0}^{k-1} \cot \left( \frac{r\pi}{k} + \frac{3\pi - 4\theta}{8k} + \frac{\pi}{2} \right)
\]

\[
= -k \cot \left( k \left( \frac{4\theta - \pi}{8k} + \frac{\pi}{2} \right) \right) - k \cot \left( k \left( \frac{3\pi - 4\theta}{8k} + \frac{\pi}{2} \right) \right)
\]

\[
= -k \left( \cot \left( \frac{4\theta - \pi}{8} + \frac{k\pi}{2} \right) + \cot \left( \frac{3\pi - 4\theta}{8} + \frac{k\pi}{2} \right) \right).
\]

Using the half-angle formula \( \cot \left( \frac{\pi}{2} \right) = \frac{1 + \cos \frac{\pi}{x}}{\sin \frac{\pi}{x}} \), we then have

\[
S = -k \left( \frac{1 + \cos \left( \theta - \frac{\pi}{4} + k\pi \right)}{\sin \left( \theta - \frac{\pi}{4} + k\pi \right)} + \frac{1 + \cos \left( \frac{3\pi}{4} - \theta + k\pi \right)}{\sin \left( \frac{3\pi}{4} - \theta + k\pi \right)} \right)
\]

\[
= -k \left( \frac{1 + (-1)^k \cos \left( \theta - \frac{\pi}{4} \right)}{(-1)^k \sin \left( \theta - \frac{\pi}{4} \right)} + \frac{1 + (-1)^k \cos \left( \frac{3\pi}{4} - \theta \right)}{(-1)^k \sin \left( \frac{3\pi}{4} - \theta \right)} \right)
\]

\[
= -k \left( \frac{(-1)^k + \cos \left( \theta - \frac{\pi}{4} \right)}{\sin \left( \theta - \frac{\pi}{4} \right)} + \frac{(-1)^k + \sin \left( \theta - \frac{\pi}{4} \right)}{\cos \left( \theta - \frac{\pi}{4} \right)} \right)
\]

\[
= -k \left( \frac{(-1)^k \left[ \sin \left( \theta - \frac{\pi}{4} \right) + \cos \left( \theta - \frac{\pi}{4} \right) \right] + 1}{\sin \left( \theta - \frac{\pi}{4} \right) \cos \left( \theta - \frac{\pi}{4} \right)} \right). \tag{3}
\]

Since

\[
\sin \left( \theta - \frac{\pi}{4} \right) \cos \left( \theta - \frac{\pi}{4} \right) = \frac{1}{2} \sin \left( 2\theta - \frac{\pi}{2} \right) = -\frac{1}{2} \cos (2\theta)
\]

and

\[
\sin \left( \theta - \frac{\pi}{4} \right) + \cos \left( \theta - \frac{\pi}{4} \right)
\]

\[
= \sqrt{2} \left( \sin \left( \theta - \frac{\pi}{4} \right) \cos \frac{\pi}{4} + \cos \left( \theta - \frac{\pi}{4} \right) \sin \frac{\pi}{4} \right)
\]

\[
= \sqrt{2} \sin \theta,
\]

we get from (3) that

\[
S = 2k \left( \frac{(-1)^k \sqrt{2} \sin \theta + 1}{\cos 2\theta} \right). \tag{4}
\]

But

\[
\cos 2\theta = 1 - 2 \sin^2 \theta = 1 - (\sqrt{2} \sin \theta)^2
\]

\[
= \left( 1 - (-1)^k \sqrt{2} \sin \theta \right) \left( 1 + (-1)^k \sqrt{2} \sin \theta \right)
\]
so that we finally obtain from (4) that

\[ S = \frac{2k}{1 - (-1)^k \sqrt{2} \sin \theta} = \frac{2k}{1 + (-1)^{k+1} \sqrt{2} \sin \theta}. \]

Reference


Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Both Bataille and Woo used complex numbers in their solutions which are shorter than, but not as elementary as the one published above. Liu was the only one who pointed out explicitly that the proposed identity makes sense only if \( \theta \neq \frac{(8m-1)\pi}{4} \), or \( \frac{(8m-3)\pi}{4} \) when \( k \) is odd, and \( \theta \neq \frac{(8m+1)\pi}{4} \), or \( \frac{(8m+3)\pi}{4} \) when \( k \) is even, where \( m \in \mathbb{Z} \).


In \( \triangle ABC \), the side \( BC \) is fixed. \( A \) is a variable point. Assume that \( AC > AB \). Let \( M \) be the midpoint of \( BC \), let \( O \) be the circumcentre of \( \triangle ABC \), let \( R \) be the circumradius, let \( G \) be the centroid and \( H \) the orthocentre. Assume that the Euler line, \( OH \), is perpendicular to \( AM \).

1. Determine the locus of \( A \).
2. Determine the range of \( \angle BGC \).

Solution by Toshio Seimiya. Kawasaki, Japan.

1. Since \( G \) is the intersection of \( AM \) and \( OH \), we have \( \angle AGO = 90^\circ \). Since \( OA = OC \) and \( OM \perp MC \), we also have

\[ AG^2 - GM^2 = AO^2 - OM^2 = OC^2 - OM^2 = MC^2. \]

Since \( AG = 2GM \) it follows that \( 3GM^2 = MC^2 \). Thus, \( \sqrt{3} GM = MC \). Therefore, \( AM = 3GM = \sqrt{3} MC \). Let \( \triangle PBC \) and \( \triangle QBC \) be equilateral (with \( P \) and \( Q \) on opposite sides of \( BC \)). Then \( MP = MQ = \sqrt{3} MC \), and \( MP \perp BC \), \( MQ \perp BC \), whence \( P \), \( M \), \( Q \) are collinear. Thus, we have \( MA = MP = MQ \), which means that \( A \) is a point on the circle with diameter \( PQ \). Since \( AB < AC \), \( A \) is a point on the same side as \( B \) with respect to the line \( PQ \). Therefore, the locus of \( A \) is a semi-circle with diameter \( PQ \), which lies on the same side of \( PQ \) as \( B \), excluding the intersection with \( BC \) (say \( R \), as in figure 1 below).
2. We assume that $A$, $P$ lie on the same side of $BC$. Let $S$ be the centroid of $\triangle PBC$. Then $\angle BSC = 120^\circ$ and $MS = \frac{1}{3}MC = MG$. Let $T$ be the circumcenter of $\triangle SBC$. (See figure 2 above.) Then $S$, $M$, $T$ are collinear and
\[
GT < MG + MT = MS + MT = TS.
\]
Thus, $G$ is an interior point of segment $BSC$, whence
\[
\angle BGC > \angle BSC = 120^\circ.
\]
Therefore, $120^\circ < \angle BGC < 180^\circ$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOQ, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Almost all solutions were variations of the above. Some solvers chose not to eliminate the degenerate triangle which occurs when $A = R$; others ignored the condition $AC > AB$.

2661. [2001 : 337] Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Let $H$ be the orthocentre of acute-angled $\triangle ABC$ in which $\tan \left( \frac{A}{2} \right) = \frac{1}{2}$. Show that the sum of the radii of the incircles of $\triangle AHB$ and $\triangle AHC$ is equal to the inradius of $\triangle ABC$.

Is the converse true?
Solution by Michel Bataille, Rouen, France.

Let \( r, r', \) and \( r'' \) denote the inradii of \( \triangle ABC, \triangle AHB, \) and \( \triangle AHC, \) respectively. Recall the following formulas (in which \( R \) is the circumradius and \( s \) the semiperimeter of \( \triangle ABC \)):

\[
r = (s - a) \tan \left( \frac{A}{2} \right), \quad \text{and} \quad (1)
\]

\[
r = 4R \sin \left( \frac{A}{2} \right) \sin \left( \frac{B}{2} \right) \sin \left( \frac{C}{2} \right). \quad \text{(2)}
\]

Moreover, \( \triangle AHB \) has circumradius \( R \) (equal to the circumradius of the original triangle) and angles \( 90^\circ - B, \) \( 180^\circ - C, \) \( 90^\circ - A \) (in the order \( A, \) \( H, \) \( B \)). Thus (by the Sine Law), \( AH = 2R \cos A, BH = 2R \cos B, \) and \( \angle AHB = 180^\circ - C. \) It follows (using (1) and \( c = 2R \sin C \)) that

\[
r' = \left( \frac{AH + BH + c}{2} - c \right) \tan (90^\circ - C/2) = \frac{R(\cos A + \cos B - \sin C)}{\tan \left( \frac{C}{2} \right)}.
\]

Similarly,

\[
r'' = \frac{R(\cos A + \cos C - \sin B)}{\tan \left( \frac{B}{2} \right)}.
\]

With the help of the appropriate trigonometric formulas we calculate

\[
r' + r'' = R \left( \frac{2 \sin \left( \frac{C}{2} \right) \cos \left( \frac{A - B}{2} \right) - 2 \sin \left( \frac{C}{2} \right) \cos \left( \frac{C}{2} \right)}{\tan \left( \frac{C}{2} \right)} \right.
\]

\[
+ \left. \frac{2 \sin \left( \frac{B}{2} \right) \cos \left( \frac{A - C}{2} \right) - 2 \sin \left( \frac{B}{2} \right) \cos \left( \frac{B}{2} \right)}{\tan \left( \frac{B}{2} \right)} \right) \right)
\]

\[
= R \left( 2 \cos \left( \frac{C}{2} \right) \cos \left( \frac{A - B}{2} \right) - \cos \left( \frac{C}{2} \right) \right)
\]

\[
+ 2 \cos \left( \frac{B}{2} \right) \cos \left( \frac{A - C}{2} \right) - \cos \left( \frac{B}{2} \right) \right) \right)
\]

\[
= R \left[ 4 \cos \left( \frac{C}{2} \right) \sin \left( 45^\circ - \frac{A}{2} \right) \sin \left( 45^\circ - \frac{B}{2} \right)
\]

\[
+ 4 \cos \left( \frac{B}{2} \right) \sin \left( 45^\circ - \frac{A}{2} \right) \sin \left( 45^\circ - \frac{C}{2} \right) \right]
\]

\[
= 4R \sqrt{2} \left[ \cos \left( \frac{A}{2} \right) - \sin \left( \frac{A}{2} \right) \right] \frac{\sqrt{2}}{2} \left[ 2 \cos \left( \frac{B}{2} \right) \cos \left( \frac{C}{2} \right) - \sin \left( \frac{B + C}{2} \right) \right]
\]

\[
= 2R \left[ \cos \left( \frac{A}{2} \right) - \sin \left( \frac{A}{2} \right) \right] \left[ \cos \left( \frac{B - C}{2} \right) + \sin \left( \frac{A}{2} \right) \right] - \cos \left( \frac{A}{2} \right) \right)
\]

Now let \( t = \tan \left( \frac{A}{2} \right). \) Using (2) we easily obtain

\[
\frac{r'}{r''} = \frac{(1 - t) \cos \left( \frac{B - C}{2} \right) + (t - 1) \cos \left( \frac{A}{2} \right)}{t \cos \left( \frac{B - C}{2} \right) - t \cos \left( \frac{A}{2} \right)}.
\]
Therefore,
\[
\frac{r' + r''}{r} = 1 \quad \text{if and only if} \quad (2t - 1) \left[ \cos \left( \frac{B - C}{2} \right) - \cos \left( \frac{A}{2} \right) \right] = 0.
\]
Note that \( \cos \left( \frac{B - C}{2} \right) = \cos \left( \frac{A}{2} \right) \) holds for \( \triangle ABC \) if and only if \( B \) or \( C \) is a right angle. Consequently, we conclude that
\[
r' + r'' = r \quad \text{if and only if} \quad B \text{ or } C \text{ is a right angle or } \tan \left( \frac{A}{2} \right) = 1/2.
\]
Since our problem asks what happens when all angles of \( \triangle ABC \) are acute, the answer is yes, the converse is true: for an acute triangle, \( r' + r'' = r \) if and only if \( \tan \left( \frac{A}{2} \right) = 1/2 \).

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALther Janous, Ursulengymnasium, Innsbruck, Austria; D. Kipp Johnson, Beaverton, OR, USA; Gergy Leversha, St. Paul’s School, London, England; Henry Liu, student, University of Memphis, Memphis, TN, USA; David Loeffler, student, Trinity College, Cambridge, UK; Toshio Seimiya, Kawasaki, Japan; D. J. SmEnk, Zaltbommel, the Netherlands; Peter Y. WOO, Biola University, La Mirada, CA, USA, and the proposer.

2662. [2001 : 337] Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.
Suppose that \( \triangle ABC \) is acute-angled, has inradius \( r \) and has area \( \Delta \).
Prove that
\[
\left( \sqrt{\cot A + \cot B + \cot C} \right)^2 \leq \frac{\Delta}{r^2}.
\]
Solution by David Loeffler, student, Trinity College, Cambridge, UK.
Note that
\[
\cot A = \frac{\cos A}{\sin A} = \frac{b^2 + c^2 - a^2}{2bc \sin A} = \frac{R}{abc} (b^2 + c^2 - a^2).
\]
We must show that
\[
\frac{R}{abc} \left( \sqrt{b^2 + c^2 - a^2} + \sqrt{c^2 + a^2 - b^2} + \sqrt{a^2 + b^2 - c^2} \right)^2 \leq \frac{\Delta}{r^2},
\]
or,
\[
\left( \sqrt{b^2 + c^2 - a^2} + \sqrt{c^2 + a^2 - b^2} + \sqrt{a^2 + b^2 - c^2} \right)^2 \leq \frac{\Delta abc}{r^2 R} = (2s)^2,
\]
or,
\[
\sqrt{b^2 + c^2 - a^2} + \sqrt{c^2 + a^2 - b^2} + \sqrt{a^2 + b^2 - c^2} \leq a + b + c.
\]
The last inequality is identical to the first of the April issue’s Five Klamkin Quickies [2001 : 166]; the proof is in [2001 : 299]. Equality holds for an equilateral triangle.
Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENZCE, Brasov, Romania; RICHARD EDEN, Ateneo de Manila University, Philippines; VINAYAK GANESHWAN, student, University of Waterloo, Waterloo, Ontario; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; D. KIPF JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HENRY LIU, student, University of Memphis, Memphis, TN, USA; VEDULA N. MURTY, Dover, PA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; D. J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Klamkin and Benzce proved the stronger inequality

\[ 3(\cot A + \cot B + \cot C) \leq \frac{\Delta}{r^2}. \]

2663. [2001 : 337] Proposed by Andreas P. Hatzipolakis, Athens, Greece; and Paul Yu, Florida Atlantic University, Boca Raton, FL, USA.

Suppose that the incircle of \( \triangle ABC \) is tangent to the circle with \( BC \) as diameter. Show that the excircle on \( BC \) has radius equal to \( BC \).

1. Solution by Michel Bataille, Rouen, France.

Let \( D \) and \( D' \) be the points of tangency of \( BC \) with the incircle and the excircle on \( BV \), respectively.

Then \( BD = \frac{r}{\tan(B/2)} \) and \( BD' = \frac{r_a}{\tan((\pi - B)/2)} = r_a \tan(B/2) \).

(Here, \( r \) and \( r_a \) denote the inradius and the radius of the excircle on \( BC \), respectively.) Making use of signed distances, it follows that

\[ DB \cdot DC = DB \cdot BD' = -r \cdot r_a. \]  \hspace{1cm} (1)

Now, under the inversion with centre \( D \) and (negative) power \(-r \cdot r_a\), the circle with diameter \( BC \) is invariant (because of (1)) and the incircle is transformed into a line \( L \) perpendicular to \( DI \); hence, parallel to \( BC \). The distance between \( L \) and \( BC \) is \( d(D, L) = \frac{|-r \cdot r_a|}{2r} = \frac{r_a}{2} \), and the hypothesis implies that \( L \) is tangent to the circle with diameter \( BC \). Thus, \( \frac{r_a}{2} = \frac{a}{2} \); that is, \( r_a = a \).

II. Solution by Vinayak Ganeshan, student, University of Waterloo, Waterloo, Ontario. [Ed.: The solver uses standard triangle notation.]

Since the two circles touch [internally], the difference of their radii must be equal to the distance between their centres:

\[ \frac{a}{2} - r = \sqrt{(s - c - \frac{a}{2})^2 + r^2}. \]

Squaring yields

\[ \frac{a^2}{4} - ar = \left(\frac{b - c}{2}\right)^2, \]  so that \( ar = (s - c)(s - b) \).
But, \( rs = r_a(s - a) = \Delta = \sqrt{s(s - a)(s - b)(s - c)} \), so that
\[
ar = \frac{\Delta^2}{s(s - a)} = r r_a, \text{ giving } r_a = a = BC.
\]

Also solved by MICHEL BATAILLE, Rouen, France (2nd solution); FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul’s School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LQUEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; D. J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

2664. [2001: 403] Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.

Let \( a, b \) and \( c \) be positive real numbers such that \( a + b + c = abc \). Prove that \( a^5(b - 1) + b^5(ca - 1) + c^5(ab - 1) \geq 54\sqrt{3} \).

1. Solution by Kee-Wai Lau, Hong Kong, China.

By the AM–GM inequality, we have \( abc = a + b + c \geq 3(abc)^{\frac{1}{3}} \) so that \( abc \geq 3\sqrt{3} \).

Hence,
\[
\begin{align*}
a^5(b - 1) + b^5(ca - 1) + c^5(ab - 1) &= abc(a^4 + b^4 + c^4) - a^5 - b^5 - c^5 \\
&= (a + b + c)(a^4 + b^4 + c^4) - a^5 - b^5 - c^5 \\
&= a(b^4 + c^4) + b(c^4 + a^4) + c(a^4 + b^4) \\
&\geq a(2b^2c^2) + b(2c^2a^2) + c(2a^2b^2) \\
&\geq 6(abc^2)(bc^2a^2)(ca^2b^2)^{\frac{1}{3}} \\
&= 6(abc)^{\frac{8}{3}} \geq 6(3\sqrt{3})^{\frac{8}{3}} = 54\sqrt{3}.
\end{align*}
\]

11. Generalization by Murray S. Klamkin, University of Alberta, Edmonton, Alberta (expanded slightly by the editor).

We establish the following more general result:

Let \( n \in \mathbb{N}, n > 1 \) and let \( a_k > 0, k = 1, 2, \ldots, n \). Suppose that \( S = P \)
where \( S = \sum_{k=1}^{n} a_k \) and \( P = \prod_{k=1}^{n} a_k \). Then, for all real numbers \( m \) and \( r \), we have
\[
\sum_{k=1}^{n} a_k^m \left( \frac{P}{a_k} - 1 \right)^r \geq n(n - 1)^r \cdot n^{\frac{m-1}{n}}
\]
with equality if and only if all the \( a_k \)'s are equal.
To prove this, we apply the AM–GM inequality twice to get

\[
\sum_{k=1}^{n} a_k^n \left( \frac{P}{a_k} - 1 \right)^r \geq \frac{n}{r} \left( \prod_{k=1}^{n} (P - a_k) \right)^r \geq n \left( P^{m-r} \cdot \prod_{k=1}^{n} (P - a_k)^r \right)^{\frac{1}{n}} \geq nP^{\frac{m-r}{n}} \cdot \prod_{k=1}^{n} (S - a_k)^{\frac{r}{n}} \geq nP^{\frac{m-r}{n}} (n-1)^r \cdot P^{\frac{r}{n}} = n(n-1)^r P^{\frac{m}{n}}
\]

(1)

Now, by AM–GM inequality again, we have \( P = S \geq nP^{\frac{r}{n}} \) or \( P^{n-1} \geq n^n \). Hence,

\[
P \geq n^{\frac{n-r}{n-1}}.
\]

(2)

From (1) and (2), our claim follows.

Note that the given inequality is the special case when \( n = 3, m = 5 \) and \( r = 1 \).

III. Generalization by Walther Janous, Ursulengymnasium, Innsbruck, Austria (modified slightly by the editor).

We prove the following more general result which contains the given inequality as a special case.

**Theorem:** Let \( a, b, c \) be non-negative reals and let \( r \in \mathbb{R} \) with \( r \geq 2 \).

Then

\[
a^r(b + c) + b^r(c + a) + c^r(a + b) \geq \frac{2}{3(r-1)}(ab + bc + ca)^{\frac{r+3}{r}}.
\]

**Proof:** If \( ab + bc + ca = 0 \) then clearly equality holds.

Thus, we may assume that \( ab + bc + ca > 0 \).

Since \( r \geq 2 \) the function \( f(x) = x^{r-1} \) is convex on \( [0, \infty) \).
Hence, by Jensen’s Inequality, we have
\[
\frac{a^r(b + c) + b^r(c + a) + c^r(a + b)}{2(ab + bc + ca)}
= \frac{(ab + ca)a^{r-1} + (bc + ab)b^{r-1} + (ca + bc)c^{r-1}}{2(ab + bc + ca)}
\geq \left( \frac{(ab + ca)a + (bc + ab)b + (ca + bc)c}{2(ab + bc + ca)} \right)^{r-1}
= \left( \frac{a^2(b + c) + b^2(c + a) + c^2(a + b)}{2(ab + bc + ca)} \right)^{r-1}
\] (3)

Next, we establish the following inequality
\[
a^2(b + c) + b^2(c + a) + c^2(a + b) \geq \frac{2}{\sqrt{3}} (ab + bc + ca)^{\frac{3}{2}}
\] (4)

Note that (4) is equivalent to
\[
(a + b + c)(ab + bc + ca) \geq \frac{2}{\sqrt{3}} (ab + bc + ca)^{\frac{3}{2}} + 3abc
\] (5)

By the AM–GM inequality, we have
\[
\frac{1}{3} (a + b + c)(ab + bc + ca) \geq 3abc
\] (6)

Furthermore, from \((a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0\) we have
\[
(a + b + c)^2 \geq 3(ab + bc + ca)
\]
or
\[
a + b + c \geq \sqrt{3}(ab + bc + ca)^{\frac{1}{3}}.
\]

Hence,
\[
\frac{2}{3} (a + b + c)(ab + bc + ca) \geq \frac{2}{\sqrt{3}} (ab + bc + ca)^{\frac{3}{2}}
\] (7)

Adding up (6) and (7) yields (5) and hence, (4). Substituting (4) into (3) we then have
\[
a^r(b + c) + b^r(c + a) + c^r(a + b)
\geq \left( \frac{2}{\sqrt{3}} \frac{(ab + bc + ca)^{\frac{3}{2}}}{2(ab + bc + ca)} \right)^{r-1}
= \frac{2}{3^{\frac{r-1}{2}}} (ab + bc + ca)^{\frac{r+1}{2}},
\]
completing the proof.
Finally, we show that the given inequality is a special case of our theorem. Let \( r = 4 \) and assume that \( a + b + c = abc \). Note first that
\[
(a + b + c)(ab + bc + ca) \geq 9abc \quad \text{from (6)} \quad \text{and hence,} \quad ab + bc + ca \geq 9.
\]

Using the theorem above, we then have
\[
a^5(b - 1) + b^5(c - 1) + c^5(a - 1) = a^4(b + c) + b^4(c + a) + c^4(a + b) \geq 2 \left( \frac{2}{3} \right)^2 (9) \frac{2}{3} = 2 \cdot 3^2 = 54\sqrt{3}.
\]

**Remark:** Clearly equality holds in our theorem if \( r = 1 \). This leads to the natural question: What happens for \( r \in (1, 2) \)?

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ATENEO PROBLEM-SOLVING GROUP, Ateneo de Manila University, The Philippines; DIONNE T. BAILEY, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENČÉ, Brașov, Romania; PIERRE BÖRNÖSZTÉIN, Coudrec, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, student, Balliol College, Oxford, UK; MARCELO R. DE SOUZA, Rio de Janeiro, Brazil; NATALIO H. GUÉRENZAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA; PAUL JEFFEYRYS, student, Berkhamsted Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; VEDULA N. MURTY, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzischule Graz, Austria; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most of the solutions are either variations of or very similar to, solution 1 above. Benez, Börnösztén, and Seiffert also gave various generalizations, but they are all special cases of the more general result obtained by Klaman in solution 2 above.

2665. [2001 : 403] Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.

In \( \triangle ABC \), we have \( \angle ACB = 90^\circ \) and sides \( AB = c \), \( BC = a \) and \( CA = b \). In \( \triangle DEF \), we have \( \angle EFD = 90^\circ \), \( EF = (a + c) \sin \left( \frac{B}{2} \right) \) and \( FD = (b + c) \sin \left( \frac{A}{2} \right) \). Show that \( DE \geq c \).

**Solution by Christopher J. Bradley, Clifton College, Bristol, UK; and Heinz-Jürgen Seiffert, Berlin, Germany.**

The claim of the problem is incorrect. We prove that
\[
\frac{c}{\sqrt{2}} < DE \leq c \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}.
\]
with equality on the right-hand side only when $A = 45^\circ$. Without loss of generality, take $c = 1$. Then $a = \sin A$ and $b = \cos A$. In addition, 

$$\sin^2 \frac{A}{2} = \frac{1}{2} (1 - \cos A)$$

and 

$$\sin^2 \frac{B}{2} = \frac{1}{2} (1 - \sin B) = \frac{1}{2} (1 - \sin A).$$

It follows that 

$$DE^2 = EF^2 + FD^2$$

$$= \frac{1}{2} (1 + \sin A)^2 (1 - \sin A) + \frac{1}{2} (1 + \sin A)^2 (1 - \cos A)$$

$$= \frac{1}{2} \cos^2 A (1 + \sin A) + \frac{1}{2} \sin^2 A (1 + \cos A)$$

$$= \frac{1}{2} + \frac{1}{2} \sin A \cos A (\sin A + \cos A)$$

$$= \frac{1}{2} + \frac{\sqrt{2}}{4} \sin 2A \cos (45^\circ - A).$$

Clearly, 

$$\frac{1}{\sqrt{2}} < DE \leq \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}},$$

with equality only if $A = 45^\circ$, as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ATENEO PROBLEM-SOLVING GROUP, Ateneo de Manila University, The Philippines; ELSIE CAMPBELL, Angelo State University, San Angelo, TX, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinenimnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; VEDULA N. MURTY, Dover, PA, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

The range of $DE$ has also been found by Hess and Janous. All of the other solvers have noted that the stated inequality is incorrect and either proved that $DE < \epsilon$ or found a counterexample to the statement $DE \geq \epsilon$.


You are given a circle $\Gamma$ and two points $A$ and $B$ outside of $\Gamma$ such that the line through $A$ and $B$ does not intersect $\Gamma$. Let $X$ be any point on $\Gamma$.

Determine at which point $X$ on $\Gamma$ the sum $AX + XB$ attains its minimum value.

Solution by Toshio Seimiyia, Kawasaki, Japan.

Let $O$ be the centre of the circle $\Gamma$ and let $Y$ be the point on $\Gamma$ such that:

1. the line $OY$ is the bisector of $\angle AYB$;
2. if $t$ is the tangent line to $\Gamma$ at $Y$, then $A$ and $B$ lie on the opposite side of $\Gamma$ with respect to $t$. 


We claim that $AY + YB < AX + XB$ for every point $X$ of $\Gamma$, $X \neq Y$.

First, we note that the angles between $t$ and $AY$ and between $t$ and $BY$ are equal. Let $X$ be any point on $\Gamma$ other than $Y$. Let $Z$ be the foot of the perpendicular from $X$ to the tangent line $t$. Then $\angle AZX > 90^\circ$ and $\angle BZX > 90^\circ$, so that $AZ + BZ < AX + BX$. It is well-known that $AY + BY < AZ + BZ$ for every point $Z$ on the line $t$.

Hence, $AY + BY < AX + BX$, which proves our claim. Therefore, the point $Y$ is the desired one.

The problem of constructing the point $Y$ is a classical problem, also known as the Alhasen's Problem. We refer the reader to [1] for further discussion.

Reference

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; VICTOR PAMBUCCIAN, ASU West, Phoenix, Arizona, USA. There were also two incorrect (incomplete) solutions submitted.

Congratulations

We note that the MAA has announced its 2002 awards for distinguished teaching. Included in the list are two Cruxers; Andy Liu and Li Zhou. Our hearty congratulations to both.

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