Polya's Paragon

Paul Ottaway

I would like to begin by examining a problem that I left at the end of last month's article. Here is the problem again to refresh your memory:

In the given diagram, triangle $ABC$ is isosceles with $AB = AC$. We extend $AC$ to $D$ and construct $F$ on $AB$ such that $CD = BF$. $E$ is the intersection of $DF$ and $BC$. We would like to show that $EF = ED$.

\[\text{Diagram with labels: } A, B, C, D, E, F, AB = AC, CD = BF, EF = ED.\]

The reason I am revisiting this problem is because of the variety of solutions. One in particular uses a very clever technique. Here are a couple of the solutions that I have discovered:

**Solution #1:**

Extend $CB$ to $E'$ such that triangle $ECD$ is congruent to triangle $E'BF$. Now, $\angle FE'E = \angle DEC = \angle FEE'$. Therefore, triangle $FEE'$ is isosceles with $FE = FE' = ED$.

\[\text{Diagram with labels: } A, B, C, D, E', E, AB = AC, CD = BF, EF = ED.\]
Solution #2:

Using the sine law multiple times, we discover the following:

\[
\frac{\sin(\angle EBF)}{EF} = \frac{\sin(\angle BEF)}{BF} = \frac{\sin(\angle CED)}{CD} = \frac{\sin(\angle ECD)}{ED}
\]

We also know that \( \angle EBF = \angle ECA \), \( \angle ECD = 180^\circ - \angle ECA \) and that \( \sin(x) = \sin(180^\circ - x) \) so that \( \sin(\angle EBF) = \sin(\angle ECD) \). Plugging that into our previous relation we see that

\[
1 = \frac{\sin(\angle EBF)}{\sin(\angle ECD)} = \frac{EF}{ED}
\]

Therefore, \( EF = ED \) as desired.

Solution #3:

Drop perpendiculars from \( D \) and \( F \) to the line \( BC \) extended as shown. Note that since \( \angle FBG = \angle HCD \) and \( FB = CD \) then it must be the case that the triangles \( FBG \) and \( DCH \) are congruent. Triangle \( EGF \) has all its angles the same as in triangle \( EHD \) and \( FG = HD \) from our previous work, so that we know these two triangles are also congruent. It therefore follows that \( FE = ED \).

There are many other solutions possible, each with its own uniqueness. The next (and last) solution I will present needs a little bit of background first. A high school student I met over the summer showed this solution to me. He used a truly beautiful method most commonly known as "mass points".

The idea is simple. We assign a "mass" to a vertex and balance, like a teeter-totter, the other vertices by assigning them "masses" as well. To begin, we need to know a little about physics and how to balance a teeter-totter. The following diagram shows two masses \( M_1 \) and \( M_2 \) balanced perfectly about a pivot point (fulcrum) \( P \). In order for the system to be balanced \( M_1 \times d_1 = M_2 \times d_2 \). This should hopefully make some intuitive sense. If one object is heavier than the other it would have to be placed closer to
the fulcrum in order for the balance to be maintained. The other piece of information that can be gained is the mass at \( P \). If we were holding this contraption at the point \( P \), it would naturally seem to have the mass \( M_1 + M_2 \). For simplicity, I will ensure that masses appear in boxes so that they are not confused with distances or labels for points.

I will not prove these results, but they are true and can be used in a variety of geometry problems, making them much easier to solve.

Here is an example of how mass points can be used to solve a rather difficult problem:

In the given diagram, we know that \( AP : PB = 3 : 4 \) and \( AQ : QC = 3 : 2 \). Prove that \( C \) is the mid-point of \( BD \).

\[
\text{Solution:}
\]

We begin by assigning a mass of 4 at the point \( A \). We can say that \( AP = 3x \) and \( PB = 4x \). From this, in order to have \( BPA \) balanced at \( P \), we need a mass of 3 at \( B \). We can also conclude that \( P \) has a mass of 7 (the sum of the masses at \( A \) and \( B \)). If we repeat this process with the line \( AQC \) we let \( AQ = 3y \) and \( QC = 2y \). Now, we see that to have balance we need a mass of 6 at \( C \) and therefore, a mass of 10 at \( Q \). Since we know that any point between two others must have the same mass as the sum of the other two we can examine the line \( PQD \) to find that the mass at \( D \) must be 3. Finally, looking along \( BCD \), we see that all of the masses have already been determined and that \( B = 3 \), \( C = 6 \) and \( D = 3 \). Since the mass at \( B \) and \( D \) are equal and \( C \) is their sum, we must conclude that \( BC = CD \) and thus, \( C \) is at the mid-point.

This solution seems much simpler to me than any alternative that comes to mind. Can you see another way to solve the problem? I assure you there is!
Let us now return to our original problem to see how it can be solved using mass points.

**Solution #4:**

Since we do not know specific distances or ratios, we begin by assigning the variable $a$ to lengths $CD$ and $BF$. We also let $AF = b$ and therefore, $AC = a + b$. Now, we assign a mass of $a$ at point $A$. We continue by finding the mass at $B$ is $b$ and the mass at $F$ is $a + b$ using the line $AFB$. Now, using the line $ACD$ we find that the mass at $D$ is $a + b$. At this point, we can stop and examine the line $EFD$. Since both endpoints have the same mass, they must both be the same distance from the fulcrum at $E$. Therefore, $FE = ED$ as required.

![Diagram](image)

If you feel so inclined, continue to fill in the masses of the remaining points just to make sure the system as a whole is sound.

Overall, this is a technique that is rarely seen these days and has the power to deliver surprisingly simple solutions. It is an excellent tool that can help approach geometry problems that can sometimes seem too tough to tackle. What I have shown here is only a very brief introduction to the topic and those that find this interesting should look for more information from further sources.

Here are a couple more problems for you to try on your own:

1. The medians of a triangle concur at a point called the centroid or centre of mass. Show that this point divides each of the medians into $2 : 1$ ratios.

2. A tetrahedron is a three dimensional object that has 4 triangular faces. On each face, construct the centroid. From each centroid, construct the line connecting it to the opposite vertex of the tetrahedron. These 4 lines intersect at a point. Find the ratio in which these 4 lines are divided by this point.

3. **(AHSME 1964 #35)** - The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point $H$. If $AD$ is the altitude to the side of length 14, what is the ratio $HD : HA$?