An Industrial Application of Spherical Inversion

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A local engineer, Noel Stephens, built an instrument to locate the position of a solitary light source of unknown brightness and position in three-dimensional space. The instrument consisted of an arrangement in the plane of \( n \) fixed brightness sensors; the light source was always to lie above this plane. In this problem there are four real unknowns: the brightness of the source and three position co-ordinates, so that at least four sensors are necessary. Noel knew that it would be pointless to place the sensors all in a line, for then rotating the light source about that line would not alter the sensor readings. Thus, for his first experiment, he placed the sensors at the vertices of a square. To Noel's surprise, his computer was unable to determine the position of the source. He tried a rectangle, but again the computer was unable to fix the source. The sensors were working, and there was no mistake in the code — the data were simply insufficient to determine the position of the source uniquely! After some computer experimentation, Noel conjectured the following theorem, which we prove below.

**Theorem 1.** Let \( Q_0, \ldots, Q_{n-1} \) lie on a circle \( C \) in \( \mathbb{R}^3 \), and let \( P \) be any other point of \( \mathbb{R}^3 \). For each \( i \neq j \) let \( T_{ij} \) be the centre of the Apollonian sphere \( S_{ij} \) consisting of the points \( X \) such that \( \frac{Q_iP}{Q_jP} = \frac{Q_iX}{Q_jX} \). Then

1. the centres \( T_{ij} \) lie on a line, and
2. there is a circle through \( P \) that is contained in all the spheres \( S_{ij} \).

Although the brightness of the light source at \( P \) is unknown, if we know the ratio of intensities measured by two different sensors at \( Q_i \) and \( Q_j \) then we know a sphere of Apollonius on which \( P \) must lie. The theorem says that if you place all of your sensors on a circle then you have no hope of locating a light source at \( P \), regardless of how many sensors you have, because two different sources on the common circle of those spheres and of appropriate brightnesses will give exactly the same data from the sensors. We assume that the source emits light isotropically (the same intensity in all directions), and that the sensors respond isotropically.

The proof uses spherical inversion. Inversion in a plane means reflection in that plane. In this article we write sphere to mean "sphere or plane" and circle to mean "circle or line". A line or plane may be
characterised as a circle or sphere that passes through $\infty$, the point at infinity. For example, some of the Apollonian spheres $S_{ij}$ could be planes; the centre in such a case is $T_{ij} = \infty$.

It is natural to use spherical inversion because of the following well-known lemma:

**Lemma 1.**

1. The sphere or plane $S$ is a sphere of Apollonius of two points $A$ and $B$ if and only if inversion in $S$ swaps $A$ with $B$.

2. If $A$ and $B$ are inverse points with respect to a sphere $S$ and $I$ is any spherical inversion, then $I(A)$ and $I(B)$ are inverse points with respect to $I(S)$.

It follows that any spherical inversion sends Apollonian spheres to Apollonian spheres.

**Proof.**

1. A simple exercise using similar triangles. See [1], p. 89 (section 6.6) and p. 92 (section 6.82).

2. $I(A), I(B)$ are inverse with respect to $I(S)$ if and only if every sphere $T$ passing through both $I(A)$ and $I(B)$ is orthogonal to $I(S)$. Since inversion is its own inverse and preserves angles, the image spheres $I(T)$ all pass through $A$ and $B$ and are orthogonal to $S$.

Now we can prove the theorem.

**Proof.** Since we know that all the spheres $S_{ij}$ contain $P$, it is clear that the two parts of the conclusion are equivalent: the spheres contain the circle through $P$ in the plane perpendicular to the line of their centres.

Invert the figure about some point $X$ on $C$: we will denote images under this inversion by dashes. The points $Q'_{ij}$ all lie on the line $C'$. By the lemma, the image sphere $S'_{ij}$ is the Apollonian sphere for $Q'_{ij}$ and $Q'_{ij}$ that passes through $P'$, whose centre lies on the line $Q'_{ij}Q'_{ij}$, which is $C'$. Since the centres of the $S'_{ij}$ are collinear, the $S'_{ij}$ all contain a certain circle $E'$, passing through $P'$. Therefore, the $S_{ij}$ all contain the image of $E'$ under inversion about $X$, which is a circle through $P$.

**Remark:** the centre of $S_{ij}$ never inverts to the centre of $S'_{ij}$, which is why for an inversive proof it is best to aim for the second part of the conclusion. We may sum the proof up as follows:

Because the original problem has a description that is invariant under the Möbius group (generated by all inversions), the set of useless positionings of the sensors must also be invariant under the Möbius group.
Here is an alternative proof, using a different inversion, in which we aim
directly for the first conclusion of the theorem.

**Proof.** Call the plane of the $n$-gon $H$, and let $R$ be the reflection of $P$ in $H$. Then all the spheres $S_i$ pass through $P$ and also through $R$. Invert the figure about $P$; again we will use dashes to denote components of the inverse figure. $H'$ is a sphere through $P = \infty'$. By the lemma, $P$ and $R$ are inverses with respect to $H$, so that $P' = \infty$ and $R'$ are inverses with respect to $H'$, so that $R'$ is the centre of $H'$. The spheres of Apollonius invert to spheres $S'_{ij}$, which all pass through both $R'$ and $P' = \infty$, so that they are planes through $R'$. The points $Q_i$ and $Q_j$ are inverse with respect to $S_{ij}$, so that $Q'_{ij}$ are inverse with respect to $S'_{ij}$: they are reflections of each other in the plane $S'_{ij}$.

Now, the points $Q_i$ all lie on a circle $C$ in the plane $H$, which inverts to a circle $C'$ on the sphere $H'$. Since $C'$ contains a point and its reflection in the plane $S'_{ij}$, (namely $Q'_{ij}$ and $Q'_{ij}$), the plane of $C'$ must be perpendicular to each $S_{ij}$. Now we know that the planes $S'_{ij}$ all pass through the centre $R'$ of the sphere $H'$, and that they have a common perpendicular plane (the plane of $C'$). Therefore, they all contain a certain diameter of $H'$, say $L'$.

The points $T_{ij}$ are the centres of the $S_{ij}$, which are the points inverse to infinity with respect to $S_{ij}$. Thus, $P = \infty$ and $T_{ij}$ are inverse with respect to $S_{ij}$, so that they are each other's reflections in $S_{ij}$. The sphere $H'$ passes through $P$, and the planes $S'_{ij}$ all contain a diameter $L'$ of $H'$, so that the reflections $T_{ij}$ of $P$ in those planes all lie in a circle $D'$ passing through $P = \infty'$ whose plane is perpendicular to $L'$. Thus, the circle $D'$ inverts to a line $D$, containing all the $T_{ij}$. ■

**References**


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