Pólya's Paragon

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"Beauty" and "Mathematics" are two words that you do not often hear in the same sentence — unless, of course, you are talking to a mathematician. Many people will call mathematics useful, difficult, mechanical or even boring — but beautiful? I hope to show through a couple of examples where I think this beauty lies.

One of the most famous mathematical theorems in the world is one attributed to Pythagoras dealing with the relationship between the sides of a right-angled triangle. No doubt the phrase "$a^2+b^2=c^2$" has a permanent place in the mind of most of my readers. This theorem has been examined by a countless number of people ever since it was first conceived. Moreover, dozens of proofs to his theorem have been found. You might ask yourself, why after all this time do people still examine this theorem and try to find other unique proofs. To a mathematician, this would be equivalent to asking a painter why he would bother painting a sunset, since so many have been painted in the past. Each proof has its own beauty (or lack thereof) due to its elegance, succinctness, imagination or any number of other traits.

To further illustrate my point, here is the problem that was the inspiration for this article:

Prove that, for all $n \in \mathbb{N}$, the sum of the elements in the n^{th} row of Pascal's triangle is 2^n.

Pascal's Triangle

Row 1			1		1		
Row 2		1		2		1	
Row 3	1		3		3		1
Etc.							

Each row begins and ends with a 1. Every other element is formed by taking the sum of the two numbers above to the right and above to the left of its position.

Solution # 1:

We know that the numbers in the n^{th} row of Pascal's triangle are the elements

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}.$$

Therefore, the problem is equivalent to showing that their sum is 2^n.

From the binomial theorem, we know that

$$(1+x)^n = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n .$$

If we now let $x = 1$, we get the desired result.

Solution # 2:

Imagine we have n objects and we want to create a subset of a particular size.

There are $\binom{n}{0}$ ways of making a subset of 0 elements.

There are $\binom{n}{1}$ ways of making a subset of 1 element.

$\vdots$

There are $\binom{n}{n}$ ways of making a subset of n elements.

By summing these terms, we will have counted all possible ways of making an arbitrary subset of the n objects. Another way of counting the total number of subsets would be as follows: Each object is either in the subset or it is not, so that we have 2 choices for each of the n elements. Therefore, there are 2^n subsets possible. We have now counted these subsets two different ways to get the following relationship:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n ,$$

and we arrive at the desired result.

Solution # 3:

When constructing Pascal's Triangle, each number is added twice to the following row to help form the numbers down and to the right and down and to the left (you can think of the 1's as being $(0+1)$ from the previous row). Therefore, the sum of the elements of any row must be twice the sum of the numbers in the previous row. Here is an example showing how row 4 is constructed from row 3:

Row 3		1		3		3		1	
Summations	$(0+1)$		$(1+3)$		$(3+3)$		$(3+1)$		$(1+0)$
Row 4	1		4		6		4		1

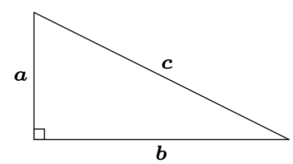
It is easy to see that the sum of the numbers in the first row is $2 = 2^1$, so that, by induction, it must be the case that the n^{th} row has a sum of 2^n.

These three proofs use almost entirely unrelated methods to come up with the same result. I am sure that there are at least two or three other really amazing proofs that I have never seen. This is one of the aspects of mathematics that I find the most aesthetically pleasing. It is also the reason why people will pursue a problem even when a solution is known. It would be a very boring world indeed if artists only ever painted things that had never been painted before.

Problems for you to try:

The following problems have many different solutions, so that, even if you find a good proof, try to find another! If you get stuck on a problem put it aside for a while and try again later with a different approach. Some solutions are much easier to come by than others.

1. (Pythagorean Theorem) In the right angle triangle shown, prove that $a^2 + b^2 = c^2$.



(Try to find at least three different proofs. Hundreds have been found so far. It is not cheating to look them up as long as you understand the solution!)

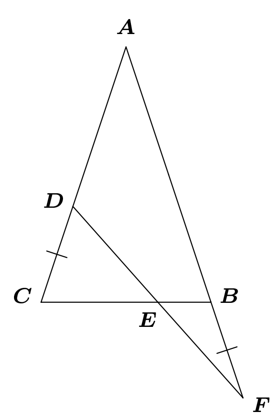
2. For all positive values of n we define $h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Prove that $n + h(1) + h(2) + h(3) + \cdots + h(n-1) = nh(n)$.

(I have discovered a nice direct proof as well as one that uses induction — try to find them both and more!)

3. In the given diagram, $AC = AB$ and $CD = BF$.

Show that $DE = EF$.



So that you can try the problem without the diagram, triangle ABC is isosceles with $AC = AB$. Extend AB to a point F. Draw a line from F to a point D on AC such that $CD = BF$. Label the intersection of this line with BC as the point E.

(I have found at least five or six proofs to this one using trigonometry, constructions, rotations or even co-ordinate geometry. Needless to say, some proofs are more elegant than others!)