Some generalizations of an inequality from IMO 2001

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The purpose of this paper is to consider some natural generalizations of Problem 2 from IMO 2001 which states:

Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ac}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1,$$

where $a$, $b$ and $c$ are arbitrary positive numbers.

Many different proofs of this inequality were given during the Olympiad and it was also shown by the first author that

$$\frac{a}{\sqrt{a^2 + \lambda bc}} + \frac{b}{\sqrt{b^2 + \lambda ac}} + \frac{c}{\sqrt{c^2 + \lambda ab}} \geq \frac{3}{\sqrt{1 + \lambda}}$$

for arbitrary $a$, $b$, $c > 0$ and $\lambda \geq 8$. It is easy to see that the latter inequality is not true for $0 < \lambda < 8$. Moreover, it can be shown that in this case

$$\frac{a}{\sqrt{a^2 + \lambda bc}} + \frac{b}{\sqrt{b^2 + \lambda ac}} + \frac{c}{\sqrt{c^2 + \lambda ab}} > 1,$$

and the lower bound is sharp.

We now prove a general inequality that encompasses all of these results.

**Proposition 1.** For any positive integers $n$ and $m$, and any positive numbers $x_1, x_2, \ldots, x_n$ with $x_1 x_2 \ldots x_n = \lambda^n$ ($\lambda > 0$), we have the following sharp inequality:

$$\sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\frac{m}{n}}} \geq \min \left(1, \frac{n}{(1 + \lambda)^{\frac{m}{n}}} \right).$$

(1)
\textbf{Proof.} Set
\[ d = \min \left( 1, \frac{n}{(1 + \lambda)^{\frac{1}{m}}} \right). \]

Multiplying both sides of (1) by \( \prod_{i=1}^{n} (1 + x_i)^{\frac{1}{m}} \) and then taking the \( m \)-th power we see that (1) is equivalent to the inequality
\[ \sum_{i=1}^{n} \prod_{k=1, k \neq i}^{n} (1 + x_k) + T \geq d^m \prod_{i=1}^{n} (1 + x_i), \tag{2} \]
where
\[ T = \left[ \sum_{i=1}^{n} \prod_{k=1, k \neq i}^{n} (1 + x_k)^{\frac{1}{m}} \right]^m - \sum_{i=1}^{n} \prod_{k=1, k \neq i}^{n} (1 + x_k). \]

Denote by \( \sigma_1, \sigma_2, \ldots, \sigma_n \), the elementary symmetric functions of the \( x_i \) and set \( \sigma_0 = 1 \). Then it is easy to check that
\[ \prod_{i=1}^{n} (1 + x_i) = \sum_{i=0}^{n} \sigma_i \quad \text{and} \quad \prod_{i=1}^{n} (1 + x_k) = \sum_{i=0}^{n-1} (n - i) \sigma_i. \]

Hence, (2) can be rewritten as
\[ \sum_{i=0}^{n-1} (n - i - d^m) \sigma_i + T \geq d^m \sigma_n. \]

By the AM-GM inequality we have
\[ \sigma_i \geq \left( \frac{n}{i} \right)^{\frac{1}{i}} \left( \sigma_n \right)^{\frac{1}{i}} = \left( \frac{n}{i} \right)^{\lambda_i}, \quad 0 \leq i \leq n, \tag{3} \]
and, therefore,
\[ \prod_{i=1}^{n} (1 + x_i) = \sum_{i=0}^{n} \sigma_i \geq \sum_{i=0}^{n} \left( \frac{n}{i} \right)^{\lambda_i} = (1 + \lambda)^n. \tag{4} \]

To estimate the term \( T \) we use the following inequality
\[ \left( \sum_{i=1}^{n} a_i \right)^m \geq \sum_{i=1}^{n} a_i^m + (n^m - n) \left( \prod_{i=1}^{n} a_i \right)^{\frac{m}{n}} \quad \text{for} \quad a_i > 0, \tag{5} \]
which follows easily by induction on \( m \). Setting
\[ a_i = \prod_{k=1, k \neq i}^{n} (1 + x_k)^{\frac{1}{m}} \]
in (5) gives
\[ T \geq (n^m - n) \prod_{i=1}^{n} (1 + x_i)^{\frac{n-1}{m}}, \]
and, therefore, (4) implies that
\[ T \geq (n^m - n)(1 + \lambda)^{n-1}. \]  
(6)

In view of (3), (6) and the fact that \( d \leq 1 \), to prove (2), it is sufficient to show that
\[ d^m \lambda^n - (n^m - n)(1 + \lambda)^{n-1} - \sum_{i=0}^{n-1} (n - i - d^m) \binom{n}{i} \lambda^i \leq 0. \]  
(7)

But the left hand side of (7) is equal to \((1 + \lambda)^{n-1} (d^m (1 + \lambda) - n^m)\) (this can be seen, for example, by comparing the coefficients of the powers of \( \lambda \) in both expressions) and the inequality (7) follows since
\[ d \leq \frac{n}{(1 + \lambda)\frac{1}{m}}. \]

Note that, if \( \lambda \geq n^m - 1 \), then \( d = n(1 + \lambda)^{-\frac{1}{m}} \) and (1) tells us that
\[ \sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\frac{1}{m}}} \geq \frac{n}{(1 + \lambda)^{\frac{1}{m}}} \]
with equality if and only if \( x_1 = x_2 = \cdots = x_n = \lambda \). On the other hand, if \( \lambda < n^m - 1 \), then \( d = 1 \) and (1) takes the form
\[ \sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\frac{1}{m}}} > 1. \]

To see that the latter inequality is sharp, set \( x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{t} \) and \( x_n = t^{n-1}\lambda^n \), where \( t \to 0 \).

Now, we shall show that the inequality (1) still holds if we replace the power \( \frac{1}{m} \) by any real number \( \alpha \in (0, 1] \). In this case, however, it is not possible to proceed as in the proof of Proposition 1, since inequality (5) is not true for any real number \( m > 1 \) and any positive integer \( n \) (take, for example, \( m = \frac{3}{2}, n = 2, x_1 = 1, x_2 = \frac{1}{10} \)). Instead, we shall use the powerful Lagrange multiplier criterion.

**Proposition 2.** For any \( \alpha \in (0, 1] \) and any positive numbers \( x_1, x_2, \ldots, x_n \) with \( x_1 x_2 \ldots x_n = \lambda^n \) \((\lambda > 0)\), we have the following sharp inequality:
\[ \sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\alpha}} \geq \min \left( 1, \frac{n}{(1 + \lambda)^{\alpha}} \right). \]  
(8)
Proof. Denote by $d$ the infimum of the function

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \frac{1}{(1 + x_i)^\alpha}$$

on the set

$$A = \{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 x_2 \cdots x_n = \lambda^n, x_1, x_2, \ldots, x_n > 0 \}.$$ 

Suppose first that this infimum is not attained at a point of $A$. Then, $d = \lim_{k \to \infty} f(x_1^{(k)}, \ldots, x_n^{(k)})$, where, for example, $\lim_{k \to \infty} x_i^{(k)} = 0$ or $+\infty$.

Then, for example, $\lim_{k \to \infty} x_1^{(k)} = +\infty$ or 0 and, in both cases, we see that $d \geq 1$. Note that if $\lim_{k \to \infty} x_s^{(k)} = +\infty$ for $s = 1, 2, \ldots, n - 1$ and $\lim_{k \to \infty} x_n^{(k)} = 0$, then $\lim_{k \to \infty} f(x_1^{(k)}, \ldots, x_n^{(k)}) = 1$, which shows that $d = 1$.

Now, let $d$ be attained at a point of $A$. Consider the function

$$F(x_1, x_2, \ldots, x_n, \mu) = f(x_1, x_2, \ldots, x_n) + \mu(x_1 x_2 \cdots x_n - \lambda^n).$$

Then the Lagrange multiplier criterion says that $d$ is attained at a point $(x_1, x_2, \ldots, x_n) \in A$ such that

$$\frac{\partial F}{\partial x_i} = -\frac{\alpha}{(1 + x_i)^{\alpha+1}} + \frac{\mu x_1 \cdots x_n}{x_i} = 0;$$

that is, when

$$\frac{x_i}{(1 + x_i)^{\alpha+1}} = \frac{x_j}{(1 + x_j)^{\alpha+1}}, \quad 1 \leq i, j \leq n. \quad (9)$$

Consider the function

$$g(x) = \frac{x}{(1 + x)^{\alpha+1}}.$$ 

Then,

$$g'(x) = \frac{1 - \alpha x}{(1 + x)^{\alpha+2}},$$

and, therefore, $g(x)$ takes each of its values at most twice. Hence, (9) shows that $x_1 = \cdots = x_k = x$ and $x_{k+1} = \cdots = x_n = y$ for some $1 \leq k \leq n$. If $k = n$, then $x_1 = x_2 = \cdots = x_n = \lambda$ and

$$f(x_1, x_2, \ldots, x_n) = \frac{n}{(1 + \lambda)^\alpha}. $$

If $k < n$, then

$$f(x_1, x_2, \ldots, x_n) = \frac{k}{(1 + x)^\alpha} + n - k \geq \frac{1}{(1 + x)^\alpha} + \frac{1}{(1 + y)^\alpha}. $$
To prove Proposition 2 it is sufficient to show that
\[
\frac{1}{(1 + x)^\alpha} + \frac{1}{(1 + y)^\alpha} > 1
\]  
provided that
\[
\frac{x}{(1 + x)^{\alpha+1}} = \frac{y}{(1 + y)^{\alpha+1}}, \quad x \neq y. 
\]  
Set \( \beta = 1/\alpha \geq 1 \), \( z = (1 + x)^\alpha \) and \( t = (1 + y)^\alpha \). Then (10) and (11) can be written, respectively, as \( z + t > zt \) and
\[
(zt)^\beta = \frac{z^{\beta+1} - t^{\beta+1}}{z - t}.
\]  
Thus, we have to prove that
\[
(z + t)^\beta > \frac{z^{\beta+1} - t^{\beta+1}}{z - t}. 
\]  
Assume that \( z < t \) and set \( u = z/t < 1 \). Applying Bernoulli's inequality twice, we obtain
\[
(1 + u)^\beta \geq 1 + \beta u > \frac{1 - u^{\beta+1}}{1 - u},
\]  
which is just the inequality (12).

**Remark.** Using similar arguments to the ones used in the proof of Proposition 2, one can show that the inequality (8) holds also in the case \( \alpha > 1 \) and \( n \geq \alpha + 1 \). Note that if \( \alpha > 1 \) but \( n < \alpha + 1 \), then this inequality is not true in general (take, for example, \( \alpha = n = 2, \ x_1 = 8, \ x_2 = \frac{1}{50} \)).

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