THE ACADEMY CORNER

No. 47

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Here, we present the official solutions of the 8th International Mathematics Competition, held at the Charles University, Prague, Czech Republic, on 19-25 July 2001. See [2002: 3]. This competition is for university students completing up to their fourth year, and consists of two sessions, each of five hours. Thanks to Moubinool Omarjee for sending them to us.

8th International Mathematics Competition

Day 1 Problems

Problem 1. Let \( n \) be a positive integer. Consider an \( n \times n \) matrix with entries \( 1, 2, \ldots, n^2 \) written in order starting top left and moving along each row in turn left to right. We choose \( n \) entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Solution. Since there are exactly \( n \) rows and \( n \) columns, the choice is of the form

\[ \{(j, \sigma(j)) : j = 1, \ldots, n\}, \]

where \( \sigma \in S_n \) is a permutation. Thus, the corresponding sum is equal to

\[
\sum_{j=1}^{n} n(j - 1) + \sigma(j) = \sum_{j=1}^{n} nj - \sum_{j=1}^{n} n + \sum_{j=1}^{n} \sigma(j) \\
= n \sum_{j=1}^{n} j - n + \sum_{j=1}^{n} j \\
= (n + 1) \frac{n(n + 1)}{2} - n^2 = \frac{n(n^2 + 1)}{2},
\]

which shows that the sum is independent of \( \sigma \).
Problem 2. Let \( r, s, t \) be positive integers which are pairwise relatively prime. If \( a \) and \( b \) are elements of a commutative multiplicative group with unity element \( e \), and \( a^r = b^s = (ab)^t = e \), prove that \( a = b = e \).

Does the same conclusion hold if \( a \) and \( b \) are elements of an arbitrary non-commutative group?

Solution.

1. There exist integers \( u \) and \( v \) such that \( us + vt = 1 \). Since \( ab = ba \), we obtain

\[
ab = (ab)^{us+vt} = (ab)^{us} ((ab)^t)^v = (ab)^{us} (b^s)^u = a^{us} e = a^{us}.
\]

Therefore, \( b^r = e b^r = a^r b^r = (ab)^r = a^{usr} = (a^r)^{us} = e \). Then

\[
b = b^{ser+us} = (b^r)^{us} (b^s)^u = e.
\]

It follows similarly that \( a = e \) as well.

2. This is not true. Let \( a = (123) \) and \( b = (34567) \) be cycles of the permutation group \( S_7 \) of order 7. Then, \( ab = (1234567) \) and \( a^3 = b^5 = (ab)^7 = e \).

Problem 3. Find \( \lim_{t \to 1} \left( 1 - t \right) \sum_{n=1}^{\infty} \left( \frac{t^n}{1 + t^n} \right) \), where \( t \to 1 \) means that \( t \) approaches 1 from below.

Solution.

\[
\lim_{t \to 1^+} \left( 1 - t \right) \sum_{n=1}^{\infty} \left( \frac{t^n}{1 + t^n} \right) = \lim_{t \to 1^-} \left( \frac{1 - t}{- \ln t} \right) \cdot \left( - \ln t \right) \sum_{n=1}^{\infty} \left( \frac{t^n}{1 + t^n} \right) = \lim_{t \to 1} \left( - \ln t \right) \sum_{n=1}^{\infty} \left( \frac{1}{1 + e^{-n \ln t}} \right) = \lim_{h \to 0^+} h \sum_{n=1}^{\infty} \left( \frac{1}{1 + e^{n h}} \right) = \int_0^{\infty} \frac{dx}{1 + e^x} = \ln 2.
\]

Problem 4. Let \( q \) be a positive integer. Let \( p(x) \) be a polynomial of degree \( q \), each of whose coefficients is \(-1, 1 \) or \( 0 \), and which is divisible by \((x - 1)^k \). Let \( q \) be a prime such that \( \frac{q}{\ln q} < \frac{k}{\ln (n + 1)} \). Prove that the complex \( q^{th} \) roots of unity are roots of the polynomial \( p(x) \).

Solution. Let \( p(x) = (x - 1)^k r(x) \) and \( \epsilon_j = e^{2\pi i j/q} \) (\( j = 1, 2, \ldots, q - 1 \)). As is well known, the polynomial

\[
x^{q-1} + x^{q-2} + \cdots + x + 1 = (x - \epsilon_1) \cdots (x - \epsilon_{q-1})
\]

is irreducible. Thus, all \( \epsilon_1, \ldots, \epsilon_{q-1} \) are roots of \( r(x) \), or none of them are.
Suppose that none of them are roots of \( r(x) \). Then \( \prod_{j=1}^{q-1} r(\epsilon_j) \) is a rational integer, which is not 0, and

\[
(n+1)^{q-1} \geq \prod_{j=1}^{q-1} |p(\epsilon_j)| = \left| \prod_{j=1}^{q-1} (1 - \epsilon_j)^k \right| \cdot \prod_{j=1}^{q-1} r(\epsilon_j) = \left| \prod_{j=1}^{q-1} (1 - \epsilon_j) \right|^k = (1^{q-1} + 1^{q-2} + \cdots + 1^1 + 1)^k = q^k.
\]

This contradicts the condition \( \frac{q}{\ln q} < \frac{k}{\ln(n+1)} \).

**Problem 5** Let \( A \) be an \( n \times n \) matrix such that \( A \neq \lambda I \) for all \( \lambda \in \mathbb{C} \). Prove that \( A \) is similar to a matrix having at most one non-zero entry on the main diagonal.

**Solution.** The statement will be proved by induction on \( n \). For \( n = 1 \), there is nothing to do.

In the case \( n = 2 \), write \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If \( b \neq 0 \) and \( c \neq 0 \), or if \( b = c = 0 \), then \( A \) is similar to

\[
\begin{bmatrix} 1 & 0 \\ a/b & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a/b & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ c - ad/b & a + d \end{bmatrix},
\]

or

\[
\begin{bmatrix} 1 & -a/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & a/c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b - ad/c \\ c & a + d \end{bmatrix},
\]

respectively. If \( b = c = 0 \) and \( a \neq d \), then \( A \) is similar to

\[
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & d - a \\ 0 & d \end{bmatrix},
\]

and we can perform the step seen in the case \( b \neq 0 \) again.

Assume now that \( n > 3 \) and that the problem has been solved for all \( n' < n \). Let \( A = \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix} \), where \( A' \) is an \((n-1) \times (n-1)\) matrix. Clearly, we may assume that \( A' \neq \lambda I \), so that the induction provides a \( P \) with, say, \( P^{-1}A'P = \begin{bmatrix} 0 & * \\ * & \alpha \end{bmatrix} \). But then, the matrix

\[
B = \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P^{-1}A'P & * \\ * & \beta \end{bmatrix}
\]

is similar to \( A \), and its diagonal is \((0, 0, \ldots, 0, \alpha, \beta)\). On the other hand, we may also view \( B \) as \( \begin{bmatrix} 0 & * \\ * & C \end{bmatrix} \), where \( C \) is an \((n-1) \times (n-1)\) matrix with
diagonal \((0, 0, \ldots, 0, \alpha, \beta)\). If the inductive hypothesis is applicable to \(C\), we would have \(Q^{-1} CQ = D\), with \(D = \begin{bmatrix} 0 & * \\ * & \gamma \end{bmatrix}_n\), so that, finally, the matrix
\[
E = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} B \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} 0 & * \\ * & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & D \end{bmatrix}
\]
is similar to \(A\), and its diagonal is \((0, 0, \ldots, 0, 0, \gamma)\) as required.

The inductive argument can fail only when \(n - 1 = 2\), and the resulting matrix applying \(P\) has the form
\[
P^{-1} AP = \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix},
\]
where \(d \neq 0\). The numbers \(a, b, c, e\) cannot be 0 at the same time. If, say, \(b \neq 0\), then \(A\) is similar to
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -b & a & b \\ e & d & 0 \\ -b - d & a & b + d \end{bmatrix}.
\]
Performing the first half of the induction step again, the diagonal of the resulting matrix will be \((0, d - b, d + b)\) [the trace is the same] and the induction step can be finished. The cases \(a \neq 0, c \neq 0\) and \(e \neq 0\) are similar.

**Problem 6.** Suppose that the differentiable functions \(a, b, f, g : \mathbb{R} \to \mathbb{R}\) satisfy
\[
f(x) \geq 0, \quad f'(x) \geq 0, \quad g'(x) > 0 \quad \text{for all} \quad x \in \mathbb{R},
\]
\[
\lim_{x \to -\infty} a(x) = A > 0, \quad \lim_{x \to \infty} b(x) = B > 0, \quad \lim_{x \to -\infty} f(x) = \lim_{x \to \infty} g(x) = \infty,
\]
and
\[
f'(x) + a(x) \frac{f(x)}{g(x)} = b(x).
\]
Prove that
\[
\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 2 \frac{B}{A + 1}.
\]

**Solution.** Let \(0 < \epsilon < A\) be an arbitrary real number. If \(x\) is sufficiently large, then \(f(x) > 0, g(x) > 0, |a(x) - A| < \epsilon, |b(x) - B| < \epsilon, \) and
\[
B - \epsilon < b(x) = \frac{f'(x)}{g'(x)} + a(x) \frac{f(x)}{g(x)} < \frac{f'(x)}{g'(x)} + (A + \epsilon) \frac{f(x)}{g(x)} < \frac{(A + \epsilon)(A + 1)}{A} \frac{f'(x) (g(x))^A + A f(x) (g(x))^{A-1} g'(x)}{(A + 1) (g(x))^A g'(x)} = \frac{(A + \epsilon)(A + 1)}{A} \left( \frac{f'(x) (g(x))^A}{(g(x))^{A+1}} \right).
Thus,
\[
\frac{(f(x) (g(x))^A)''}{(g(x))^{A+1}} > \frac{A(B - \epsilon)}{(A + \epsilon)(A + 1)}.
\]
Similarly, it can be obtained that, for sufficiently large \( x \),
\[
\frac{(f(x) (g(x))^A)''}{(g(x))^{A+1}} < \frac{A(B + \epsilon)}{(A - \epsilon)(A + 1)}.
\]
From letting \( \epsilon \to 0 \), we have
\[
\lim_{\epsilon \to 0} \frac{(f(x) (g(x))^A)''}{(g(x))^{A+1}} = \frac{B}{A + 1}.
\]
By l'Hôpital's Rule, this implies that
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f(x) (g(x))^A}{(g(x))^{A+1}} = \frac{B}{A + 1}.
\]

**Day 2 Problems**

**Problem 1** Let \( r, s \geq 1 \) be integers and \( a_0, a_1, \ldots, a_r-1, b_0, b_1, \ldots, b_{s-1} \) be real non-negative numbers such that
\[
(a_0 + a_1 x + a_2 x^2 + \cdots + a_{r-1} x^{r-1} + x^r)(b_0 + b_1 x + b_2 x^2 + \cdots + b_{s-1} x^{s-1} + x^s)
\]
\[
= 1 + x + x^2 + \cdots + x^{r+s-1} + x^{r+s}.
\]
Prove that each \( a_i \) and each \( b_j \) equals either 0 or 1.

**Solution.** Multiply the left hand side polynomials. We obtain the following equalities:
\[
a_0 b_0 = 1, \quad a_0 b_1 + a_1 b_0 = 1, \quad \ldots.
\]
Amongst them, one can find the equations \( a_0 + a_1 b_{r-1} + a_2 b_{s-2} + \cdots = 1 \) and \( b_0 + b_1 a_{r-1} + b_2 a_{r-2} + \cdots = 1 \). From these equations, it follows that \( a_0, b_0 \leq 1 \).

Using \( a_0 b_0 = 1 \), we can see that \( a_0 = b_0 = 1 \).

Now, looking at the following equations, we notice that all \( a \)'s must be less than or equal to 1. The same holds for the \( b \)'s. It follows from the equation \( a_0 b_1 + a_1 b_0 = 1 \) that one of the numbers \( a_1, b_1 \) equals 0, while the other must be 1.

The rest of the proof is by induction.
Problem 2  Let \( a_0 = \sqrt{2}, \ b_0 = 2, \) and \( a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}, \) \( b_{n+1} = \frac{2b_n}{2 + \sqrt{4 + b_n^2}}. \)

(a) Prove that the sequences \( \{a_n\}, \ {b_n}\) are decreasing and converge to 0.

(b) Prove that the sequence \( \{2^n a_n\} \) is increasing, the sequence \( \{2^n b_n\} \) is decreasing and that these two sequences converge to the same limit.

(c) Prove that there is a positive constant \( C \) such that for all \( n \) the following inequality holds: \( 0 < b_n - a_n < \frac{C}{8^n}. \)

Solution. Clearly \( a_1 = \sqrt{2 - \sqrt{2}} < \sqrt{2}. \) Since the function \( f(x) = \sqrt{2 - \sqrt{4 - x^2}} \) is increasing on the interval \( [0, 2], \) the inequality \( a_1 > a_2 \) implies that \( a_2 > a_3. \) Simple induction ends the proof of the monotonicity of \( \{a_n\}. \) In the same way, we can prove that \( \{b_n\} \) decreases

\[
\text{just notice that } g(x) = \frac{2x}{2 + \sqrt{4 + x^2}} = \frac{2}{\frac{2}{x} + \sqrt{1 + \frac{4}{x^2}}}. \]

It is a matter of simple manipulation to prove that \( 2f(x) > x \) for all \( x \in (0, 2). \) This implies that the sequence \( \{2^n a_n\} \) is strictly increasing. The inequality \( 2g(x) < x \) for all \( x \in (0, 2) \) implies that the sequence \( \{2^n b_n\} \) is strictly decreasing. By an easy induction, one can show that \( a_n^2 = \frac{4b_n^2}{4 + b_n^2} \) for positive integers \( n. \) Since the limit of the decreasing sequence \( \{2^n b_n\} \) of positive numbers is finite, we have

\[
\lim 4^n a_n^2 = \lim \frac{4 \cdot 4^n b_n^2}{4 + b_n^2} = \lim 4^n b_n^2.
\]

Thus, we know that the limits \( \lim 2^n a_n \) and \( \lim 2^n b_n \) are equal. The first of the two is positive because the sequence of positive numbers, \( \{2^n a_n\}, \) is strictly increasing. The existence of a number \( C \) follows from the equalities

\[
2^n b_n - 2^n a_n = \frac{4^n b_n^2 - 4^{n+1} b_n^2}{2^n b_n + 2^n a_n} = \frac{(2^n b_n)^2}{4 + b_n^2} \cdot \frac{1}{4^n} \cdot \frac{1}{2^n (b_n + a_n)}
\]

and from the existence of positive limits \( \lim 2^n b_n \) and \( \lim 2^n a_n. \)

Remark. The last problem may be solved in a much simpler way by someone who is able to make use of the sine and cosine functions. It is sufficient to notice that \( a_n = \sin \left( \frac{\pi}{2^{n+1}} \right) \) and \( b_n = \tan \left( \frac{\pi}{2^{n+1}} \right). \)
Problem 3. Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater than $\sqrt{2}$.

Solution. The unit sphere in $\mathbb{R}^n$ is defined by

$$S_{n-1} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{k=1}^{n} x_k^2 = 1 \right\}.$$

The distance, $d(X, Y)$ between the points $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ is given by

$$d^2(X, Y) = \sum_{k=1}^{n} (x_k - y_k)^2.$$

We have

$$d(X, Y) > \sqrt{2} \iff d^2(X, Y) > 2 \iff \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 - 2 \sum_{k=1}^{n} x_k y_k > 2 \iff \sum_{k=1}^{n} x_k y_k < 0. \quad (1)$$

Because of the symmetry of the sphere, we may suppose that

$$A_1 = (-1, 0, \ldots, 0).$$

For $X = A_1$, condition (1) implies that $y_1 > 0 \forall Y \in M_n$.

Let $X = (x_1, \overline{x}), Y = (y_1, \overline{y}) \in M_n \setminus \{A_1\}, \overline{x}, \overline{y} \in \mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^{n} x_k y_k < 0 \iff x_1 y_1 + \sum_{k=1}^{n-1} \overline{x}_k \overline{y}_k < 0 \iff \sum_{k=1}^{n-1} \overline{x}_k \overline{y}_k < 0,$$

where

$$x'_k = \frac{\overline{x}_k}{\sqrt{\sum \overline{x}_k^2}}, \quad y'_k = \frac{\overline{y}_k}{\sqrt{\sum \overline{y}_k^2}}.$$

Therefore, $(x'_1, \ldots, x'_{n-1}), (y'_1, \ldots, y'_{n-1}) \in S_{n-2}$, and this verifies condition (1).

If $a_n$ is the number of points in $\mathbb{R}^n$ sought, we have $a_n \leq 1 + a_{n-1}$, and $a_1 = 2$ implies that $a_n \leq n + 1$. 
We show that $a_n = n + 1$, giving an example of a set in $M_n$ with $(n+1)$ elements satisfying the conditions of the problem.

\[
A_1 = (-1, 0, 0, \ldots, 0, 0) \\
A_2 = \left( \frac{1}{n}, -c_1, 0, \ldots, 0, 0 \right) \\
A_3 = \left( \frac{1}{n}, \frac{1}{n-1}c_1, -c_2, 0, \ldots, 0, 0 \right) \\
A_4 = \left( \frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, -c_3, \ldots, 0, 0 \right) \\
\vdots \\
A_{n-1} = \left( \frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \ldots, -c_{n-2}, 0 \right) \\
A_n = \left( \frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \ldots, \frac{1}{2}c_{n-2}, -c_{n-1} \right) \\
A_{n+1} = \left( \frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \ldots, \frac{1}{2}c_{n-2}, c_{n-1} \right),
\]

where

\[
c_k = \sqrt{\left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{n-k+1}\right)}, \quad k = 1, 2, \ldots, n - 1.
\]

We have

\[
\sum_{k=1}^{n} x_k y_k = -\frac{1}{n} < 0 \quad \text{and} \quad \sum_{k=1}^{n} x_k^2 = 1 \quad \forall \ X, Y \in \{A_1, \ldots, A_{n+1}\}.
\]

These points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points is equal to

\[
d = \sqrt{2} \sqrt{1 + \frac{1}{n}} > \sqrt{2}.
\]

Remark. For $n = 2$, the points form an equilateral triangle in the unit circle; for $n = 3$, the four points form a regular tetrahedron; and in $\mathbb{R}^n$, the points form an $n$-dimensional regular simplex.

Problem 4. Let $A = (a_{k,t})_{k,t=1,\ldots,n}$ be an $n \times n$ complex matrix such that for each $m \in \{1, \ldots, n\}$ and $1 \leq j_1 < \cdots < j_m \leq n$ the determinant of the matrix $(a_{j_0,j_i})_{k,t=1,\ldots,m}$ is zero. Prove that (1) $A^n = 0$ and (2) that there exists a permutation $\sigma \in S_n$ such that the matrix

\[
(a_{\sigma(k),\sigma(t)})_{k,t=1,\ldots,n}
\]

has all of its non-zero elements above the diagonal.

Solution. We shall prove only (2), since it implies (1).

Consider a directed graph $G$ with $n$ vertices $V_1, \ldots, V_n$, and a directed edge from $V_k$ to $V_t$, where $a_{k,t} \neq 0$. We shall prove that it is acyclic.
Assume that there exists a cycle and take one of minimum length $m$. Let $j_1 < \cdots < j_m$ be the vertices that the cycle goes through, and let $\sigma_0 \in S_n$ be a permutation such that $a_{j_k, j_{\sigma_0}(k)} \neq 0$ for $k = 1, \ldots, m$. Observe that for any other $\sigma \in S_n$, we have $a_{j_k, j_{\sigma}(k)} = 0$ for some $k \in \{1, \ldots, m\}$; for otherwise, we would obtain a different cycle through the same set of vertices and, consequently, a shorter cycle. Finally,

$$0 = \det (a_{j_k, j_t})_{k, t=1, \ldots, m}$$

$$= (-1)^{\text{sgn} \sigma_0} \prod_{k=1}^{m} a_{j_k, j_{\sigma_0}(k)} + \sum_{\sigma \neq \sigma_0} (-1)^{\text{sgn} \sigma} \prod_{k=1}^{m} a_{j_k, j_{\sigma}(k)} \neq 0,$$

which is a contradiction.

Since $G$ is acyclic, there exists a topological ordering; that is, a permutation $\sigma \in S_n$ such that $k < t$ whenever there is an edge from $V_{\sigma(k)}$ to $V_{\sigma(t)}$. It is easy to see that this permutation solves the problem.

**Problem 5.** Let $\mathbb{R}$ be the set of real numbers. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x + y) \geq f(x) + yf(f(x)) \text{ for all } x, y \in \mathbb{R}.$$

**Solution.** Suppose that there exists a function satisfying the inequality. If $f(f(x)) \leq 0$ for all $x$, then $f$ is a decreasing function because of the inequalities $f(x + y) \geq f(x) + yf(f(x)) \geq f(x)$ for any $y \leq 0$. Since $f(0) > 0 \geq f(f(x))$, we have $f(x) > 0$ for all $x$, which is a contradiction.

Hence, there is a $z$ such that $f(f(z)) > 0$. Then, the inequality $f(z + x) \geq f(z) + xf(f(z))$ shows that $\lim_{x \to \infty} f(x) = +\infty$, and, therefore, $\lim_{x \to \infty} f(f(x)) = +\infty$. In particular, there exist $x, y > 0$ such that $f(x) \geq 0, f(f(x)) > 1, y \geq \frac{x+1}{f(f(x))-1}$ and $f(f(x) + y + 1) \geq 0$. Then

$$f(f(x) + y) \geq f(x + y + 1) + (f(x + y) - (x + y + 1)) f(f(x) + y + 1) \geq f(x + y + 1) \geq f(x + y) + f(f(x + y)) \geq f(x) + yf(f(x)) + f(f(x + y)) \geq f(f(x) + y).$$

This contradiction completes the solution of the problem.

**Problem 6.** For each positive integer $n$, let $f_n(\theta) = \sin \theta \cdot \sin(2\theta) \cdot \sin(4\theta) \cdots \sin(2^n \theta)$.

For all real $\theta$ and all $n$, prove that

$$|f_n(\theta)| \leq \frac{2}{\sqrt{3}} |f_n(\pi/3)|.$$
Solution. We first prove that \( g(\theta) = |\sin \theta| |\sin(2\theta)|^{\frac{1}{2}} \) attains its maximum value \( \left( \frac{\sqrt{3}}{2} \right)^{\frac{3}{2}} \) at points \( 2^k \frac{\pi}{3} \) (where \( k \) is a positive integer). This can be seen by using derivatives of a classical bound like

\[
|g(\theta)| = |\sin \theta| |\sin(2\theta)|^{\frac{1}{2}} \\
= \frac{\sqrt{2}}{\sqrt{3}} \left( \sqrt{|\sin \theta| |\sin \theta| |\sin(2\theta)| |\sqrt{3} \cos \theta|} \right)^2 \\
\leq \frac{\sqrt{2}}{\sqrt{3}} \frac{3 \sin^2 \theta + 3 \cos^2 \theta}{4} = \left( \frac{\sqrt{3}}{2} \right)^{\frac{3}{2}}.
\]

Hence,

\[
\frac{f_n(\theta)}{f_n(\frac{\pi}{3})} = \frac{g(\theta) g(2\theta)^{\frac{1}{2}} g(4\theta)^{\frac{1}{2}} \ldots g(2^{n-1}\theta)^{E}}{g \left( \frac{\pi}{3} \right) g \left( \frac{2\pi}{3} \right)^{\frac{1}{2}} g \left( \frac{4\pi}{3} \right)^{\frac{1}{2}} \ldots g \left( \frac{2^{n-1}\pi}{3} \right)^{E}} \frac{\sin(2^n\theta)}{\sin \left( \frac{2^n\pi}{3} \right)}^{1-\frac{E}{2}} \\
\leq \frac{\sin \left( \frac{2^n\theta}{3} \right)}{\sin \left( \frac{2^n\pi}{3} \right)}^{1-\frac{E}{2}} \leq \left( \frac{1}{\sqrt{3}} \right)^{1-\frac{E}{2}} \leq \frac{2}{\sqrt{3}},
\]

where \( E = \frac{2}{3} \left( 1 - \left( -\frac{1}{2} \right)^n \right) \). This is exactly the bound that we were required to obtain.


4. Show that \( \lim_{n \to \infty} \frac{1}{n} \sqrt{\frac{(2n)!}{n!}} = \frac{4}{e} \).