SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2608*. [2001 : 49] Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Suppose that $x, y, z \geq 0$ and $x^2 + y^2 + z^2 = 1$. Prove or disprove that

(a) $1 \leq \frac{x}{1 - yz} + \frac{y}{1 - zx} + \frac{z}{1 - xy} \leq \frac{3\sqrt{3}}{2}$;

(b) $1 \leq \frac{x}{1 + yz} + \frac{y}{1 + zx} + \frac{z}{1 + xy} \leq \sqrt{2}$.

I. Solution to (a) by Michel Bataille, Rouen, France.

Let $S = \frac{x}{1 - yz} + \frac{y}{1 - zx} + \frac{z}{1 - xy}$. We show that $S \leq \frac{3\sqrt{3}}{2}$.

If one of $x, y$ or $z$ is 0, say, $x = 0$, then $S = y + z < 2 < \frac{3\sqrt{3}}{2}$.

Now, suppose that $xyz \neq 0$, so that $x, y, z \in (0, 1)$.

Note that $\frac{x}{1 - yz} = x + \frac{xyz}{1 - yz}$. Hence,

$$S = x + y + z + xyz \left(\frac{1}{1 - yz} + \frac{1}{1 - zx} + \frac{1}{1 - xy}\right).$$

Since

$$1 - yz \geq 1 - \frac{1}{2}(y^2 + z^2) = \frac{1}{2}(1 + x^2)$$

$$= \frac{1}{2}\left(2x^2 + y^2 + z^2\right) \geq 2\sqrt[4]{x^2y^2z^2} = 2x\sqrt{yz},$$

we have, using the AM–GM Inequality several times, that

$$xyz \left(\frac{1}{1 - yz} + \frac{1}{1 - zx} + \frac{1}{1 - xy}\right)$$

$$\leq \frac{xyz}{2} \left(\frac{1}{x\sqrt{yz}} + \frac{1}{y\sqrt{zx}} + \frac{1}{z\sqrt{xy}}\right)$$

$$= \frac{1}{2} \left(\sqrt{yz} + \sqrt{zx} + \sqrt{xy}\right) \leq \frac{1}{2} \left(\frac{y + z}{2} + \frac{z + x}{2} + \frac{x + y}{2}\right)$$

$$= \frac{1}{2}(x + y + z).$$

From (1) and (2), we have, using the Cauchy–Schwarz Inequality, that

$S \leq \frac{3}{2}(x + y + z) \leq \frac{3}{2} \left(1^2 + 1^2 + 1^2\right)^{\frac{1}{2}} (x^2 + y^2 + z^2)^{\frac{1}{2}} = \frac{3\sqrt{3}}{2}.$
II. Composite of essentially the same solution to (a) by Walther Janous, Ursulengymnasium, Innsbruck, Austria, and the proposers.

Let \( f(x, y, z) = \frac{x}{1 - yz} + \frac{y}{1 - zx} + \frac{z}{1 - xy} \). The left inequality is trivial since the given assumptions imply that \( 0 \leq x, y, z \leq 1 \), and thus, \( f(x, y, z) \geq x + y + z \geq x^2 + y^2 + z^2 = 1 \). Clearly, equality holds if and only if \( (x, y, z) = (1, 0, 0), (0, 1, 0) \) or \( (0, 0, 1) \).

For the right inequality, first note that \( 1 - yz \geq 1 - \frac{1}{2} (y^2 + z^2) = \frac{1}{2} (1 + x^2) \) and, hence, \( \frac{x}{1 - yz} \leq \frac{2x}{1 + x^2} \). Therefore,

\[
\tag{3}
f(x, y, z) \leq \frac{2x}{1 + x^2} + \frac{2y}{1 + y^2} + \frac{2z}{1 + z^2}.
\]

Next, we claim that

\[
\tag{4}
\frac{2x}{1 + x^2} \leq \frac{3}{8} \sqrt{3} (1 + x^2).
\]

To show that (4) holds, note that

\[
(1 + x^2)^2 - \frac{16}{9} \sqrt{3} x = \frac{1}{9} \left( 9x^4 + 18x^2 - 16\sqrt{3} x + 9 \right)
\]

\[
= \frac{1}{9} \left( 3x^2 - 2\sqrt{3} x + 1 \right) \left( 3x^2 + 2\sqrt{3} x + 9 \right)
\]

\[
= \frac{1}{9} \left( \sqrt{3} x - 1 \right)^2 \left( 3x^2 + 2\sqrt{3} x + 9 \right) \geq 0.
\]

Hence, (4) holds, and we have equality if and only if \( x = \frac{\sqrt{3}}{3} \).

Summing (4) with the two corresponding inequalities in \( y \) and \( z \) then yields

\[
\frac{2x}{1 + x^2} + \frac{2y}{1 + y^2} + \frac{2z}{1 + z^2} \leq \frac{3}{8} \sqrt{3} (3 + x^2 + y^2 + z^2) = \frac{3}{2} \sqrt{3} \tag{5}
\]

From (3) and (5), we conclude that \( f(x, y, z) \leq \frac{3}{2} \sqrt{3} \), with equality if and only if \( x = y = z = \frac{\sqrt{3}}{3} \).

III. Solution to the left inequality in (b) by Walther Janous, Ursulengymnasium, Innsbruck, Austria.

Note first that

\[
x + xyz \leq x + \frac{1}{2} x (y^2 + z^2) = x + \frac{1}{2} x (1 - x^2) = \frac{1}{2} (3x - x^3) \leq 1,
\]

since \( x^3 - 3x + 2 = (x - 1)^2(x + 2) \geq 0 \).
Hence, with all summations being cyclic, we have
\[
\sum \frac{x}{1+yz} = \sum \frac{x^2}{x+xyz} \geq \sum x^2 = 1.
\]

IV. Solution to the right inequality in (b) by the proposers, modified slightly by the editor.

Due to complete symmetry in \(x, y\) and \(z\), we may assume, without loss of generality, that \(x \leq y \leq z\). Then,
\[
\frac{x}{1+yz} + \frac{y}{1+zx} + \frac{z}{1+xy} \leq \frac{x+y+z}{1+xy}.
\]

Hence, it suffices to prove that \(\frac{x+y+z}{1+xy} \leq \sqrt{2}\), or
\[
x + y + z - \sqrt{2}xy \leq \sqrt{2}.
\]

We now use the method of Lagrange Multipliers to determine the extreme values of the function \(g(x, y, z) = x + y + z - \sqrt{2}xy\) in the region
\[
B = \{(x, y, z) \mid x, y, z \geq 0 \text{ and } x^2 + y^2 + z^2 = 1\}.
\]

We let \(G(x, y, z) = x + y + z - \sqrt{2}xy - \lambda (x^2 + y^2 + z^2 - 1)\), and set
\[
\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = \frac{\partial G}{\partial z} = 0.
\]

Then, we have
\[
\begin{align*}
1 - \sqrt{2}y - 2\lambda x &= 0, \quad (7) \\
1 - \sqrt{2}x - 2\lambda y &= 0, \quad (8) \\
1 - 2\lambda z &= 0, \quad (9) \\
x^2 + y^2 + z^2 &= 1. \quad (10)
\end{align*}
\]

From (7) and (8), we get \((\sqrt{2} - 2\lambda)(x - y) = 0\), and hence, either \(\lambda = \frac{\sqrt{2}}{2}\) or \(x = y\).

If \(\lambda = \frac{\sqrt{2}}{2}\), then \(z = \frac{\sqrt{2}}{2}\) from (9). Hence, from (10), we get
\[
x^2 + y^2 = \frac{1}{2}. \quad (11)
\]

On the other hand, from (7), we have \(1 - \sqrt{2}(x + y) = 0\), and hence,
\[
x + y = \frac{\sqrt{2}}{2}. \quad (12)
\]

From (11) and (12), we easily have \(x = 0\) and \(y = \frac{\sqrt{2}}{2}\).

Clearly, equality holds in (6) when \(x = 0\) and \(y = z = \frac{\sqrt{2}}{2}\).
If \(x = y\), then (6) becomes
\[
2x + z - \sqrt{2}x^2 \leq \sqrt{2}.
\] (13)
Since \((2x + z)^2 \leq 2(4x^2 + z^2)\), we have
\[
2x + z - \sqrt{2}x^2 \leq \sqrt{2}\sqrt{4x^2 + z^2} - \sqrt{2}x^2,
\]
and hence, (13) is true if
\[
\sqrt{4x^2 + z^2} \leq x^2 + 1,
\]
or
\[
2x^2 + x^2 + y^2 + z^2 \leq (x^2 + 1)^2,
\]
or
\[
2x^2 + 1 \leq x^4 + 2x^2 + 1,
\]
which is clearly true. This shows that (13) is true, and hence, (6) holds.

Finally, we check the points on the boundary of \(B\). Without loss of generality, we may assume that \(x = 0\). Then,
\[
x + y + z - \sqrt{2}xy = y + z \leq \sqrt{2}(y^2 + z^2) = \sqrt{2}.
\]
Hence, (6) holds.

Therefore, the given inequality is valid, and it is easy to see that equality holds for \((x, y, z) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\), or \(\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\), or \(\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)\).

Also solved (both parts) by MANUEL BENITO and EMILIO FERNÁNDEZ, I. B. Práxedes Mateo Sagasta, Logroño, Spain. Part (a) only was also solved by NIKOLAOS DERGIADIES, Thessaloniki, Greece; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

There were also two incomplete solutions, both of which employed the method of Lagrange Multipliers, but claimed, without proof, that the solutions to \(\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = \frac{\partial G}{\partial z} = 0\) must be \(x = y = z\) by symmetry.


For natural numbers \(n\), define functions \(f\) and \(g\) by \(f(n) = \left\lfloor \frac{n}{\sqrt{n}} \right\rfloor\)
and \(g(n) = \left\lfloor \frac{n}{\sqrt{n}} \right\rfloor\). Determine all possible values of \(f(n) - g(n)\), and characterize all those \(n\) for which \(f(n) = g(n)\). [See [2000 : 197], Q. 8.]
Solution by David Loeffler, student, Cotham School, Bristol, UK.

If \( n \) is a perfect square, say \( n = m^2 \), then \( f(m^2) = g(m^2) = m \). If this is not the case, set \( n = m^2 + k \), where \( 0 < k < 2m + 1 \). We then have

\[
\begin{align*}
    f(m^2 + k) &= \left\lfloor \frac{m^2 + k}{\sqrt{m^2 + k}} \right\rfloor = \left\lfloor \frac{m^2 + k}{m} \right\rfloor = m + \left\lfloor \frac{k}{m} \right\rfloor \\
    g(m^2 + k) &= \left\lceil \frac{m^2 + k}{\sqrt{m^2 + k}} \right\rceil = \left\lceil \frac{m^2 + k}{m + 1} \right\rceil = m - 1 + \left\lfloor \frac{k+1}{m+1} \right\rfloor 
\end{align*}
\]

If \( k < m \), then \( \left\lfloor \frac{k}{m} \right\rfloor = 0 \) and \( \left\lfloor \frac{k+1}{m+1} \right\rfloor = 1 \) and we get \( f(m^2 + k) = g(m^2 + k) = m \).

If \( k = m \), then both fractions are 1, which implies \( f(m^2 + k) = m + 1 \) and \( g(m^2 + k) = m \).

If \( m < k < 2m \), then we have \( 1 < \frac{k}{m} < 2 \), so that \( \left\lfloor \frac{k}{m} \right\rfloor = 1 \) and \( f(m^2 + k) = m + 1 \); this also implies \( m + 1 < k + 1 < 2m + 1 < 2(m + 1) \), implying \( 1 < \frac{k+1}{m+1} < 2 \). Thus \( \left\lfloor \frac{k+1}{m+1} \right\rfloor = 2 \) and \( g(m^2 + k) = m + 1 \).

If \( k = 2m \), then we have \( \left\lfloor \frac{k}{m} \right\rfloor = 2 \) and \( \left\lfloor \frac{k+1}{m+1} \right\rfloor = 2 \), yielding \( f(m^2 + k) = m + 2 \) and \( g(m^2 + k) = m + 1 \).

Thus, \( f(n) - g(n) \) is always either 0 or 1. Also, \( f(n) = g(n) \) for all integers \( n \) which are not of the form \( m^2 + m \) or \( m^2 + 2m \) for some integer \( m \), (that is, those \( n \) for which neither \( n + 1 \) nor \( 4n + 1 \) are perfect squares).

**Comment:** We may extend the problem by defining

\[
h(n) = \left\lfloor \frac{n}{\sqrt{n}} \right\rfloor \quad \text{and} \quad j(n) = \left\lceil \frac{n}{\sqrt{n}} \right\rceil.
\]

This case may be analysed in exactly the same way as above; we find that \( h(n) - j(n) \in \{0, 1, 2\} \), with \( h(n) = j(n) \) if and only if \( n \) is a perfect square. Also \( h(n) - j(n) = 1 \) if and only if \( n = m^2 + m \) for some \( m \).

Also solved by SEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHARLES R. DILMINIEN, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; OLEG IVRIU, Cumber Valley Middle School, Toronto, Ont, WALTHER JANOUSH, Ursulinegymnasium, Innsbruck, Austria; PAUL JEFFREYS, student, Berkhamsted Collegiate School, UK; KEE-WAI LAU, Hong Kong; CRAIG CHAPMAN, RICHARD CRAMMER, students, and CARL LIBIS, Richard Stockton Collegiate of HJ, Pomona, NJ; HENRY LIU, student, University of Memphis, TN; WILLIAM MOSER, McGill University, Montreal, Que; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TREY SMITH, Angelo State University, San Angelo, TX; SOUTHWEST MISSOURI STATE PROBLEM SOLVING GROUP; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; LI ZHOU, Polk Community College, Winter Haven, FL; and the proposer.

Zhou points out that the above solution also shows that \( f(n) > f(n + 1) \) if and only if \( n = k^2 - 1 \), which answers problem 8 from the Olympiad Corner from 2000, p. 197. Seiffert also considered the extended problem defined in Loeffler's comment.
2621. [2001: 137] Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan, and Bruce Shawyer, Memorial University of Newfoundland, St. John's, Newfoundland, dedicated to Murray S. Klamkin, on his 80th birthday.

You are given:

(a) fixed real numbers \( \lambda \) and \( \mu \) in the open interval \((0, 1)\);

(b) circle \( ABC \) with fixed chord \( AB \), variable point \( C \), and points \( L \) and \( M \) on \( BC \) and \( CA \), respectively, such that \( BL : LC = \lambda : (1 - \lambda) \) and \( CM : MA = \mu : (1 - \mu) \);

(c) \( P \) is the intersection of \( AL \) and \( BM \).

Find the locus of \( P \) as \( C \) varies around the circle \( ABC \). (If \( \lambda = \mu = \frac{1}{2} \), it is known that the locus of \( P \) is a circle.)

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let the lines \( CP \) and \( AB \) intersect at \( N \). Then by Ceva's Theorem,

\[
\frac{AN}{NB} = \frac{AM}{MC} \cdot \frac{CL}{LB} = \frac{(1 - \mu)(1 - \lambda)}{\mu \lambda},
\]

which is a constant. Hence, \( N \) is a fixed point; moreover,

\[
\frac{AN}{AB} = \frac{(1 - \mu)(1 - \lambda)}{(1 - \mu)(1 - \lambda) + \mu \lambda}.
\]

By Menelaus's Theorem applied to \( \triangle NBC \) with line \( AL \),

\[
\frac{NP}{PC} = \frac{NA}{AB} \cdot \frac{BL}{LC} = \frac{(1 - \mu) \lambda}{(1 - \mu)(1 - \lambda) + \mu \lambda}.
\]

It follows that

\[
\frac{NP}{NC} = \frac{(1 - \mu) \lambda}{(1 - \mu)(1 - \lambda) + \mu \lambda + (1 - \mu) \lambda} = \frac{(1 - \mu) \lambda}{1 - \mu + \mu \lambda}.
\]
is a constant. Hence, as $C$ travels along circle $ABC$, $P$ will travel along a circle whose circumference is divided by line $AB$ into the same ratio as is the original circle; specifically, the locus of $P$ is obtained from the locus of $C$ by a dilatation centred at $N$ with ratio of magnitude

$$\frac{NP}{NC} = \frac{(1-\mu)\lambda}{1-\mu + \mu \lambda}.$$ 

QED.

**Editor's comments.** Technically speaking, $P$ is not defined for positions of $C$ at $A$ or $B$, so the locus of $P$ is a circle minus its two points on $AB$. Two solvers noted that more generally, were $C$ constrained to move about an arbitrary curve, the above argument establishes that $P$ would trace out a homothetic image of that curve shrunk by the factor

$$\frac{NP}{NC} = \frac{(1-\mu)\lambda}{1-\mu + \mu \lambda}.$$ 

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEEK, Zaltbommel, the Netherlands; and by the proposers.

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2622. [2001 : 139] **Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.**

Find the exact value of \(\sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)
(\begin{scriptsize}n\end{scriptsize})}.\)

A combination of almost identical solutions by Jan Ciach, Ostrowiec Świętokrzyski, Poland and Kee-Wai Lau, Hong Kong, China.

From the identity \((\arcsin x)^2 = \sum_{n=0}^{\infty} \frac{2^{n+1}(n!)^2 x^{2n+2}}{(2n+1)!(n+1)}\), which is valid for \(|x| \leq 1\), we have, by differentiation,

\[
4x \frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{2^{n+2}(n!)^2 x^{2n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(2x)^{2n+2}}{(2n+1)(\begin{scriptsize}2n\end{scriptsize})}.
\]

By taking \(x = \frac{\sqrt{2}}{2}\), we conclude that \(\sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)(\begin{scriptsize}2n\end{scriptsize})} = \pi\).

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; MURRAY S. KLAAMKIN, University of Alberta, Edmonton, Alberta; CARL LIFS, Richard Stockton College of NJ, Pomona, NJ, USA; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; DAVID E. MANES, SUNY at Oneonta,

Let $n$ black objects and $n$ white objects be placed on the circumference of a circle, and define any set of $m$ consecutive objects from this cyclic sequence to be an $m$-chain.

(a) Prove that, for each natural number $k \leq n$, there exists at least one $2k$-chain consisting of $k$ black objects and $k$ white objects.

(b) Prove that, for each natural number $k \leq \sqrt{2n + 5} - 2$, there exist at least two such disjoint $2k$-chains.

Solution by Elsie Campbell and Trey Smith, Angelo State University, San Angelo, TX, USA.

For ease of notation, a $2k$-chain will be called good if it consists of $k$ black and $k$ white objects. We may label the objects $M(1), M(2), \ldots, M(2n)$ where $M(1)$ is arbitrarily chosen and the objects are taken in a clockwise order. Then we observe that for $k \leq n$ there are $2n$ different $2k$-chains which will be labelled

$$C(1) = (M(1), M(2), \ldots, M(2k))$$
$$C(2) = (M(2), M(3), \ldots, M(2k + 1))$$
$$\vdots$$
$$C(2n) = (M(2n), M(1), \ldots, M(2k - 1)).$$

For each chain $C(i)$ let $W(i)$ and $B(i)$ be the number of white objects and the number of black objects, respectively, in the chain.

Notice that each object will be in exactly $2k$ different $2k$-chains. Since there are $n$ white objects we have that

$$W(1) + W(2) + \cdots + W(2n) = n \cdot 2k = 2kn.$$ 

We may now prove:

Theorem (a): Fix natural numbers $n$ and $k$ with $k \leq n$. Then there exists at least one good $2k$-chain.
Proof: Suppose that no good $2k$-chain exists. We must derive a contradiction. Arbitrarily fix a $2k$-chain $C(1)$. We may assume, without loss of generality, that $W(1) > B(1)$. Since there are no good $2k$-chains, we conclude that for all natural numbers $i$ such that $1 \leq i \leq 2n$ we have $W(i) > B(i)$. If this were not the case, then we could let $j$ be the least natural number such that $W(j) < B(j)$. Then $W(j - 1) > B(j - 1)$, but this is impossible since $C(j)$ could have at most 1 white object less than, and 1 more black object, than $C(j - 1)$, and

$$W(j) < B(j) \implies W(j) + 2 \leq B(j),$$

since $W(i)$ and $B(i)$ have the same parity for all $i$, $1 \leq i \leq 2n$. Thus, we have $W(i) \geq k + 1$ for every $i$, $1 \leq i \leq 2n$. This implies that

$$W(1) + W(2) + \cdots + W(2n) \geq 2n(k + 1) > 2kn,$$

which is a contradiction. Therefore, there is a good $2k$-chain.

Theorem (b): Fix natural numbers $k$ and $n$ with $k \leq \sqrt{2n + 5} - 2$. Then there exist at least two good disjoint $2k$-chains.

Proof: If $2n - 4k + 1 < 0$, then

$$2n < 4k - 1 \implies 2n + 5 \leq 4k + 4$$

$$\implies 2n + 5 < k^2 + 4k + 4$$

$$\implies \sqrt{2n + 5} < k + 2$$

$$\implies \sqrt{2n + 5} - 2 < k,$$

which is a contradiction. Thus we may assume (since $k$ and $n$ are integers) $2n - 4k + 1 > 0$.

By the previous theorem we may fix a good $2k$-chain and label it $C(1)$. Observe that there are $2n - 4k + 1$ many $2k$-chains which are disjoint from $C(1)$. They are, in particular,

$$C(2k + 1), C(2k + 2), \ldots, C(2n - 2k), C(2n - 2k + 1).$$

Now suppose that none of these are good. We must derive a contradiction. By the same argument as that used in the previous theorem, it cannot be the case that some of the chains have more black objects than white objects while others have more white objects than black objects. We may assume, without loss of generality, that all of the chains have at least $k + 1$ many whites. Thus,

$$W(2k + 1) + W(2k + 2) + \cdots + W(2n - 2k + 1) \geq (k + 1)(2n - 4k + 1).$$

In addition, each of the $k$ white objects in $C(1)$ will be in $2k$ many $2k$-chains. Hence the sum of the white objects from $C(1)$ that are in the chains

$$C(2n - 2k + 2), \ldots, C(2n), C(1), \ldots, C(2k)$$

is $k \cdot 2k = 2k^2$. 


We must, finally, sum up all of the white objects that fall outside of
\(C(1)\) from the chains
\[C(2n - 2k + 2), \ldots, C(2n), C(1), \ldots, C(2k).\]
Since \(W(2k + 1) \geq k + 1\), then
\[
\begin{align*}
C(2k) & \quad \text{contains at least} \ k \ \text{white objects outside of} \ C(1), \\
C(2k - 1) & \quad \text{contains at least} \ k - 1 \ \text{white objects outside of} \ C(1), \\
& \quad \ldots \quad \ldots \\
C(2k - (k - 1)) & \quad \text{contains at least} \ 1 \ \text{white object outside of} \ C(1),
\end{align*}
\]
for a total of at least \(\frac{1}{2}k(k + 1)\). Similarly, the sum of the white objects
outside of \(C(1)\) from
\[C(2n - 2k + 2), C(2n - 2k + 3), \ldots, C(2n)\]
is at least \(\frac{1}{2}k(k + 1)\).
Using the fact that \(W(1) + W(2) + \cdots + W(2n) = 2kn\), we get
\[
\begin{align*}
(k + 1)(2n - 4k + 1) + 2k^2 + \frac{1}{2}k(k + 1) + \frac{1}{2}k(k + 1) & \leq 2kn \\
2kn + 2n - k^2 - 2k + 1 & \leq 2kn \\
2n & \leq k^2 + 2k - 1 \\
2n & < k^2 + 4k - 1 \\
2n + 5 & < k^2 + 4k + 4 \\
\sqrt{2n + 5} - 2 & < k,
\end{align*}
\]
which is a contradiction. Thus, there are at least two good disjoint \(2k\)-chains.

Also solved by HENRY LIU, student, University of Memphis, Memphis, TN, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Part (a) only was solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and JOEL SCHLOSBERG, student, New York University, NY, USA.

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